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AN APPLICATION OF FINE POTENTIAL THEORY TO PROVE A PHRAGMEN LINDELÖF THEOREM

by **Terry J. LYONS**

In response to a conjecture of Newman, Fuchs [2] proved a Phragmen Lindelöf theorem valid for an arbitrary open subset of $\mathbf{C}$. More recently Gehring, Hayman and Hinkkanen [4] have extended these arguments and given sufficient conditions for Hölder estimates on the function at the boundary of the open set to propagate to the interior. In this paper we give a simple alternative proof of Fuchs' result. Our methods extend to give a proof of the result in [4].

We establish some notation. U is an arbitrary unbounded open connected proper subset of $\mathbf{C}$; ∂U is the topological boundary of U and U^c the complement of U. We say that U is a neighbourhood of ∞ if U contains the complement of some disc. Let W be an open set containing U^c.

THEOREM (Fuchs). — *Let f be an analytic function on U which is bounded on $W \cap U$ and which has at most polynomial growth as z tends to ∞ in U. (So $|f(z)| = o(|z|^n)$ for some $n > 0$ as $|z| \to \infty$ in U). If U is not a neighbourhood of ∞ then f is bounded.*

If U is a neighbourhood of ∞ then the hypotheses on growth imply easily that f has a pole at ∞ and although f does not need to be bounded it must grow like $|z|^n$ for some positive integer $n \geqslant 0$. We therefore have the following.

THEOREM (Conjectured by Newman). — *Let U, W, f be as above and suppose further that $|f(z)| = o(|z|)$ as z tends to infinity in U. Then f is bounded.*

It is clear from examples that if no restriction is placed on U then these theorems cannot be much improved.

The simple statement of the main theorem suggests that one should argue differently according to whether U is a neighbourhood of ∞ or not. In fact the argument will depend on whether U is a fine neighbourhood of ∞. In more classical language – the argument depends on whether ∞ is a regular boundary point of U or not.

If ∞ is a regular boundary point the proof is relatively standard potential theory and would follow from [5]. However, we will outline a probabilistic proof of this case here. To do this we first make some preliminary remarks about h-transforms [1]. In the last two paragraphs we will return to the case where ∞ is irregular.

Let $h > 0$ be harmonic on V. Then one may define h-transformed Brownian motion on V by defining its Radon Nikodyn derivative with respect to usual Brownian motion. Let $T_K(w)$ denote the first time $Z_t(w)$ leaves a compact set K in V and suppose $A \in \mathscr{F}_{T_K}$ where $\mathscr{F}_{T_K}$ is the σ-algebra of events up to time T_k. Then

$$\mathbf{P}^{x,h}(w \in A) = h(x)^{-1}\mathbf{E}(h(Z_{T_K}).\chi_A).$$

Because $h(Z_{t\wedge T})$ is a martingale it follows that if $A \in \mathscr{F}_{T_{K'}}$ and $K' \subset K$ then the two definitions of $P^{x,h}(w \in A)$ are consistent. Suppose u is superharmonic then one knows that $u(Z_{t\wedge K})$ is a $\mathbf{P}^x$ supermartingale for each K and among l.s.c. functions this determines superharmonicity. Using the definition of $\mathbf{P}^{x,h}$ one sees that a l.s.c. function u is superharmonic if and only if $u/h(Z_{t\wedge K})$ is a $\mathbf{P}^{x,h}$ local supermartingale. We now return to the main thrust of our argument.

Let U, f, W be as in the theorem. We may without loss of generality assume $U \subset \{|z| > 1\}$. Put $V = \left\{z : \frac{1}{z} \in U\right\}$, $g(z) = f\left(\frac{1}{z}\right)$ and $W' = \left\{z : \frac{1}{z} \in W\right\}$. To say that infinity is a regular point for U is to say that if Brownian motion Z_t is started from zero then it will almost surely leave $V \cup \{0\}$ immediately.

Consider then the Brownian bridge W_t^a terminating at $a \in D = \{|z| < 1\}$ obtained by h-transforming [1] the usual Brownian motion on $D\backslash\{a\}$ using the positive harmonic function $g(\,, a)$ where g is the Green function for D. For $a \neq 0$ the Brownian bridge W_t^a will leave V instantaneously when started at zero, because the h-transform construction ensures that the laws of W_t^a and Z_t are mutually absolutely continuous on paths stopped when they first get within ε of a of ∂D.

