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A C*-ALGEBRAIC SCHOENBERG THEOREM

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0. Introduction.

Let $\mathfrak{A}$ be a C*algebra. A linear operator $\delta\,;\ D(\delta) \longmapsto \mathfrak{A}$ is defined to be a *derivation* if is satisfies the following three properties:

a) the domain $D(\delta)$ is a norm-dense *-subalgebra of $\mathfrak{A}$,

b) $\delta(X^*) = \delta(X)^*\,,\ X \in D(\delta)$,

c) $\delta(XY) = \delta(X)\,Y + X\delta(X)\,,\ X, Y \in D(\delta)$.

Similarly δ is defined to be a *dissipation* if it satisfies properties a) and b) together with

c′) $\delta(X^*X) \leqslant \delta(X^*)\,X + X^*\delta(X)\,,\ X \in D(\delta)$.

The operator δ is defined to be a *complete dissipation* if it satisfies properties a) and b) together with the matrix inequalities

c″) $[\delta(X_i^*X_j)] \leqslant [\delta(X_i)^*X_j + X_i^*\delta(X_j)]$

for all finite sequences $X_1, \ldots, X_n$ in $D(\delta)$. The study of derivations and dissipations is mainly motivated by the following facts.

If δ is a bounded linear operator on $\mathfrak{A}$, then:

1) $t \in \mathbf{R} \mapsto \alpha_t = \exp\{-t\delta\}$ is a uniformly continuous one-parameter group of *-automorphisms of $\mathfrak{A}$ if, and only if, δ is a derivation,

2) $t \in \mathbf{R}_+ \longmapsto \alpha_t = \exp\{-t\delta\}$ is a uniformly continuous one-parameter contraction semigroup of symmetric maps, i.e., $\alpha_t(X)^* = \alpha_t(X^*)$, $X \in \mathfrak{A}$, which are strongly positive, i.e., $\alpha_t(X^*X) \geqslant \alpha_t(X)^*\alpha_t(X)$, $X \in \mathfrak{A}$, if, and only if, δ is a dissipation,

3) $t \in \mathbf{R}_+ \longmapsto \alpha_t = \exp\{-t\delta\}$ is a uniformly continuous semigroup of completely positive contractions, i.e., $\alpha_t \otimes \iota_n$; $\mathfrak{A} \otimes M_n \longmapsto \mathfrak{A} \otimes M_n$ is positive for all $n \in \mathbf{N}$, and $\|\alpha_t\| \leqslant 1$, if, and only if, δ is a complete dissipation.

The first statement is easily verified. The second is less obvious; it is a result of Evans and Hanche-Olsen [15]. The third follows from the second together with results by Evans [14].

The principal problem in the theory of derivations is to characterize those derivations which generate strongly continuous one-parameter groups of *-automorphisms. This problem has attracted a great deal of attention in the last decade (see, for example, [9] Chapter 3 or [24] [7]). An analogous problem is to characterize those complete dissipations which generate strongly continuous one-parameter contraction semigroups of completely positive maps. In this paper we examine this latter problem in a more restricted framework of "non-commutative harmonic analysis" and extend some results obtained in [6].

The extra ingredient for this restricted setting is a compact abelian group G represented by a strongly continuous group τ_G of *-automorphisms of $\mathfrak{A}$. If $\mathfrak{A}^\tau$ denotes the fixed point algebra of τ in $\mathfrak{A}$ then one has the following result for derivations.

THEOREM [8]. – *If δ is a closed derivation on $\mathfrak{A}$ such that*

1) $D(\delta)$ *is τ-invariant and*

$$\delta(\tau_g(X)) = \tau_g(\delta(X)), X \in D(\delta), \ g \in G,$$

2) $\mathfrak{A}^\tau$ *is contained in* $D(\delta)$ *and* $\delta(X) = 0$, $X \in \mathfrak{A}^\tau$,

*then δ is the generator of a strongly continuous one-parameter group of *-automorphisms of $\mathfrak{A}$.*

Our main result is a direct analogue of this theorem for complete dissipations and completely positive semigroups. It can also be viewed as a generalization of Schoenberg's theorem [25] [4], in classical harmonic analysis. This classical setting is given by choosing $\mathfrak{A} = C(G)$, the continuous functions over G with the supremum norm, and considering a negative definite function $\psi : \hat{G} \longrightarrow \mathbf{C}$ where $\hat{G}$ is the dual of G. The function ψ defines a linear operator δ_ψ on C(G) by the definition

$$\widehat{(\delta_\psi(f))}(\gamma) = \psi(\gamma)\hat{f}(\gamma), \ \gamma \in \hat{G},$$

where $\hat{f}$ is the Fourier transform of $f \in \mathfrak{A}$. Then

$$(\delta_\psi(\bar{f})f + \bar{f}\delta_\psi(f) - \delta_\psi(\bar{f}f))(g)$$
$$= \sum_{\xi,\eta\in\hat{G}} \overline{\hat{f}(\xi)\langle\xi,g\rangle}\,[\overline{\psi(\xi)} + \psi(\eta) - \psi(\eta-\xi)]\,\hat{f}(\eta)\langle\eta,g\rangle$$
$$\geqslant 0, \; g \in G,$$

because ψ is negative-definite. But Schoenberg's theorem states that if ψ is a function with $\psi(0) = 0$ then ψ is negative-definite if, and only if, $\gamma \longmapsto \exp\{-t\psi(\gamma)\}$ is positive-definite for all $t \geqslant 0$. A straightforward calculation shows, however, that this latter condition is equivalent to positivity of the semigroup $t \longmapsto \exp\{-t\delta_\psi\}$, or, by the generalized Schwarz inequality, to the strong positivity requirements

$$(e^{-t\delta_\psi}(\bar{f}f))(g) \geqslant (e^{-t\delta_\psi}(\bar{f}))(g)\,(e^{-t\delta_\psi}(f))(g).$$

In fact these inequalities follow from negative-definiteness of ψ and Schur's theorem (see the proof of Lemma 1.7), and conversely negative-definiteness of ψ follows from the inequalities by differentiation.

In fact negative definiteness of ψ implies that δ_ψ is a complete dissipation as expressed by the matrix inequalities

$$[(\delta_\psi(\bar{f}_i)f_j + \bar{f}_i\delta_\psi(f_j) - \delta_\psi(\bar{f}_i f_j))(g)] \geqslant 0$$

where $f_1, f_2, \ldots, f_n$ is any finite sequence in the domain of δ_ψ. This in turn implies complete positivity of the semigroup $t \longmapsto \exp\{-t\delta_\psi\}$, i.e.,

$$[(e^{-t\delta_\psi}(\bar{f}_i f_j) - e^{-t\delta_\psi}(\bar{f}_i)\,e^{-t\delta_\psi}(f_j))(g)] \geqslant 0.$$

Our main theorem proves an equivalence between such conditions for operators and semigroups acting on non-abelian C*-algebras. In this context it should be remarked that every positive map on an abelian C*-algebra is automatically completely positive and each dissipation is a complete dissipation, (see Remark 1 after the main theorem) but this is not necessarily the case for non-abelian C*-algebras, as our Theorem 2 indicates.

Finally we note that Akemann and Walter [1] have studied negative definite functions on non-abelian groups G. In particular they associated to each such function ψ a dissipation δ_ψ, by an extension of the above construction for abelian G, and used the existence of this operator to obtain structural results for G.

1. A C*-algebraic Schoenberg Theorem.

We adopt the notation of [8], except the representation of G as automorphisms of $\mathfrak{A}$ is denoted by τ instead of α. For example $\mathfrak{A}^\tau(\gamma)$ denotes the spectral subspace of τ corresponding to $\gamma \in \hat{G}$,

$$\mathfrak{A}^\tau(\gamma) = \{X \in \mathfrak{A} \, ; \, \tau_g(X) = \langle \gamma, g \rangle X \text{ for all } g \in G\},$$

and $\mathfrak{A}^\tau = \mathfrak{A}^\tau(0)$ is the fixed point subalgebra. Moreover $\mathfrak{A}_F$ denotes the linear span of the $\mathfrak{A}^\tau(\gamma)$.

THEOREM 1. – *Let* G *be a compact abelian group,* τ *an action of* G *on the* C*-*algebra* $\mathfrak{A}$, *and* $\mathcal{D}$ *a* τ-*invariant dense*-subalgebra of* $\mathfrak{A}$ *such that* $\mathfrak{A}^\tau \subseteq \mathcal{D}$ *and* $\mathcal{D}$ *is the linear span of the subspaces* $\mathcal{D} \cap \mathfrak{A}^\tau(\gamma)$, $\gamma \in \hat{G}$. *Further let* $H : \mathcal{D} \longrightarrow \mathfrak{A}$ *be a linear operator which is symmetric, i.e.,* $H(X^*) = H(X)^*$, $X \in \mathcal{D}$, *which satisfies* $H(X) = 0$ *for all* $X \in \mathfrak{A}^\tau$, *and is such that* $H(\tau_g(X)) = \tau_g(H(X))$ *for all* $X \in \mathcal{D}$, $g \in G$.

The following conditions are equivalent:

1) The matrix inequality

$$[H(X_i)^*X_j + X_i^* H(X_j) - H(X_i^* X_j)] \geqslant 0$$

holds in $\mathfrak{A} \otimes M_n$, *for all finite sequences* $X_1, X_2, \ldots, X_n \in \mathcal{D}$.

2) The operator H *is closable and its closure* $\bar{H}$ *generates a* C_0-*semigroup* S *of contractions which is completely positive, i.e., the matrix inequality*

$$[S_t(X_i^* X_j) - S_t(X_i)^* S_t(X_j)] \geqslant 0$$

holds for all finite sequences $X_1, X_2, \ldots, X_n \in \mathcal{D}$.

A special case of this result was proved in [6]. Before turning to the proof we make a series of comments on its assumptions and conclusions.

Remark 1. – If $\mathfrak{A}$ is an abelian C*-algebra and $H : \mathcal{D} \longrightarrow \mathfrak{A}$ a dissipation defined on a dense*-subalgebra $\mathcal{D}$ then H is automatically a complete dissipation. To prove this one first argues that one may reduce to the case that $\mathfrak{A} = C(\Omega)$, where Ω is a compact Hausdorff space, and $1 \in \mathcal{D}$. Then if

$$Q_{ij} = H(X_i)^*X_j + X_i^*H(X_j) - H(X_i^*X_j) \text{ and } X = \sum_i \lambda_i X_i$$
with $\lambda_i \in \mathbf{C}$ one has

$$\sum_{i,j} \bar{\lambda}_i \lambda_j Q_{ij}(\omega) = (H(X)^*X + X^*H(X) - H(X^*X))(\omega) \geqslant 0,$$

i.e., the scalar-valued matrix $Q(\omega) = [Q_{ij}(\omega)]$ is positive for each $\omega \in \Omega$. Therefore $\rho(Q(\omega)) \geqslant 0$ for each state ρ on M_n. But every pure state σ on $\mathfrak{A} \otimes M_n$ has the form $\sigma = \delta_\omega \otimes \rho$ where δ_ω is the point measure at $\omega \in \Omega$. Consequently $\sigma([Q_{ij}]) \geqslant 0$, and $[Q_{ij}] \geqslant 0$ in $\mathfrak{A} \otimes M_n$. (The basic idea is in [19], [29], and [3].)

Remark 2. – Let
$$P_\gamma : X \longmapsto \int_G dg\, \overline{\langle \gamma, g \rangle}\, \tau_g(X)$$
denote the projection from $\mathfrak{A}$ onto $\mathfrak{A}^\tau(\gamma)$. Since $\mathcal{D}$ is assumed to be the linear span of $\mathcal{D} \cap \mathfrak{A}^\tau(\gamma)$, it follows, from the uniqueness of the Fourier decomposition, that $P_\gamma(\mathcal{D}) \subset \mathcal{D}$ for each $\gamma \in \hat{G}$ and hence $\mathcal{D} \cap \mathfrak{A}^\tau(\gamma) = P_\gamma(\mathcal{D})$ is dense in $\mathfrak{A}^\tau(\gamma)$ for each $\gamma \in \hat{G}$.

Remark 3. – It would be interesting if one could replace the hypothesis on $\mathcal{D}$ by the alternative hypothesis; $\mathcal{D}$ is a τ-invariant dense *-subalgebra of $\mathfrak{A}$, or $\mathfrak{A}_F$, and H is closable. Indeed from this latter hypothesis and the commutation condition, Condition 2 of the theorem, it follows that $P_\gamma(D(\bar{H})) \subseteq D(\bar{H})$ for all $\gamma \in \hat{G}$, and thus $D(\bar{H}) \cap \mathfrak{A}^\tau(\gamma)$ is dense in $\mathfrak{A}^\tau(\gamma)$ for each $\gamma \in \hat{G}$. It is not clear, however, that the matrix inequalities of Condition 1 extend to $D(\bar{H}) \cap \mathfrak{A}_F$, and it is not even clear that $D(\bar{H}) \cap \mathfrak{A}_F$ is an algebra.

Remark 4. – Note that if $\mathfrak{A}$ has an identity $\mathbf{1}$ and if the ideals $\mathfrak{A}^\tau(\gamma)^*\mathfrak{A}^\tau(\gamma)$ are dense in $\mathfrak{A}^\tau$ for each $\gamma \in \hat{G}$ then a simple argument shows that any $\mathcal{D}$ satisfying the hypothesis of the theorem must be equal to $\mathfrak{A}_F$. This is no longer the case if $\mathfrak{A}$ does not have an identity; a counterexample is given by a minor modification of Examples 6.1 and 6.2 in [8].

Remark 5. – In the proof of the theorem we also give a classification of all completely positive C_0-semigroups on $\mathfrak{A}$ which commute with τ_G and restrict to the identity on the fixed point algebra $\mathfrak{A}^\tau$. This classification is in terms of maps from $\hat{G}$ onto unbounded operators affiliated with the centres of the multiplier algebras of the ideals $\mathfrak{A}^\tau(\gamma)\mathfrak{A}^\tau(\gamma)^*$ in $\mathfrak{A}^\tau$ [22]. If $\mathfrak{A}^\tau(\gamma)\mathfrak{A}^\tau(\gamma)^*$ is dense in $\mathfrak{A}^\tau$ for each $\gamma \in \hat{G}$ these maps can be described as follows. Let $\mathfrak{Z}$ denote the centre of the multiplier algebra $M(\mathfrak{A}^\tau)$ of $\mathfrak{A}^\tau$. In Lemma 1.5 we prove that there is a representation $\gamma \in \hat{G} \longmapsto \alpha_\gamma$ of $\hat{G}$ as *-automorphisms of $\mathfrak{Z}$ such that $\alpha_\gamma(A)X = XA$ for all $X \in \mathfrak{A}^\tau(\gamma)$ and $A \in \mathfrak{Z}$. Given the semigroup S satisfying the above requirements there exists a map $\gamma \in \hat{G} \longmapsto \exp\{-tL(\gamma)\}$ such that

1) $e^{-tL(\gamma)} \in \mathfrak{Z}$, $t \geqslant 0$.

2) $t \longmapsto e^{-tL(\gamma)}$ is a contraction semigroup, continuous in the strict topology on $M(\mathfrak{A}^\tau)$, i.e., $t \longmapsto \exp\{-tL(\gamma)\}X$ is norm continuous for each $X \in \mathfrak{A}^\tau$.

3) The matrix inequality

$$[\alpha_{\gamma_i}(e^{-tL(\gamma_j-\gamma_i)})] \geqslant [e^{-tL(\gamma_i)^*}e^{-tL(\gamma_j)}]$$

is valid for all finite sequences $\gamma_1, \ldots, \gamma_n \in \hat{G}$.

4) $e^{-tL(0)} = 1$, $t \geqslant 0$.

The connection between S and L is given by

$$S_t(X) = e^{-tL(\gamma)}X \tag{*}$$

for all $X \in \mathfrak{A}^\tau(\gamma)$, all $\gamma \in \hat{G}$, and all $t \geqslant 0$. Conversely given a map $\gamma \longmapsto L(\gamma)$ satisfying these four conditions the relation (*) defines a C_0-semigroup of completely positive maps on $\mathfrak{A}$ commuting with τ_G and restricting to the identity on $\mathfrak{A}^\tau$. Moreover if the automorphisms α_γ are equal to the identity ι, or, equivalently, if the centre of $M(\mathfrak{A}^\tau)$ is contained in the centre of $M(\mathfrak{A})$, then Condition 3 reduces to positivity of the matrices $[\exp\{-tL(\gamma_j-\gamma_i)\}]$ and, by using Bochner's theorem as in [6], one concludes that $S_t(X) = \int_G d\mu_t(g)\,\tau_g(X)$ where μ_t is a $\mathfrak{Z}$-valued convolution semigroup of probability measures on G. In this case, the Levy-Khinchin theorem, [21], gives an explicit representation of H, (see, for example, [6] Corollary 5.8).

Now we turn to the proof of Theorem 1.

Condition 2 of the theorem implies Condition 1 by differentiation. The principal part of the proof of the converse is contained in the following series of eight lemmas. The first four lemmas are very similar to results used to prove Theorem 5.1 in [6].

LEMMA 1.1. – *Let* $\mathfrak{A}$ *be a* C^**algebra,* $\mathcal{D}$ *a* $*$*-subalgebra of* $\mathfrak{A}$*, and* $\mathcal{D}_0$ *a* $*$*-subalgebra of* $\mathcal{D}$*. Let* $H : \mathcal{D} \longrightarrow \mathfrak{A}$ *be a symmetric linear operator such that* $H(X) = 0$ *for all* $X \in \mathcal{D}_0$ *, and*

$$H(X^*)X + X^*H(X) - H(X^*X) \geqslant 0$$

for all $X \in \mathcal{D}$.

It follows that $H(XY) = H(X)Y$, $H(YX) = YH(X)$ *for all* $X \in \mathcal{D}$, $Y \in \mathcal{D}_0$.

This is an immediate consequence of the Cauchy-Schwarz inequality applied to the positive sesquilinear forms

$$X, Y \in \mathcal{D} \longmapsto \omega(X(X^*)Y + X^*H(Y) - H(X^*Y))$$

where ω ranges over all states on $\mathfrak{A}$ (see, for example, the proof of Lemma 5.3 in [6]).

In the sequel we assume that $\mathfrak{A}$ is represented faithfully, and non-degenerately, on a Hilbert space $\mathcal{H}$. Let $\mathfrak{Z} = \mathfrak{A}^{\tau''} \cap \mathfrak{A}^{\tau'}$. Since $\mathfrak{A}^{\tau}(\gamma)\mathfrak{A}^{\tau}(\gamma)^*$, $\gamma \in \hat{G}$, are ideals in $\mathfrak{A}^{\tau}$ the subalgebras $\overline{\mathfrak{A}^{\tau}(\gamma)\mathfrak{A}^{\tau}(\gamma)^*}$, where the bar denotes σ-weak closure on $\mathcal{H}$, are ideals in $\mathfrak{A}^{\tau''}$ and thus there exist projections $E(\gamma) \in \mathfrak{Z}$ such that

$$\begin{aligned}\overline{\mathfrak{A}^{\tau}(\gamma)\mathfrak{A}^{\tau}(\gamma)^*} &= \mathfrak{A}^{\tau''}E(\gamma)\\ &= E(\gamma)\mathfrak{A}^{\tau''} = (\mathfrak{A}^{\tau''})E(\gamma).\end{aligned}$$

Now define $\mathcal{D}(\gamma) = \mathfrak{A}^{\tau}(\gamma) \cap \mathcal{D}$ where $\mathcal{D}$ is the domain specified in the main theorem.

LEMMA 1.2. – *For each* $\gamma \in \hat{G}$ *there exists a closed, densely defined, possibly unbounded operator* $L(\gamma)$ *on* $E(\gamma)\mathcal{H}$ *such that*

1) $\mathcal{D}(\gamma)\mathcal{H} \subseteq D(L(\gamma))$, $\mathcal{D}(\gamma)\mathcal{H}$ *is a core for* $L(\gamma)$ *and* $H(X) = L(\gamma)X$ *for all* $X \in \mathcal{D}(\gamma)$, $\gamma \in \hat{G}$.

2) The operator $-L(\gamma)$ *is dissipative, i.e.,* $\mathrm{Re}\langle \xi, L(\gamma)\xi\rangle \geqslant 0$ *for all* $\xi \in D(L(\gamma))$.

3) $L(\gamma)$ *is affiliated with the abelian von Neumann algebra* $\mathfrak{Z}E(\gamma)$.

Proof. – This is deduced from Lemma 1.1. by the same argument used to deduce Lemma 5.4 from Lemma 5.3 in [6]. Roughly, one constructs an approximate identity for the C*-closure of $\mathfrak{A}^\tau(\gamma)\,\mathfrak{A}^\tau(\gamma)^*$ of the form $E_\alpha = \Sigma\, X_i^\alpha X_i^{\alpha *}$ where $X_i^\alpha \in \mathcal{D}(\gamma)$ and each sum is finite. The E_α converges σ-weakly to $E(\gamma)$ on $\mathcal{H}$, and one defines $L(\gamma)$ such that

$$L(\gamma)\, X = \lim_\alpha \sum_i H(X_i^\alpha)\, X_i^{\alpha *} X$$

for each $X \in \mathcal{D}(\gamma)$. where the limit exists in norm. The remaining details of the proof are given in Lemma 5.4 of [6].

LEMMA 1.3. – *1) The operator* $L(\gamma)$ *is the generator of a* C_0*-semigroup* $t \longmapsto \exp\{-tL(\gamma)\}$ *of contractions on* $E(\gamma)\,\mathcal{H}$ *such that* $\exp\{-tL(\gamma)\} \in \mathfrak{Z}\,E(\gamma)$ *for all* $t \geqslant 0$.

2) If $X \in \mathfrak{A}^\tau(\gamma)$, *then* $e^{-tL(\gamma)}\, X = e^{-tL(\gamma)}\, E(\gamma)\, X \in \mathfrak{A}^\tau(\gamma)$ *for all* $t \geqslant 0$, *and the map* $t \geqslant 0 \longmapsto e^{-tL(\gamma)}\, X$ *in norm-continuous.*

Proof. – 1) Again the reasoning closely resembles the proof of Lemma 5.6 in [6]. By Lemma 1.2, $L(\gamma)$ is dissipative and affiliated with the abelian von Neumann algebra $\mathfrak{Z}\,E(\gamma)$. Therefore $L(\gamma)$ is the generator of a contraction semigroup in $\mathfrak{Z}\,E(\gamma)$ by spectral theory, [27] [30]. Alternatively, one can see that $L(\gamma)$ is a generator by the analytic element method employed in proving part 2.