Because the Green function is symmetric [7] one knows that a Brownian bridge W_t^a from zero to a when run backwards from its arrival time at a is identical in law to W_t^0 started at a. Therefore a Brownian bridge to zero W_t^0 will almost surely leave V through Y before hitting zero. This is the vital fact.

Suppose $|f| < 1$ on $Y \cap W'$. Then we know that $\dfrac{\log|f|}{\log|z|}$ is lower bounded on V and positive on $W' \cap Y$.

Moreover because $-\log|f|$ is superharmonic and $-\log|z| = g(0, z)$ it follows that $\dfrac{\log|f(W_t^0)|}{\log|W_t^0|}$ is a lower bounded local supermartingale for $t \leqslant T_{V\setminus W'}$. Lower bounded local supermartingales are supermartingales so we know that for any $a \in V$

$$\frac{\log|f(a)|}{\log|a|} \geqslant \mathbf{E}^{x,h}\left[\frac{\log|f(W^0_{T_{V\setminus W'}})|}{\log|W^0_{T_{V\setminus W'}}|}\right]\cdot\chi(T_{V\setminus W'} < T_V)$$
$$+ \mathbf{E}^{(x,h)}\left(\lim_{t\to T_{V\setminus W'}} \frac{\log|f(W_t^0)|}{\log|W_t^0|}\cdot\chi(T_{V\setminus W'} = T_V)\right).$$

But we have already seen that W^0 almost surely leaves V before it gets to zero so $\mathbf{P}^{(x,h)}(T_{V\setminus W'} = T_V) = 0$. The first term of the right hand expression is positive and so $\log|f(a)| \leqslant 0$ for all $a \in A$ as required.

We remark that the classical Phragmen Lindelöf theorem can be proved very readily using this sort of technique. (Consider $\dfrac{-\log|f(x+iy)|}{y}$ on a wedge in the upper half plane.)

Suppose that g is analytic on V, and grows at most as fast as $\dfrac{1}{|z|^{-n}}$ at the origin, and that V is a fine neighbourhood of zero. (So Brownian motion started at zero does stay in V for some small random interval of time). If g is not bounded we may consider $\tilde{V} = \{z : g(z) > \lambda\}$, and by choosing λ large enough we may assume $|g(z)| = \lambda$ for $z \in \partial\tilde{V}$. By what we have just proved we may also assume that $\tilde{V}$ is a fine neighbourhood of zero. By a well known estimate of Beurling it follows that any fine neighbourhood of 0 in the complex plane – in particular $\tilde{V}$ – must contain circles of arbitrarily small radius centred on zero. We will prove that $\tilde{V}^c$ has only finitely many components and hence prove the theorem.

For some n $g(z)z^n$ is bounded and hence extends finely continuously to a finely continuous and finely holomorphic function at zero. Now it is shown in [6] that such functions have fine derivatives of all orders and in [3] that these derivatives cannot all be zero at a point unless g is identically zero. It follows from [3] that there is a $j \geqslant 0$ such that

$$\lim_{\substack{z \to 0 \\ \text{in fine} \\ \text{topology}}} \frac{g(z)z^n}{z^j} = c$$

exists and is not zero. In particular we may find circles Γ_k of arbitrarily small radius centered on zero with

$$N(g(\Gamma_k),0) = \frac{1}{2\pi i} \int_{\Gamma_k} \frac{g'(z)\,dz}{g(z)} = j - n$$

where j and n are constants. On the other hand let γ_ℓ be a boundary curve of $\tilde{V}$. Then $N(g(\gamma_\ell),0) = \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_k} \frac{g'(z)\,dz}{g(z)} \leqslant -1$, and because g does not take the value zero we have $\Sigma N(g(\gamma_\ell),0) = N(g(\Gamma_k),0)$ where the left hand summation is taken over those curves γ_ℓ lying outside Γ_k. In particular there can be no more than n of them. The number of components of $\tilde{V}^c$ is at most n.

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