2) If $X \in \mathcal{D}(\gamma)$, and $f_\epsilon : [0, +\infty\rangle \longrightarrow [0, +\infty\rangle$ is a continuous function such that $f_\epsilon(x) = 0$ for $x \leqslant \epsilon$, then the argument used in Lemma 5.6 in [6] shows that $X_\epsilon = f_\epsilon(XX^*)\, X \in \mathcal{D}(\gamma)$, and X_ϵ is an entire analytic element for H. The relations $H(Y) = L(\gamma)\, Y$, for $Y \in \mathcal{D}(\gamma)$, then show that $X_\epsilon \xi$ is entire analytic for $L(\gamma)$, for any $\xi \in \mathcal{H}$, and one has

$$\sum_{n=0}^{\infty} \frac{(-tH)^n}{n!}\,(X_\epsilon) = e^{-tL(\gamma)}\, X_\epsilon\,.$$

In particular $\exp\{-tL(\gamma)\}\, X_\epsilon \in \mathfrak{A}^\tau(\gamma)$ and $t \longmapsto \exp\{-tL(\gamma)\}\, X_\epsilon$ is norm continuous. Since any $X \in \mathcal{D}(\gamma)$ can be approximated in norm by elements X_ϵ, and since $\mathcal{D}(\gamma)$ is norm dense in $\mathfrak{A}^\tau(\gamma)$, part 2 of Lemma 1.3 follows by closure.

LEMMA 1.4. – *The restriction of* H *to* $\mathcal{D}(\gamma)$ *is norm-closable as an operator on the Banach space* $\mathfrak{A}^\tau(\gamma)$ *and its closure is the generator of a* C_0*-semigroup of contractions* S *on* $\mathfrak{A}^\tau(\gamma)$. *Furthermore* $S_t(X) = e^{-tL(\gamma)} X$ *for all* $t \geqslant 0$ *and* $X \in \mathfrak{A}^\tau(\gamma)$.

Proof. – If one defines S on $\mathfrak{A}^\tau(\gamma)$ by the above relation then S is a C_0-semigroup of contractions by Lemma 1.3. Let $\widetilde{H}$ denote its generator.

If $X \in \mathcal{D}(\gamma)$ and $\xi \in \mathcal{H}$ then $X\xi \in D(L(\gamma))$ and hence

$$S_t(X)\xi - X\xi = (e^{-tL(\gamma)} - I) X\xi$$
$$= -\int_0^t ds\, e^{-sL(\gamma)} L(\gamma) X\xi .$$

Alternatively stated $S_t(X) - X = -\int_0^t ds\, e^{-sL(\gamma)} H(X)$.

It follows that $X \in D(\widetilde{H})$ and $\widetilde{H}(X) = H(X)$, i.e., $\widetilde{H}$ is an extension of H. But the proof of Lemma 1.3 showed that H has a dense set of analytic elements in $\mathcal{D}(\gamma)$, and it follows that $\mathcal{D}(\gamma)$ is a core for $\widetilde{H}$ (see, for example, [9] Corollary 3.1.20). This shows that H is closable with closure $\widetilde{H}$.

We now define a semigroup S of linear maps from $\mathfrak{A}_F$ into $\mathfrak{A}_F$ by $S_t(X) = \sum_{\gamma \in \hat{G}} e^{-tL(\gamma)} X_\gamma$ where $X = \Sigma X_\gamma$ is the Fourier decomposition of X. Our aim is to show that S extends to a completely positive contraction semigroup on $\mathfrak{A}$. The main obstacle is to show that S is contractive, or, equivalently, that S is positive on $\mathfrak{A}_F$. To this end, we first need the following results concerning the structure of the spectral subspaces $\mathfrak{A}^\tau(\gamma)$.

LEMMA 1.5. – *There exists a unique *-isomorphism*

$$\alpha_\gamma : \mathfrak{Z}E(-\gamma) \longrightarrow \mathfrak{Z}E(\gamma)$$

such that $\alpha_\gamma(A) X = XA$ *for all* $X \in \mathfrak{A}^\tau(\gamma)$ *and* $A \in \mathfrak{Z}E(-\gamma)$. *Furthermore* $AX^* = X^*\alpha_\gamma(A)$ *for all* $X \in \mathfrak{A}^\tau(\gamma)$ *and* $A \in \mathfrak{Z}E(-\gamma)$.

Remark. – If $\mathfrak{A}^\tau(\gamma)$ contains a unitary operator U, one has $E(\gamma) = E(-\gamma) = I$ and $\alpha_\gamma(A) = UAU^*$ for $A \in \mathfrak{Z}$.

Proof. – Assume first that $A \in \mathfrak{Z}E(-\gamma)$ is given. As $E(\gamma)$ is the range projection of the subspace $[\mathfrak{A}^\tau(\gamma)\mathcal{H}]$ of $\mathcal{H}$, the operator B defined on this subspace by

$$B\Big(\sum_i X_i\xi_i\Big) = \sum_i X_i A\xi_i$$

is unique whenever the definition is consistent. In fact, we will show that $\|\sum_i X_i A\xi_i\| \leqslant \|A\|\,\|\sum_i X_i\xi_i\|$ for all finite sums, where $X_i \in \mathfrak{A}^\tau(\gamma)$ and $\xi_i \in \mathcal{H}$, and hence the definition is indeed consistent. Moreover $\|B\| \leqslant \|A\|$.

To derive the inequality, first assume that the finite sum contains only one element. Then

$$\begin{aligned}\|XA\xi\|^2 &= (A\xi, X^*XA\xi)\\ &= (A^*A\xi, X^*X\xi)\\ &= (A^*A\xi, |X|^2\xi)\\ &= (A^*A|X|\xi, |X|\xi)\\ &= \|A|X|\xi\|^2\\ &\leqslant \|A\|^2\,\||X|\xi\|^2\\ &= \|A\|^2\,\|X\xi\|^2\end{aligned}$$

where we have used X^*X, $|X| \in \mathfrak{A}^{\tau''}E(-\gamma)$ and $A \in \mathfrak{Z}\,E(-\gamma)$. To derive the inequality for general finite sums, one then employs the previous reasoning on the matrices

$$\widetilde{X} = \begin{bmatrix} X_1 & \dots & X_n\\ 0 & \dots & 0\\ \cdot & & \cdot\\ \cdot & & \cdot\\ \cdot & & \cdot\\ 0 & \dots & 0\end{bmatrix}, \quad \widetilde{A} = \begin{bmatrix} A & 0 & \dots & 0\\ 0 & A & \dots & 0\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ 0 & 0 & \dots & A\end{bmatrix}, \quad \widetilde{\xi} = \begin{bmatrix}\xi_1\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \xi_n\end{bmatrix}.$$

Next we show that $B \in \mathfrak{A}^{\tau''}E(\gamma)$. Let E_α be an approximate identity for $\mathfrak{A}^\tau(\gamma)\mathfrak{A}^\tau(\gamma)^*$ of the form $E_\alpha = \sum_i X_i^\alpha X_i^{\alpha *}$, where $X_i^\alpha \in \mathfrak{A}^\tau(\gamma)$ and each sum is finite. From the defining relation for B we have $BE_\alpha = \sum_i X_i^\alpha A X_i^{\alpha *}$. Since $\lim_\alpha E_\alpha = E(\gamma)$, where the limit exists in the strong operator topology, it follows that $B = \lim_\alpha BE_\alpha = \lim_\alpha \Big(\sum_i X_i^\alpha A X_i^{\alpha *}\Big)$.

But as $A \in \mathfrak{A}^{\tau''} E(-\gamma)$, we have $X_i^\alpha A X_i^{\alpha *} \in \mathfrak{A}^{\tau''} E(\gamma)$ for each i, α, and hence $B \in \mathfrak{A}^{\tau''} E(\gamma)$.

Now we show that $B \in \mathfrak{A}^{\tau'} E(\gamma)$. If $C \in \mathfrak{A}^\tau$, $X_i \in \mathfrak{A}^\tau(\gamma)$, and $\xi_i \in \mathcal{H}$, we have $CX_i \in \mathfrak{A}^\tau(\gamma)$, and hence

$$\begin{aligned} CB\left(\sum_i X_i \xi_i\right) &= C \sum_i X_i A \xi_i \\ &= \sum_i (CX_i) A \xi_i \\ &= B \sum_i (CX_i)\xi_i = BC \sum_i X_i \xi_i \ . \end{aligned}$$

It follows that $B \in \mathfrak{A}^{\tau'} E(\gamma)$, and hence

$$B = \mathfrak{A}^{\tau''} E(\gamma) \cap \mathfrak{A}^{\tau'} E(\gamma) = \mathfrak{Z} E(\gamma).$$

Now define $\alpha_\gamma : \mathfrak{Z}E(-\gamma) \longrightarrow \mathfrak{Z}E(\gamma)$ by $\alpha_\gamma(A) = B$.

Then $$\begin{aligned} \alpha_\gamma(A_1 A_2) X = XA_1 A_2 &= \alpha_\gamma(A_1) XA_2 \\ &= \alpha_\gamma(A_1)\alpha_\gamma(A_2) X \end{aligned}$$

for all $X \in \mathfrak{A}^\tau(\gamma)$, $A_1, A_2 \in \mathfrak{Z}E(-\gamma)$ so α_γ is a morphism. We next show that α_γ is a *-morphism, i.e., $\alpha_\gamma(A^*) = \alpha_\gamma(A)^*$. Let $A \in \mathfrak{Z}E(-\gamma)$, $B = \alpha_\gamma(A)$ and $B_1 = \alpha_\gamma(A^*)$. We must show that $B_1 = B^*$. But if $X, Y \in \mathfrak{A}^\tau(\gamma)$, $\xi, \eta \in \mathcal{H}$ we have

$$\begin{aligned} (B_1 X\xi, Y\eta) &= (XA^*\xi, Y\eta) \\ &= (A^*\xi, X^*Y\eta) \\ &= (\xi, X^*YA\eta) \\ &= (X\xi, YA\eta) \\ &= (X\xi, BY\eta) \end{aligned}$$

where we have used $X^*Y \in \mathfrak{A}^\tau$ and hence $AX^*Y = X^*YA$. It follows that $B_1 = B^*$.

We thus know that $\alpha_\gamma : \mathfrak{Z}E(-\gamma) \longrightarrow \mathfrak{Z}E(\gamma)$ is a *-morphism. But since $\mathfrak{A}^\tau(-\gamma) = \mathfrak{A}^\tau(\gamma)^*$, it follows from the adjoint relation $X^*B^* = A^*X^*$ that $\alpha_{-\gamma}$ is the inverse of α_γ. Therefore α_γ is a *-isomorphism.

Next we extend the operators $L(\gamma)$ affiliated with $\mathfrak{Z}E(\gamma)$ to operators affiliated with $\mathfrak{Z}$. The extended operators will also be denoted by $L(\gamma)$ and the extensions are defined by the requirements $L(\gamma)(I - E(\gamma)) = 0$. The operators $\exp\{-tL(\gamma)\}$

and $E(\gamma)$, where $t \geqslant 0$ and $\gamma \in \hat{G}$, are then all contained in $\mathfrak{Z}$ and consequently generate an abelian von Neumann algebra. If $\hat{G}$ is countable this algebra is countably decomposable and has the form $L^\infty(\Omega, d\mu)$ for some finite measure coming from a normal state with faithful restriction to the abelian algebra. If $\hat{G}$ is not countable one may replace $\hat{G}$ by countable subsets of $\hat{G}$ throughout the following reasoning and assume the existence of a similar representation.

In this spectral representation the projections $E(\gamma)$ are represented by characteristic functions of measurable subsets of Ω, and the operators $L(\gamma)$ by almost everywhere defined measurable functions with non-negative real part. Since the maps $\alpha_\gamma : \mathfrak{Z}E(-\gamma) \longrightarrow \mathfrak{Z}E(\gamma)$ are *-isomorphisms they are automatically normal. They may be extended to normal *-morphisms $\alpha_\gamma : \mathfrak{Z} \longrightarrow \mathfrak{Z}$, with range $\mathfrak{Z}E(\gamma)$, by the requirement $\alpha_\gamma(1 - E(-\gamma)) = 0$. Since they are normal they extend uniquely to unbounded operators affiliated to $\mathfrak{Z}$ so, for example, $\alpha_{\gamma_1}(L(\gamma_2))$ is a dissipative operator affiliated with $\mathfrak{Z}E(\gamma_1)$. We will also adjoin these operators to the spectral representation mentioned above.

LEMMA 1.6. – *For all* $\gamma_1, \ldots, \gamma_n \in \hat{G}$ *and* $m \in \mathbf{N}$ *one has*

$$[E(\gamma_i)\{L(\gamma_i)^* + L(\gamma_j) - \alpha_{\gamma_i}(L(\gamma_j - \gamma_i))\}^m E(\gamma_j)] \geqslant 0,$$

i.e., the matrix-valued measurable function from Ω *into* M_n *whose* (i,j)-th *matrix element is given by the foregoing expression takes values in the positive matrices, almost everywhere.*

Proof. – If $X_i \in \mathcal{D}(\gamma_i)$, $i = 1, 2, \ldots, n$, then by the hypothesis of the theorem $[H(X_i)^* X_j + X_i^* H(X_j) - H(X_i^* X_j)] \geqslant 0$.

Therefore $[X_i^* L(\gamma_i)^* X_j + X_i^* L(\gamma_j) X_j - L(\gamma_j - \gamma_i) X_i^* X_j] \geqslant 0$ by Lemma 1.2. But by Lemma 1.5 one then has

$$L(\gamma_j - \gamma_i) X_i^* X_j = X_i^* \alpha_{\gamma_i}(L(\gamma_j - \gamma_i)) X_j$$

in the sense that the densely defined operator to the right has a bounded extension, equal to the operator to the left. Thus

$$[X_i^* \{L(\gamma_i)^* + L(\gamma_j) - \alpha_{\gamma_i}(L(\gamma_j - \gamma_i))\} X_j] \geqslant 0.$$

Using the matrix methods employed in the proof of Lemma 4.5 in [6] one deduces that

$$\Big(\sum_k X_i^k X_i^{k*}\Big)\ \{L(\gamma_i)^* + L(\gamma_j) - \alpha_{\gamma_i}(L(\gamma_j - \gamma_i))\} \sum_k X_j^k X_j^{k*}] \geqslant 0$$

for all finite sequences $X_i^k \in \mathcal{D}(\gamma_i)$. But if $\sum_k X_i^k X_i^{k*}$ runs through an approximate identity for $\mathfrak{A}^\tau(\gamma)\mathfrak{A}^\tau(\gamma)^*$, then $\sum_k X_i^k X_i^{k*}$ converges strongly to $E(\gamma)$, and hence, working in the spectral representation, one finds

$$[E(\gamma_i)\ \{L(\gamma_i)^* + L(\gamma_j) - \alpha_{\gamma_i}(L(\gamma_j - \gamma_i))\}E\,(\gamma_j)] \geqslant 0\,.$$

But a classical theorem of Schur, [26], implies that if $A = [A_{ij}]$ is a positive $n \times n$ matrix, then $[(A_{ij})^m]$ is also positive, for $m = 1, 2, \ldots$. Applying Schur's theorem to the matrix-valued fonctions above we deduce that

$$[E(\gamma_i)\ \{L(\gamma_i)^* + L(\gamma_j) - \alpha_{\gamma_i}(L(\gamma_j - \gamma_i))\}^m\, E(\gamma_j)] \geqslant 0\,.$$

LEMMA 1.7. – *If* $X \in \mathfrak{A}_F$ *the generalized Schwarz inequality* $S_t(X^* X) \geqslant S_t(X)^* S_t(X)$ *is valid.*

Remark. – By tensoring with the $n \times n$ matrices, one can equally well deduce the matrix inequality

$$[S_t(X_i^* X_j)] \geqslant [S_t(X_i)^* S_t(X_j)]\,.$$

Proof. – If the matrix inequality in Lemma 1.6 is multiplied to the right with the matrix

$$\begin{bmatrix} e^{-tL(\gamma_1)} & 0 & \ldots & 0 \\ 0 & e^{-tL(\gamma_2)} & \ldots & 0 \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ 0 & 0 & \ldots & e^{-tL(\gamma_n)} \end{bmatrix}$$

and to the left with the adjoint of the matrix, we obtain

$$[E(\gamma_i)\, e^{-tL(\gamma_i)^*}\ \{t(L(\gamma_i)^* + L(\gamma_j) - \alpha_{\gamma_i}(L(\gamma_j - \gamma_i)))\}^m\, e^{-tL(\gamma_j)}\, E(\gamma_j)] \geqslant 0\,.$$

Therefore, if this matrix is divided by $m!$, and the sum is taken over $m = 0, 1, 2, \ldots,$ then the sum is larger thant the $m = 0$ term, i.e.,

$$[E(\gamma_i)\, e^{-t\alpha_{\gamma_i}(L(\gamma_j-\gamma_i))}\, E(\gamma_j)] \geqslant [E(\gamma_i)\, e^{-tL(\gamma_i)^*}\, e^{-tL(\gamma_j)}\, E(\gamma_j)]$$

and both matrix operators involved in this last inequality have finite norm. Assume that $X \in \mathfrak{A}_F$, and that X has the Fourier decomposition $X = \sum_{i=1}^{n} X_{\gamma_i} = \sum_{i=1}^{n} X_i$. Multiplying the matrices above to the right with

$$\begin{bmatrix} X_1 & 0 & \ldots & 0 \\ X_2 & 0 & \ldots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ X_n & 0 & \ldots & 0 \end{bmatrix}$$

and to the left with the adjoint of the matrix, one obtains matrices where only the upper left hand corner is nonzero. The resulting inequality becomes, after using $X_i^* E(\gamma_i) = X_i^*$ and $E(\gamma_j) X_j = X_j$,

$$\sum_{ij} X_i^*\, e^{-t\alpha_{\gamma_i}(L(\gamma_j-\gamma_i))}\, X_j \geqslant \sum_{ij} X_i^*\, e^{-tL(\gamma_i)^*}\, e^{-tL(\gamma_j)}\, X_j .$$

But
$$\begin{aligned} X_i^*\, e^{-t\alpha_{\gamma_i}(L(\gamma_j-\gamma_i))}\, X_j &= X_i^*\, \alpha_{\gamma_i}(e^{-tL(\gamma_j-\gamma_i)})\, X_j \\ &= e^{-tL(\gamma_j-\gamma_i)}\, X_i^* X_j \\ &= S_t(X_i^* X_j) \end{aligned}$$

and $e^{-tL(\gamma_j)} X_j = S_t(X_j)$.

The above inequality therefore states $S_t(X^* X) \geqslant S_t(X)^* S_t(X)$.

Now we come to the crucial part of the proof that S is positive and contractive.

LEMMA 1.8. – *Suppose that a linear mapping $\beta : \mathfrak{A}_F \longrightarrow \mathfrak{A}$ is defined with the two properties*

1) $\beta(X^* X) \geqslant 0$, *for all* $X \in \mathfrak{A}_F$, *2) For every* $\gamma \in \hat{G}$ *there is a* $C_\gamma > 0$ *such that*

$$\|\beta(X)\| \leqslant C_\gamma \|X\|, \text{ for all } X \in \mathfrak{A}^\tau(\gamma).$$

It follows that $\|\beta(X)\| \leqslant C_0 \|X\|$ *for all* $X \in \mathfrak{A}_F$ *and* β *extends to a positive linear map* $\beta : \mathfrak{A} \longrightarrow \mathfrak{A}$, *with* $\|\beta\| \leqslant C_0$.

Remark 1. – The main problem in the proof of this lemma is to show that β is positive on $\mathfrak{A}_F$. This is not a simple consequence of the first condition because positive elements in $\mathfrak{A}_F$ do not necessarily have the form X^*X with $X \in \mathfrak{A}_F$. An example showing the possible problems is as follows. Let $\mathfrak{A} = C(0,1)$ and $\mathcal{D}$ the *-subalgebra of polynomials. Each $f \in \mathcal{D}$ extends by analyticity to $\mathbf{C}$ and if one defines β by

$$(\beta(f))(x) = f(x+2), \quad f \in \mathcal{D},$$

then $\beta : \mathcal{D} \longrightarrow \mathcal{D}$ is a *-automorphism and in particular $\beta(f^* f) \geqslant 0$ for all $f \in \mathcal{D}$. But β is not positive on $\mathcal{D}$ nor is it norm-continuous.

Remark 2. – In our application of Lemma 1.8 the map β satisfies the stronger positivity requirement

$$\beta(X^*X) \geqslant \beta(X)^* \beta(X), \text{ for all } X \in \mathfrak{A}_F ,$$

and this implies $C_\gamma \leqslant 1$ for all $\gamma \in \hat{G}$. This is because

$$\|\beta(X)\|^2 \leqslant \|\beta(X^*X)\| \leqslant C_0 \|X\|^2$$

for $X \in \mathfrak{A}^\tau(\gamma)$, since $X^*X \in \mathfrak{A}^\tau$. Therefore $C_\gamma \leqslant \sqrt{C_0}$ and setting $\gamma = 0$ one has $\sqrt{C_0} \leqslant 1$. Hence Lemma 1.8 then implies $\|\beta(X)\| \leqslant \|X\|$ for all $X \in \mathfrak{A}_F$.

Remark 3. – The group G plays a somewhat spurious role in Lemma 1.8 and in fact one can give a reformulation which does not involve G. *Let* $\mathfrak{A}$ *be a* C**algebra containing an increasing net* $\mathfrak{A}_\Lambda$ *of closed subspaces indexed by a lattice such that* $\mathcal{D} = \cup_\Lambda \mathfrak{A}_\Lambda$ *is a norm-dense *-algebra of* $\mathfrak{A}$ *and assume there exist linear operators* $P_\Lambda : \mathfrak{A} \longrightarrow \mathfrak{A}_\Lambda$ *such that*

a) $P_\Lambda(\mathfrak{A}_{\Lambda'}) \subseteq \mathfrak{A}_{\Lambda \cap \Lambda'}$, *for all* Λ, Λ',

b) $P_\Lambda(X) \in \overline{\mathrm{Co}}\,\{\alpha(X)\,;\ \alpha \in \operatorname{aut} \mathcal{D},\ \alpha(\mathfrak{A}_{\Lambda'}) = \mathfrak{A}_{\Lambda'}$ *for all* $\Lambda'\}$ *for all* $X \in \mathcal{D}$, *where* $\overline{\mathrm{Co}}$ *denotes the weakly (norm) closed convex hull,*

c) $P_\Lambda(X) \longrightarrow X$ *for all* $X \in \mathcal{D}$

Further let $\beta : \mathcal{D} \longrightarrow \mathfrak{A}$ *be a linear map with the properties*

1) $\beta(X^* X) \geqslant 0$ *for all* $X \in \mathcal{D}$,

2) for each Λ *there is a* $C_\Lambda > 0$ *such that*

$$\|\beta(X)\| \leqslant C_\Lambda \|X\| \quad \textit{for all} \quad X \in \mathfrak{A}_\Lambda .$$

It follows that β *is a positive map. Moreover if* $\mathfrak{A}$ *contains an identity* $\mathbb{1} \in \mathcal{D}$ *then* β *is bounded and* $\|\beta\| = \|\beta(\mathbb{1})\|$.

The proof of this statement is a rephrasing of the following proof with P_Λ identified with the regularization operator $\tau(f_\Lambda)$.

Proof of Lemma 1.8. – The main onus of the proof is to establish that $\beta(Y) \geqslant 0$ for all positive $Y \in \mathfrak{A}_F$. Once this is done the rest of the proof is as follows.

Choose an approximate identity $\{E_\delta\}$ for $\mathfrak{A}$ such that $E_\delta \in \mathfrak{A}^\tau$ for all δ (see for example, [6] Lemma 4.1). Then

$$- \|X\| E_\delta^2 \leqslant E_\delta X E_\delta \leqslant \|X\| E_\delta^2$$

for all $X = X^* \in \mathfrak{A}_F$. Hence

$$- \|X\| \beta(E_\delta^2) \leqslant \beta(E_\delta X E_\delta) \leqslant \|X\| \beta(E_\delta^2) .$$

Therefore

$$\begin{aligned} \|\beta(E_\delta X E_\delta)\| &\leqslant \|X\| \, \|\beta(E_\delta^2)\| \\ &\leqslant \|X\| \, C_0 \|E_\delta^2\| \leqslant \|X\| \, C_0 . \end{aligned}$$

But $\beta|_{\mathfrak{A}^\tau(\Lambda)}$ is bounded for every finite subset $\Lambda \subset \hat{G}$ by the estimate

$$\begin{aligned} \|\beta(X)\| &= \Big\|\sum_{\gamma \in \Lambda} \beta(P_\gamma(X))\Big\| \\ &\leqslant \Big(\sum_{\gamma \in \Lambda} C_\gamma\Big) \|X\| , \quad X \in \mathfrak{A}^\tau(\Lambda); \end{aligned}$$

(here $\mathfrak{A}^\tau(\Lambda) = \sum_{\gamma \in \Lambda} \mathfrak{A}^\tau(\gamma)$, see [9], Definition 3.2.37). Therefore taking the limit over the approximate identity one obtains $\|\beta(X)\| \leqslant C_0 \|X\|$ for all $X = X^* \in \mathfrak{A}_F$.

Now β extends by continuity to the space $\mathfrak{A}_{sa}$ of all self-adjoint elements of $\mathfrak{A}$ and can be further extended by linearity to $\mathfrak{A} = \mathfrak{A}_{sa} + i\mathfrak{A}_{sa}$. Since β is positive on $\mathfrak{A}_F$ it follows by approximation that the extension $\widetilde{\beta}$ is positive on

$\mathfrak{A}$, e.g., this follows from the regularization method used in the sequel. Therefore $\|\widetilde{\beta}\| = \lim_{\delta} \|\widetilde{\beta}(E_\delta)\| \leqslant C_0$ by Lemma 2 of [11]. In particular $\|\beta(X)\| \leqslant C_0 \|X\|$ for all $X \in \mathfrak{A}$.

Thus to complete the proof of the lemma it remains to establish that β is positive on $\mathfrak{A}_F$. This will be achieved by approximating a positive $Y \in \mathfrak{A}_F$ with elements of the form Z^2 with $Z = Z^* \in \mathfrak{A}_F$ and using the fact that β is bounded on each $\mathfrak{A}^\tau(\Lambda)$. It is, however, necessary to keep control over the size of Λ. For this a regularization argument is essential.

Let f be a function on G whose Fourier transform $\hat{f}$ has finite support $\Lambda(f)$ and define $\tau(f) = \int_G dg\, f(g)\, \tau_g$. Then $\tau(f)$ maps $\mathfrak{A}^\tau(\Lambda)$ into $\mathfrak{A}^\tau(\Lambda \cap \Lambda(f)) \subseteq \mathfrak{A}^\tau(\Lambda(f))$ because if $X \in \mathfrak{A}^\tau(\Lambda)$ then

$$\begin{aligned}\tau(f)(X) &= \sum_{\gamma\in\Lambda} \int_G dg\, f(g)\, \tau_g(X_\gamma) \\ &= \sum_{\gamma\in\Lambda} \int_G dg\, f(g)\, (\gamma, g)\, X_\gamma \\ &= \sum_{\gamma\in\Lambda\cap\Lambda(f)} \hat{f}(\gamma)\, X_\gamma \in \mathfrak{A}^\tau(\Lambda\cap\Lambda(f)).\end{aligned}$$

Next we observe that one can choose an approximate identity $\{f_\Lambda\}$, for the convolution algebra $L^1(G)$, consisting of continuous, positive f_Λ, with $\int_G dg\, f_\Lambda(g) = 1$, and such that $\hat{f}_\Lambda$ has finite support $\Lambda(f)$. One way to achieve this is to consider an upward directed family of finite sets $\Lambda \subset \hat{G}$ with union $\hat{G}$ such that $-\Lambda = \Lambda$ and then set $f_\Lambda = (h_\Lambda)^2/|\Lambda|$ where $h_\Lambda \in C(G)$ is the Fourier transform of the characteristic function of Λ. In this case $\Lambda(f) = \Lambda + \Lambda = \{\gamma + \xi\,;\ \gamma, \xi \in \Lambda\}$.

Note that the regularization operators $\tau(f_\Lambda)$ associated with an approximate identity of this type are completely positive contractions on $\mathfrak{A}$.

Now we are prepared to prove the positivity of β on $\mathfrak{A}_F$.

Given $1 > \epsilon > 0$ and $Y \in \mathfrak{A}_F$, with $Y \geqslant 0$, choose $Z = Z^* \in \mathfrak{A}_F$ such that $\|\sqrt{Y} - Z\| < \epsilon/(2\|Y\|^{1/2} + 1)$. Then $\|Y - Z^2\| \leqslant (\|Y\|^{1/2} + \|Z\|)\, \|\sqrt{Y} - Z\| < \epsilon$.

Moreover if f_Λ is the above approximate identity
$$\|\tau(f_\Lambda)(Y) - \tau(f_\Lambda)(Z^2)\| \leqslant \|Y - Z^2\| < \epsilon$$
and also
$$\|\beta(\tau(f_\Lambda)(Y)) - \beta(\tau(f_\Lambda)(Z^2))\| \leqslant \Big(\sum_{\gamma\in\Lambda(f)} C_\gamma\Big)\epsilon .$$
Then clearly
$$\beta(\tau(f_\Lambda)(Y)) \geqslant \beta(\tau(f_\Lambda)(Z^2)) - \epsilon\Big(\sum_{\gamma\in\Lambda(f)} C_\gamma\Big)$$
$$\geqslant -\epsilon\Big(\sum_{\gamma\in\Lambda(f)} C_\gamma\Big),$$
since
$$\beta(\tau(f_\Lambda)(Z^2) = \int dg\, f_\Lambda(g)\,\beta(\tau(g)(Z^2))$$
$$= \int dg\, f_\Lambda(g)\beta((\tau(g)(Z))^2)$$
$$\geqslant 0$$
by assumption 1 in the lemma. But ϵ is arbitrary and hence
$$\beta(\tau(f_\Lambda)(Y)) \geqslant 0 .$$
Moreover $\tau(f_\Lambda)(Y) \in \mathfrak{A}^\tau(\Lambda' \cap \Lambda(f)) \subseteq \mathfrak{A}^\tau(\Lambda')$ where Λ' is independent of f_Λ. Thus $\tau(f_\Lambda)(Y) \longrightarrow Y$ as $\Lambda \longrightarrow \hat{G}$ in the closed subspace $\mathfrak{A}^\tau(\Lambda')$. But β is bounded in restriction to $\mathfrak{A}^\tau(\Lambda')$ and hence $\beta(\tau(f_\Lambda)(Y)) \longrightarrow \beta(Y)$ and we find $\beta(Y) \geqslant 0$ which is the desired conclusion.

End of proof of Theorem 1. – In the foregoing lemmas we have constructed a C_0-semigroup $S : \mathfrak{A}_F \longrightarrow \mathfrak{A}_F$ which is contractive, by Lemma 1.8, and hence extends by continuity to a C_0-semigroup of contractions S on $\mathfrak{A}$. But S then satisfies the generalized Schwarz inequality, by Lemma 1.7, and in fact satisfies the matrix inequality contained in Condition 2 of the theorem, by the remark after Lemma 1.7. But the reasoning in the proof of Lemma 1.4 shows that the generator $\widetilde{H}$ of S is an extension of H and therefore H is closable. In the course of that proof we constructed a dense set of analytic element for H and thus $\widetilde{H}$ is the closure of H (see, for example [9] Corollary 3.1.20).

2. Dissipations and Complete Dissipations.

In the implication $1 \Longrightarrow 2$ in Theorem 1 it was assumed that H was a complete dissipation, i.e., the matrix inequality $[H(X_i^*) X_j + X_i^* H(X_j) - H(X^* X_j)] \geq 0$ is valid for all finite sequences $X_1, \ldots, X_n \in D(H)$. This can not in general be replaced by the weaker assumption that H is a dissipation, i.e., $H(X^*) X + X^* H(X) - H(X^* X) \geq 0$ for all $X \in D(H)$. This is illustrated by the following theorem.

THEOREM 2. – *Let* $\mathfrak{A}$ *be a* C*-*algebra and assume there exists an ergodic action* τ, *on* $\mathfrak{A}$, *of a finite abelian group* G.

The following three conditions are equivalent:

1) All dissipations on $\mathfrak{A}_F$ *are complete dissipations,*

2) If $H : \mathfrak{A}_F \longrightarrow \mathfrak{A}$ *is a *-operator, with the properties:*

a. H *is a dissipation,*

b. $H\tau_g = \tau_g H$ *for all* $g \in G$:

c. $H(X) = 0$ *for all* $X \in$ centre $(\mathfrak{A}_F)$, *then* H *is a complete dissipation,*

3) $\mathfrak{A}$ *is abelian.*

Proof. – Since any dissipation of an abelian C*-algebra is a complete dissipation, by Remark 1 after Theorem 1, it follows immediately that $3 \Longrightarrow 1$ whilst $1 \Longrightarrow 2$ is trivial. To prove $2 \Rightarrow 3$ we will first just assume that G is a compact abelian group, and obtain characterizations of dissipations and complete dissipations in Lemmas 2.1-2.3. We then specialize to finite G in Lemmas 2.4 - 2.6.

We may assume by ergodicity that the action τ is faithful. Each spectral subspace $\mathfrak{A}^\tau(\gamma)$ is then spanned by a unitary operator $U(\gamma)$, and the $U(\gamma)$ satisfy the commutation relations $U(\gamma_1) U(\gamma_2) = \rho(\gamma_1, \gamma_2) U(\gamma_2) U(\gamma_1)$ for all $\gamma_1, \gamma_2 \in \hat{G}$ where ρ is an antisymmetric bicharacter, independent of the choice of the U [20]. Moreover $\mathfrak{A}$ is simple if, and only, if ρ is non-degenerate [28] and $\mathfrak{A}$ is abelian if, and only if, $\rho(\gamma_1, \gamma_2) = 1$ for all γ_1, γ_2. The U also satisfy the cocycle relation

$$U(\gamma_1) U(\gamma_2) = \beta(\gamma_1, \gamma_2) U(\gamma_1 + \gamma_2)$$

where β is a 2-cocycle, i.e., β is a phase factor satisfying

$$\beta(\gamma_1, \gamma_2)\beta(\gamma_1 + \gamma_2, \gamma_3) = \beta(\gamma_1, \gamma_2 + \gamma_3)\beta(\gamma_2, \gamma_3)$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \hat{G}$. The function β depends on the phase chosen for the $U(\gamma)$, if $U(\gamma)$ is replaced by $U'(\gamma) = \varphi(\gamma) U(\gamma)$, β is replaced by β' given by

$$\beta'(\gamma_1, \gamma_2) = \beta(\gamma_1, \gamma_2) \frac{\varphi(\gamma_1)\varphi(\gamma_2)}{\varphi(\gamma_1 + \gamma_2)}$$

and β is uniquely determined by τ up to these transformations. In particular, $\mathfrak{A}$ is abelian if, and only if, β is a co-boundary, i.e., $\beta(\gamma_1, \gamma_2) = \dfrac{\varphi(\gamma_1 + \gamma_2)}{\varphi(\gamma_1)\varphi(\gamma_2)}$ for some function $\varphi : \hat{G} \longrightarrow \mathbf{T}$. The connection between β and ρ is given by $\rho(\gamma_1, \gamma_2) = \dfrac{\beta(\gamma_1, \gamma_2)}{\beta(\gamma_2, \gamma_1)}$.

We first reduce the proof of $2 \Rightarrow 3$ to the case that $\mathfrak{A}$ is simple. To this end, define $\Gamma = \{\gamma \in \hat{G} ; \rho(\gamma, \xi) = 1$ for all $\xi \in \hat{G}\}$ and $G_0 = \Gamma^{\perp} = \{g \in G ; \langle \gamma, g\rangle = 1$ for all $\gamma \in \Gamma\}$.
Then

$$\begin{aligned} \text{centre}(\mathfrak{A}) &= \text{closed linear span of } \{U(\gamma) ; \gamma \in \Gamma\} \\ &= \mathfrak{A}^{G_0} \equiv \{X \in \mathfrak{A} ; \tau(g)(X) = X \text{ for all } g \in G_0\}. \end{aligned}$$

See, for example [5]. Define a nondegenerate antisymmetric bicharacter $\rho_0 ; \hat{G}_0 \times \hat{G}_0 = \hat{G}/\Gamma \times \hat{G}/\Gamma \longmapsto \mathbf{T}$ by

$$\rho_0([\gamma], [\xi]) = \rho(\gamma, \xi)$$

where $[\gamma]$ is in the coset of $\gamma \in \hat{G}$ in $\hat{G}/\Gamma$. Let $\mathfrak{A}_0$ be the simple C^*-algebra which is the closed linear span of unitary operators $U_0([\gamma])$, $[\gamma] \in \hat{G}_0$ satisfying

$$U_0([\gamma])\,U_0([\xi]) = \rho_0([\gamma], [\xi])\,U_0([\xi])\,U_0([\gamma])$$

and let τ^0 be the corresponding ergodic action of G_0 on $\mathfrak{A}_0$. The C^*-algebra $\mathfrak{A}$ is then a homogeneous C^*-algebra over centre $(\mathfrak{A}) \cong C(\hat{\Gamma}) \cong C(G/G_0)$ with fibre $\mathfrak{A}_0$.

LEMMA 2.1. – *There is a 1-1 correspondence between dissipations (resp. complete dissipation)* H *on* $\mathfrak{A}_F$ *satisfying*

b) $H\tau_g = \tau_g H$ *for all* $g \in G$,

c) $H(X) = 0$ *for all* $X \in \text{centre}\,(\mathfrak{A}_F)$
and dissipations (resp. complete dissipations) H_0 *on* $\mathfrak{A}_{0_F}$ *satisfying*

b') $H_0 \tau_g^0 = \tau_g^0 H_0$ *for all* $g \in G_0$,

c') $H_0(\mathbf{1}_0) = 0$, *where* $\mathbf{1}_0$ *is the identity element in* $\mathfrak{A}_0$.
If the functions $L : \hat{G} \longrightarrow \mathbf{C}$ *and* $L_0 : \hat{G}_0 \longrightarrow \mathbf{C}$ *are defined by* $H(U(\gamma)) = L(\gamma)\,U(\gamma), \gamma \in \hat{G}$,

$$H_0(U_0([\gamma])) = L_0([\gamma])\,U_0([\gamma]), \quad [\gamma] \in \hat{G}_0 = \hat{G}/\Gamma,$$

this correspondence is given by $L(\gamma) = L_0([\gamma])$, *all* $\gamma \in \hat{G}$.

Proof. – If H is a dissipation (resp. complete dissipation) on $\mathfrak{A}_F$ satisfying b) and c), then it follows from Lemma 1.1 that $H(XY) = H(X)\,Y = YH(X) = H(YX)$ for all $X \in \mathfrak{A}_F$, $Y \in$ centre $(\mathfrak{A}_F)$ = linear span of $\{U(\gamma)\,; \gamma \in \Gamma\}$. Thus, if $L : \hat{G} \longrightarrow \mathbf{C}$ is the function such that $H(U(\gamma)) = L(\gamma)\,U(\gamma)$, then L depends only on $[\gamma]$, and we may define a function $L_0 : \hat{G}_0 \longmapsto \mathbf{C}$ by $L_0([\gamma]) = L(\gamma)$ for all $\gamma \in \hat{G}$, and then define an operator H_0 on $\mathfrak{A}_{0_F}$ by

$$H_0(U_0([\gamma])) = L_0([\gamma])\,U_0([\gamma]) \quad \text{for} \quad [\gamma] \in \hat{G}_0\,.$$

The point $0 \in G/G_0$ (or any other point) defines a character on $C(G/G_0)$ = centre $(\mathfrak{A})$ by evaluation, and thus a morphism φ on $\mathfrak{A}$ with range canonically isomorphic to $\mathfrak{A}_0$ by the correspondence $\varphi(U(\gamma)) \sim U_0([\gamma]), \gamma \in \hat{G}$.
As $H(XY) = H(X)\,Y$ for all $X \in \mathfrak{A}_F$, $Y \in$ centre $(\mathfrak{A})$, H maps each fibre of the homogeneous C*-algebra $\mathfrak{A}$ over centre $(\mathfrak{A})$ into itself, and hence H lifts via φ to a map on $\varphi(\mathfrak{A}_F) \sim \mathfrak{A}_{0_F}$. This map identifies with H_0 via the above correspondence, and as φ is a *-morphism it follows that H_0 is a dissipation (resp. complete dissipation).

Conversely, if H_0 is a dissipation (resp. complete dissipation) on $\mathfrak{A}_0$ satisfying b'. and c'., define $L(\gamma) = L_0([\gamma])$ for $\gamma \in \hat{G}$ and $H(U(\gamma)) = L(\gamma)\,U(\gamma)$ for $\gamma \in \hat{G}$.
The H satisfies properties b and c, and in particular the action of H on $\mathfrak{A}$ decomposes under the "central decomposition" of $\mathfrak{A}$ into actions of H on each of the fibres of $\mathfrak{A}$ over centre $(\mathfrak{A})$. But as explained above, each of these actions is isomorphic

to the action of H_0 on $\mathfrak{A}_0$, and hence H is a dissipation (resp. complete dissipation) if, and only if, H_0 has this property.

Because of Lemma 2.1, we may assume that the bi-character ρ is non-degenerate and $\mathfrak{A}$ is simple in the rest of the proof. The condition $2c$) reduces to $H(\mathbf{1}) = 0$, and any $*$-map satisfying $2b$) and $2c$) is given by $H(U(\gamma)) = L(\gamma)\, U(\gamma)$ where $L(\gamma)$ is a scalar satisfying $L(-\gamma) = \overline{L(\gamma)}, L(0) = 0$. Actually, we will restrict attention to functions $L : \hat{G} \longrightarrow \mathbf{C}$ having the form $L(\gamma) = \sum_{\xi \in \hat{G} \setminus \{0\}} (1 - \rho(\xi, \gamma))\, \alpha(\xi)$ where α is a real function on $\hat{G}$ such that $\alpha(0) = 0$ and $\sum_{\gamma \in \hat{G}} |\alpha(\gamma)| < +\infty$. The space of such functions α will be denoted by $\mathcal{A}$, and the space of the corresponding functions L will be denoted by $\mathcal{L}$. The operator $H = H_\alpha$ corresponding to α is then given by

$$H(X) = - \sum_{\gamma \in \hat{G} \setminus \{0\}} \alpha(\gamma)\, U(\gamma)\, X U(\gamma)^* + \Big(\sum_{\gamma \in \hat{G} \setminus \{0\}} \alpha(\gamma) \Big) X .$$

H_α is thus a bounded operator satisfying b) and c). Note that if G is finite, it follows from [6], Example 6.4, and [18, 13] that H is a complete dissipation satisfying b) and c) if, and only if, $H = H_\alpha$ for an $\alpha \in \mathcal{A}$ such that $\alpha(\gamma) \geqslant 0$ for all γ. For general compact abelian groups G we have the following:

LEMMA 2.2. – 1) *The linear transformation* $M : \mathcal{A} \longrightarrow \mathcal{L}$ *defined by*

$$(M\alpha)(\gamma) = \sum_{\xi \in \hat{G}} (1 - \rho(\xi, \gamma))\, \alpha(\xi)$$

is a real linear isomorphism from $\mathcal{A} \longrightarrow \mathcal{L}$.

2) *The function* $\gamma \longrightarrow (M\alpha)(\gamma)$ *is negative definite on* $\hat{G}$ *if, and only if,* $\alpha(\xi) \geqslant 0$ *for all* $\xi \in \hat{G}$. *Thus* H_α *is a complete dissipation if, and only if,* $\alpha(\xi) \geqslant 0$ *for all* $\xi \in \hat{G}$.

Remark. – If G is finite, one deduces from 1. by counting dimensions that

$$\mathcal{L} = \{L : \hat{G} \longrightarrow \mathbf{C}\, ; L(0) = 0 \text{ and } L(-\gamma) = \overline{L(\gamma)} \text{ for all } \gamma \in \hat{G}\}$$

and hence any operator $H : \mathfrak{A} \longrightarrow \mathfrak{A}$ with the properties $H\tau_g = \tau_g H$ and $H(I) = 0$ has the form $H = H_\alpha$ for a unique $\alpha \in \mathfrak{A}$.

Proof. – 1) The antisymmetric bi-character ρ defines a morphism $\varphi : \hat{G} \longrightarrow G$ by $\langle \xi, \varphi(\gamma) \rangle = \rho(\xi, \gamma)$ for all $\xi, \gamma \in \hat{G}$. As ρ is non-degenerate this morphism is faithful, and $\varphi(\hat{G})$ is a dense subgroup of G. Using this morphism, we may view $\mathcal{L}$ as functions L on G by $L(g) = \sum_{\gamma \in \hat{G}} (1 - \langle \xi, g \rangle) \alpha(\xi)$, i.e., we identify the function L_1 on $\hat{G}$ with the function L_2 on G obtained by extending $L_2(\varphi(\gamma)) = L_1(\gamma)$ by continuity. Thus

$$L(g) = (M\alpha)(g) = \hat{\alpha}(0) - \hat{\alpha}(g)$$

where $\hat{}$ denotes the inverse Fourier transform. But if $M\alpha = 0$ one obtains $\hat{\alpha}(g) = \hat{\alpha}(0)$ for all g, and hence $\alpha(\xi)$ is proportional to $\delta(\xi)$. But as $\alpha(0) = 0$ it follows that $\alpha = 0$, and thus the kernel of M is zero.

2) As G is compact, G admits no nontrivial additive characters into the reals. It follows from the Lévy-Khinchin theorem [4, 21] that a function L on G is negative definite if, and only if, L has the form $L(g) = \sum_{\xi \in \hat{G}} (1 - \langle \xi, g \rangle) \alpha(\xi)$ where $\alpha \in \ell^1(G), \alpha(0) = 0$, and $\alpha(\xi) \geqslant 0$ for all $\xi \in G$. But as $\varphi(\hat{G})$ is dense in G, and L as a function on G is continuous, it follows that L is negative definite as a function on G if, and only if, it is as a function on $\varphi(\hat{G})$. Part 2. of the lemma then follows from the bijectivity of M. But H_α is a complete dissipation if, and only if, L is negative definite, see [6], Corollary 5.8, or Lemma 1.6 in this paper.

We next investigate when H is a dissipation. This means that $H(X^*) X + X^* H(X) - H(X^* X) \geqslant 0$ for all

$$X = \sum_\gamma x(\gamma) U(\gamma) \in \mathfrak{A}.$$

But

$$
\begin{aligned}
&H(X^*)\,X + X^*\,H(X) - H(X^*X) \\
&\quad = \sum_{\gamma,\xi} \overline{x(\gamma)}\,x(\xi)\,\{\overline{L(\gamma)} + L(\xi) - L(\xi-\gamma)\}\,U(\gamma)^*U(\xi) \\
&\quad = \sum_{\gamma,\xi} \overline{x(\gamma)}\,x(\xi)\,\Big\{\sum_{\eta}(1-\overline{\rho(\eta,\gamma)})\,\alpha(\eta) \\
&\quad + \sum_{\eta}(1-\rho(\eta,\xi))\,\alpha(\eta) - \sum_{\eta}(1-\rho(\eta,\xi-\gamma))\,\alpha(\eta)\Big\}\,U(\gamma)^*\,U(\xi) \\
&\quad = \sum_{\gamma,\xi,\eta} \overline{x(\gamma)}\,x(\xi)\,\{(1-\overline{\rho(\eta,\gamma)})\,(1-\rho(\eta,\xi))\}\,\alpha(\eta)\,U(\gamma)^*\,U(\xi) \\
&\quad = \sum_{\eta} X(\eta)^*\,X(\eta)\,\alpha(\eta)
\end{aligned}
$$

where $X(\eta) = \sum_{\xi} x(\xi)\,(1-\rho(\eta,\xi))\,U(\xi)$ for all $\eta \in \hat{G}$. If we use a similar identification of $\hat{G}$ with a dense subgroup $\varphi(\hat{G})$ of G as in the proof of Lemma 2.2, with $\rho(\eta,\xi) = \langle \xi, \varphi(\eta) \rangle$ for all $\eta, \xi \in \hat{G}$, then we have $X(\eta) = X - \tau_{\varphi(\eta)}(X)$.

Let Tr be the unique normalized trace state on $\mathfrak{A}$. As any positive linear functional ω on $\mathfrak{A}$, can be approximated by functionals of the form $\mathrm{Tr}(Z^* \cdot Z)$, where $Z \in \mathfrak{A}$ we thus obtain

LEMMA 2.3. – *Let* $\alpha \in \mathfrak{C}$. *The following two conditions are equivalent:*

1) H_α *is a dissipation,*

2) $\sum_{\xi \in \hat{G}} \mathrm{Tr}\,(Z^*(X - \tau_{\varphi(\xi)}(X))^*(X - \tau_{\varphi(\xi)}(X))Z)\,\alpha(\xi) \geqslant 0$ *for all* $X, Z \in \mathfrak{A}$.

Assume now that every dissipation of the form H_α is a complete dissipation. By Lemma 2.2 and 2.3 this means that if $\alpha \in \mathfrak{C}$ has the property that

$$\sum_{\xi \in \hat{G}} \mathrm{Tr}\,(Z^*(X - \tau_{\varphi(\xi)}(X))^*(X - \tau_{\varphi(\xi)}(X))\,Z)\,\alpha(\xi) \geqslant 0$$

for all $X, Z \in \mathfrak{A}$, then necessarily $\alpha(\xi) \geqslant 0$ for all $\xi \in \hat{G}$.

Identifying $\hat{\mathfrak{A}}$ with $\ell^1_{\mathbf{R}}(\hat{G}\backslash\{0\})$, this means that the convex cone in $\ell^\infty_{\mathbf{R}}(\hat{G}\backslash\{0\})$ generated by the functions

$$\xi \in \hat{G}\backslash\{0\} \longmapsto f_{X,Z}(\xi) = \mathrm{Tr}\,(E^*(X - \tau_{\varphi(\xi)}(X))^*(X - \tau_{\varphi(\xi)}(X))Z)$$

is weak*-dense in $\ell^\infty_{\mathbf{R}}(\hat{G}\backslash\{0\})_+ = \{\beta \in \ell^\infty\ ;\ \beta(\xi) \geqslant 0$ for all $\xi \in \hat{G}\backslash\{0\}\}$. But this is equivalent to the condition that the extremal points $\xi \longmapsto \delta(\xi - \xi_0)$ in $(\ell^\infty_+)_1$ can be approximated in the weak*-topology by a net of convex combinations of functions of the above type.

From now on we will assume that G is finite. In this case the cone generated by the function $f_{X,Z}$ is automatically closed because of finite-dimensionality, and hence the condition above is equivalent to the existence of $X, Z \in \mathfrak{A}$ for each $g_0 \in G$ such that $\mathrm{Tr}\,(Z^*(X - \tau_g(X))^*(X - \tau_g(X))\,Z) = \delta(g - g_0)$ for all $g \in \hat{G}\backslash\{0\}$. As the trace is faithful, we have proved the following :

LEMMA 2.4. – *Assume that* G *is finite. Hypothesis 2 in Theorem 2 is fulfilled if, and only if, for all* $g_0 \in G\backslash\{0\}$ *there exist* $X, Z \in \mathfrak{A}$ *with*

$$(X - \tau_g(X))\,Z \begin{cases} = 0 & \text{for } g \neq g_0 \\ \neq 0 & \text{for } g = g_0\,. \end{cases}$$

The if part of this lemma follows from the remark after Lemma 2.2.

We next translate the conditions in Lemma 2.4 into conditions on the Fourier coefficients $x(\gamma), z(\gamma)$ of $X.Z$. We have

$$X - \tau_g(X) = \sum_{\gamma_1} x(\gamma_1)\,(1 - \langle \gamma_1, g\rangle)\,U(\gamma_1)$$

and hence we may assume without loss of generality that $x(0) = 0$. Furthermore using $U(\gamma_1)\,U(\gamma_2) = \beta(\gamma_1, \gamma_2)\,U(\gamma_1 + \gamma_2)$ we have

$$(X - \tau_g(X))Z = \sum_{\gamma_1, \gamma_2} x(\gamma_1)\,z(\gamma_2)\,(1 - \langle \gamma_1, g\rangle)\,\beta(\gamma_1, \gamma_2)\,U(\gamma_1 + \gamma_2)$$

$$= \sum_{\xi} \Big\{ \sum_{\gamma_1} x(\gamma_1) z(\xi - \gamma_1)(1 - \langle \gamma_1, g\rangle)\,\beta(\gamma_1, \xi - \gamma_1) \Big\}\,U(\xi)$$

Thus, if X, Z satisfy the conclusion of Lemma 2.4 we have

$$\sum_{\gamma_1} x(\gamma_1) z(\xi - \gamma_1) (1 - \langle \gamma_1, g \rangle) \beta(\gamma_1, \xi - \gamma_1)$$

$$= \begin{cases} = 0 & \text{if } g \neq g_0 \\ \neq 0 & \text{for some } \xi \text{ if } g = g_0 \end{cases}$$

$$= h(\xi)\, \delta(g - g_0)$$

for some nonzero function h on $\hat{G}$. Put

$$f_\xi(\gamma_1) = x(\gamma_1)\, z(\xi - \gamma_1)\, \beta(\gamma_1, \xi - \gamma_1).$$

Then the above relation states that $\hat{f}_\xi(0) - \hat{f}_\xi(g) = h(\xi)\, \delta(g - g_0)$ where $\hat{}$ denotes the inverse Fourier transform, i.e.,

$$\hat{f}_\xi(g) = \hat{f}_\xi(0) - h(\xi)\, \delta(g - g_0).$$

Taking the Fourier transform, we obtain

$$f_\xi(\gamma) = \frac{1}{|G|^2} \sum_{g \in G} \langle -\gamma, g \rangle \hat{f}_\xi(g)$$

$$= \frac{1}{|G|} \delta(\gamma) \hat{f}_\xi(0) - \frac{1}{|G|^2} h(\xi) \langle -\gamma, g_0 \rangle$$

$$= \delta(\gamma)\, c_1(\xi) + g_0(\gamma)\, c_2(\xi)$$

where c_1, c_2, are functions on $\hat{G}$, c_2 is nonzero, and g_0 is the character on $\hat{G}$ given by $-\gamma_1$. Thus

$$\begin{aligned} x(\gamma_1)\, z(\gamma_2)\, \beta(\gamma_1, \gamma_2) &= \delta(\gamma_1)\, c_1(\gamma_1 + \gamma_2) + g_0(\gamma_1)\, c_2(\gamma_1 + \gamma_2) \\ &= \delta(\gamma_1)\, c_1(\gamma_2) + g_0(\gamma_1)\, c_2(\gamma_1 + \gamma_2). \end{aligned}$$

Now, the condition $x(0) = 0$ gives $c_1(\gamma_2) + c_2(\gamma_2) = 0$ for all γ_2, i.e., $c_1(\gamma) = -c_2(\gamma) = c(\gamma)$ for all γ where c is a function. Thus

LEMMA 2.5. – *Hypothesis 2 in Theorem 2 implies that the following set of equations*

$$x(\gamma_1)\, z(\gamma_2)\, \beta(\gamma_1, \gamma_2) = \delta(\gamma_1)\, c(\gamma_2) - g_0(\gamma_1)\, c(\gamma_1 + \gamma_2) \qquad (*)$$

has a solution x, z *with* $x(0) = 0$ *for some nonzero function* c *on* $\hat{G}$, *for each* $g_0 \in G \backslash \{0\}$.

The algebra $\mathfrak{A}$ is a full matrix algebra M_n. If $n = 1$, the implication $2 \Longrightarrow 3$ in Theorem 2 is trivial, and hence we may assume that $n \geqslant 2$. Then G has at least 4 elements. Our aim is to show that the equations (*) have no solution with $c \neq 0$.

LEMMA 2.6. – *If the equations (*) have a solution* x, z, c *with* $c \neq 0$, *one has*

$$x(\gamma) \neq 0 \text{ for all } \gamma \neq 0$$

and

$$\left.\begin{array}{l} z(\gamma) \neq 0 \\ c(\gamma) \neq 0 \end{array}\right\} \text{ for all } \gamma .$$

Proof. – If $x(\gamma_1) = 0$ for a $\gamma_1 \neq 0$, it follows from (*) that $c(\gamma_1 + \gamma_2) = 0$ for all γ_2, which is impossible. Thus $x(\gamma) \neq 0$ for all $\gamma \neq 0$.

If $z(\gamma_0) = 0$ for a $\gamma_0 \in \hat{G}$, it follows from (*) that

$$\delta(\gamma_1)\, c(\gamma_0) = g_0(\gamma_1)\, c(\gamma_1 + \gamma_0)$$

for all $\gamma_1 \in \hat{G}$, i.e., $c(\gamma) = d\delta(\gamma - \gamma_0)$ for some constant d. Thus $c \neq 0$ implies that $z(\gamma_2) \neq 0$ for all $\gamma_2 \neq \gamma_0$. The equations (*) now take the form

$$x(\gamma_1)\, z(\gamma_2)\, \beta(\gamma_1, \gamma_2) = d\,\{\delta(\gamma_1)\, \delta(\gamma_2 - \gamma_0) - g_0(\gamma_1)\, \delta(\gamma_1 + \gamma_2 - \gamma_0)\}\,.$$

Choosing $\gamma_2 \neq \gamma_0$, and $\gamma_2 \neq 0$, with $\gamma_1 + \gamma_2 \neq \gamma_0$, the right hand side of the above equation is zero, whilst $z(\gamma_2)\beta(\gamma_1, \gamma_2) \neq 0$. Thus $x(\gamma_1) = 0$, contradicting the first part of the proof. Thus $z(\gamma_2) \neq 0$ for all γ_2.

Thus $x(\gamma_1)\, z(\gamma_2)\beta(\gamma_1, \gamma_2) \neq 0$ for all $\gamma_1 \neq 0$ and all γ_2, i.e., $c(\gamma_1 + \gamma_2) \neq 0$ for the same (γ_1, γ_2), i.e, $c(\gamma) \neq 0$ for all γ.

We next investigate consequences of the 2-cocycle identity,

$$\beta(\gamma_1, \gamma_2)\, \beta(\gamma_1 + \gamma_2, \gamma_3) = \beta(\gamma_1, \gamma_2 + \gamma_3)\, \beta(\gamma_2, \gamma_3)$$

satisfied by β. Multiplying both sides of this equation by $x(\gamma_1)\, x(\gamma_2)\, x(\gamma_1 + \gamma_2)\, z(\gamma_2)\, z(\gamma_3)\, z(\gamma_2 + \gamma_3)$ and combining with Lemma 2.5 we obtain

$$\begin{aligned} &x(\gamma_2)g_0(\gamma_2)c(\gamma_2+\gamma_3)\{z(\gamma_2)c(\gamma_2+\gamma_3)-z(\gamma_2+\gamma_3)c(\gamma_2)\}\delta(\gamma_1) \\ &\quad -x(\gamma_2)z(\gamma_2+\gamma_3)g_0(\gamma_1)c(\gamma_3)c(0)\delta(\gamma_1+\gamma_2) \\ &\quad +x(\gamma_1)z(0)g_0(\gamma_1)c(\gamma_3)c(\gamma_1+\gamma_3)\delta(\gamma_2) \qquad (**) \\ &\quad +g_0(\gamma_1)g_0(\gamma_2)c(\gamma_1+\gamma_2+\gamma_3)\{x(\gamma_2)z(\gamma_2+\gamma_3)g_0(\gamma_1)c(\gamma_1+\gamma_2) \\ &\qquad\qquad -x(\gamma_1+\gamma_2)z(\gamma_2)c(\gamma_2+\gamma_3)\} = 0\,. \end{aligned}$$

Now, fixing $\gamma_1 = \gamma = 0$, and $\gamma_2 = \xi \neq 0$, such that $\gamma + \xi \neq 0$, the first three terms in (**) disappear, and combining with Lemma 2.6 we obtain

$$x(\xi)g_0(\gamma)c(\gamma+\xi)z(\xi+\gamma_3) = x(\gamma+\xi)z(\xi)c(\xi+\gamma_3)$$

for all $\gamma_3 \in \hat{G}$, i.e., z is proportional to c. By moving the factor of proportionality over to x we may therefore assume $z(\gamma) = c(\gamma)$ for all $\gamma \in \hat{G}$. Inserting this in (**) the $\delta(\gamma_1)$-term disappears, and choosing γ_1, γ_2, such that $\gamma_1 + \gamma_2 \neq 0, \gamma_2 \neq 0$, we obtain $x(\gamma_2)g_0(\gamma_1)c(\gamma_1+\gamma_2) = x(\gamma_1+\gamma_2)c(\gamma_2)$, or

$$\frac{x(\gamma_2)}{c(\gamma_2)g_0(\gamma_2)} = \frac{x(\gamma_1+\gamma_2)}{c(\gamma_1+\gamma_2)g_0(\gamma_1+\gamma_2)}\,,$$

i.e., $\gamma \neq 0 \longmapsto \dfrac{x(\gamma)}{c(\gamma)g_0(\gamma)}$ is a constant d. Thus

$$x(\gamma) = \begin{cases} 0 & \text{if } \gamma = 0 \\ dg_0(\gamma)c(\gamma) & \text{if } \gamma \neq 0\,. \end{cases}$$

From (*) we therefore obtain

$$\beta(\gamma_1,\gamma_2) = \frac{c(\gamma_1+\gamma_2)}{dc(\gamma_1)c(\gamma_2)} = \frac{\varphi(\gamma_1+\gamma_2)}{\varphi(\gamma_1)\varphi(\gamma_2)}$$

whenever $\gamma_1 \neq 0$, where $\varphi(\gamma) = dc(\gamma)$. But as $\beta(\gamma_1, 0) = 1$ for all γ_1 we obtain $\varphi(0) = 1$, and as $\beta(0,\gamma_2) = 1$ for all γ_2. we then see that the relation $\beta(\gamma_1,\gamma_2) = \dfrac{\varphi(\gamma_1+\gamma_2)}{\varphi(\gamma_1)\varphi(\gamma_2)}$ is valid for all $\gamma_1, \gamma_2 \in \hat{G}$. As $|\beta(\gamma_1,\gamma_2)| = 1$ for all γ_1,γ_2 it follows readily that $|\varphi(\gamma)| = 1$ for all γ, and hence β is a coboundary. But as $\mathfrak{A}$ is non-abelian, this is impossible.

We have thus proved that if $\mathfrak{A}$ is non-abelian, condition 2 of Theorem 2 is not valid. This ends the proof of $2 \Longrightarrow 3$.

3. Positivity Properties.

To conclude we examine various concepts of positivity for semigroups acting on $\mathfrak{A} = M_2$.

First consider the ergodic action of $G = \mathbf{Z}_2 \times \mathbf{Z}_2$ on M_2 defined in Section 6 of [6]. The eigenunitaries for this action can be taken to be $\sigma_0 = \mathbf{I}, \sigma_1, \sigma_2, \sigma_3$, where σ_i are the Pauli matrices. It follows that operators H commuting with the G-action have the form $H(\sigma_i) = \lambda_i \sigma_i, i = 0, 1, 2, 3$, and since $\mathfrak{A}^G = \mathbf{C}\mathbf{I}$ the condition $H(X) = 0$, $X \in \mathfrak{A}^G$, is equivalent to $\lambda_0 = 0$. Hence the semigroup $S_t = \exp\{-tH\}$ is determined by the vector $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. It was shown in [6] that S is positive if, and only if, $\lambda_i \geqslant 0$, and S is completely positive if, and only if $t_i \geqslant 0$ where $t_i = -\lambda_i + \sum_{j \neq i} \lambda_j$. But we now argue that S is strongly positive if, and only if, $\lambda_i \geqslant 0$ and $4\lambda_i \lambda_j - t_k^2 \geqslant 0$ for $i = j \neq k \neq i$. Thus the three concepts of positivity are distinct, even for semigroups commuting with an ergodic action.

To derive the characterization of strong positivity of S we first remark that it is equivalent to the conditions

$$Q_\lambda(X) = H(X^*)X + X^* H(X) - H(X^* X) \geqslant 0$$

for all $X \in \mathfrak{A}$. But a state ω over M_2 has the form $\omega(A) = \mathrm{Tr}(\rho A)$ where $\rho = (\mathbf{I} + \rho_1 \sigma_1 + \rho_2 \sigma_2 + \rho_3 \sigma_3)/2$ with $\rho_i \in \mathbf{R}$ and $\rho_1^2 + \rho_2^2 + \rho_3^2 \leqslant 1$. One then calculates that $\omega(Q_\lambda(X)) \geqslant 0$ for all $X \in \mathfrak{A}$ if, and only if, the matrix

$$\begin{bmatrix} 2\lambda_1 & i\rho_3 t_2 & -i\rho_2 t_2 \\ -i\rho_3 t_2 & 2\lambda_2 & i\rho_1 t_1 \\ i\rho_2 t_2 & -i\rho_1 t_1 & 2\lambda_3 \end{bmatrix}$$

is positive. Thus S is strongly positive if, and only if, these matrices are positive for all possible ρ.

By the Principal-Milnor-Theorem (see, for example, [16] Ch.X, Theorems 4 and 20) and the conditions $\rho_1^2 + \rho_2^2 + \rho_3^2 \leqslant 1$, positivity of these matrices is equivalent to the six conditions $\lambda_i \geqslant 0$ and $4\lambda_i \lambda_j - t_k^2 \geqslant 0$, $i \neq j \neq k \neq i$. In particular positivity, strong positivity, and complete positivity, are distinct properties for the semigroup.

A specific example of a strongly positive semigroup which is not completely positive is given by the choice

$$\lambda_0 = 0, \lambda_1 = 1/2, \lambda_2 = 3/2, \lambda_3 = 1/2.$$

This semigroup has the action

$$S_t(X) = (Ch(t/2)\,X + Sh(t/2)\,X^{tr})\,e^{-t} + \mathrm{Tr}(X)\,(1 - e^{-t/2})\,\mathbf{1}$$

where X^{tr} denotes the transpose of X and Tr(X) is the normalized trace. If H is the generator of this S then $\varphi = I - H$ is the strongly positive operator exhibited by Choi [12] which is not 2-positive. Similarly S_t is not 2-positive for all $t \geqslant 0$. Specifically 2-positivity fails for $0 < t < 2 \ln[(1 + \sqrt{5})/2]$. Nevertheless S_t is 2-positive for large t, e.g., asymptotically one has $\lim\limits_{t \to \infty} S_t(X) = \mathrm{Tr}(X)\,\mathbf{1}$.

Finally we mention that Choi [31] has shown that n-positivity and complete positivity are equivalent properties for operators on M_n.

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