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THE LEVI PROBLEM FOR COHOMOLOGY CLASSES

by Mihnea COLTOIU

Introduction.

The aim of this paper is to extend some of the results of Andreotti and Norguet from [4] to complex spaces.

The paper is divided into two paragraphs :

- 1) The local problem
- 2) The global problem

In the first paragraph we prove the following

THEOREM 1. — *Let X be a perfect complex space, $Y \subset X$ an open subset, $x_0 \in \partial Y$ and \mathfrak{F} a sheaf which is locally free in a neighbourhood of x_0 . Suppose Y is strongly pseudoconcave in x_0 and let $n_0 = \dim \mathcal{O}_{X, x_0} > 0$. Then $H^{n_0-1}(Y, x_0, \mathfrak{F})$ contains an infinitely dimensional vector subspace all of whose non-zero elements are not extendable in x_0 .*

When X is a complex manifold this result was proved in [4] using a generalization of an integral formula of E. Martinelli. In the proof of Theorem 1 we use elementary results of local cohomology (one needs only supports consisting of a point) and the local structure theorems of a strongly pseudoconcave domain from [2].

The second paragraph is devoted to the generalization of Theorem 3 from [4]. More precisely we prove

THEOREM 2. — *Let X be a complex space and $Y \subset\subset X$ an open subset which is strongly q -pseudoconvex. Suppose Y is strictly q -pseudoconvex in every point of $\partial Y \cap \text{Reg}(X)$ and let $\mathfrak{F} \in \text{Coh}(X)$ such that $\partial Y \subset \text{supp}(\mathfrak{F})$. Then there exists an element in $H^q(Y, \mathfrak{F})$ which is not extendable in any point of ∂Y .*

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1. The local problem.

Let us briefly recall some definitions from [4] which will be used throughout this paper.

Let \mathcal{F} be a sheaf of vector spaces on a topological space X , $Y \subset X$ an open subset and x_0 a point in ∂Y . Put :

$$\begin{aligned} H^r(Y, x_0, \mathcal{F}) &= \varinjlim_{U \in \mathcal{V}_{x_0}} H^r(Y \cap U, \mathcal{F}) \\ H_+^r(Y \cup \{x_0\}, \mathcal{F}) &= \varinjlim_{U \in \mathcal{V}_{x_0}} H^r(Y \cup U, \mathcal{F}) \\ H^r(x_0, \mathcal{F}) &= \varinjlim_{U \in \mathcal{V}_{x_0}} H^r(U, \mathcal{F}) \end{aligned}$$

where \mathcal{V}_{x_0} = the set of all open neighbourhoods U of x_0 in X .

We have $H^0(x_0, \mathcal{F}) = \mathcal{F}_{x_0}$ and $H^r(x_0, \mathcal{F}) = \{0\}$ for $r \geq 1$ (cf. [6, pp. 192-193]). Consider the natural restriction maps :

$$\begin{aligned} r_1 : H^r(x_0, \mathcal{F}) &\longrightarrow H^r(Y, x_0, \mathcal{F}) \\ r_2 : H_+^r(Y \cup \{x_0\}, \mathcal{F}) &\longrightarrow H^r(Y, \mathcal{F}). \end{aligned}$$

An element in $H^r(Y, x_0, \mathcal{F})$ (in $H^r(Y, \mathcal{F})$) will be called extendable in $x_0 \in \partial Y$ if it belongs to the image of the map r_1 (r_2 respectively).

Suppose now that X is a complex space. We say that Y is strongly pseudoconcave in x_0 if there exist an open neighbourhood U of x_0 in X and $\varphi \in C^\infty(U, \mathbb{R})$ a strongly plurisubharmonic function such that $U \cap Y = \{x \in U \mid \varphi(x) > \varphi(x_0)\}$.

If $x_0 \in \text{Reg}(X)$ we say that Y is strictly q -pseudoconvex in x_0 if there exist an open neighbourhood U of x_0 and $\varphi \in C^\infty(U, \mathbb{R})$ such that :

- i) $(d\varphi)_{x_0} \neq 0$
- ii) $U \cap Y = \{x \in U \mid \varphi(x) < \varphi(x_0)\}$

iii) the restriction of the Levi form $\mathcal{L}(\varphi)$ to the analytic tangent hyperplane to ∂Y at x_0 is nondegenerate and admits precisely q strictly negative eigenvalues.

Let us also recall that a complex space X is called perfect if $\mathcal{O}_{X,x}$ is Cohen-Macaulay for any $x \in X$. We denote by $H_{x_0}^i(X, \cdot)$ the cohomology groups with support in $\{x_0\}$. In order to prove Theorem 1 we shall need the following statement

PROPOSITION 1. — *Let X be a perfect complex space (not necessarily reduced), $x_0 \in X$ and $n_0 = \dim \mathcal{O}_{X,x_0} > 0$. put $L_{x_0} = \varinjlim_{U \in \mathfrak{V}_{x_0}} H_{x_0}^{n_0}(U, \mathcal{O}_X)$. Then $\dim_{\mathbb{C}} L_{x_0} = \infty$.*

The above proposition is an immediate consequence of [5, pp. 86, Corollaire 4.5.].

Remark 1. — If $U \in \mathfrak{V}_{x_0}$ we have the exact sequence

$$H^{n_0-1}(U, \mathcal{O}_X) \longrightarrow H^{n_0-1}(U \setminus \{x_0\}, \mathcal{O}_X) \longrightarrow H_{x_0}^{n_0}(U, \mathcal{O}_X) \longrightarrow H^{n_0}(U, \mathcal{O}_X).$$

Taking inductive limit we get

$$L_{x_0} \cong H^{n_0-1}(X \setminus \{x_0\}, \mathcal{O}_X) \text{ for } n_0 \geq 2.$$

THEOREM 1. — *Let X be a perfect complex space, $Y \subset X$ an open subset, $x_0 \in \partial Y$ and \mathfrak{F} a sheaf which is locally free in a neighbourhood of x_0 . Suppose Y is strongly pseudoconcave in x_0 and let $n_0 = \dim \mathcal{O}_{X,x} > 0$. Then $H^{n_0-1}(Y, x_0, \mathfrak{F})$ contains an infinitely dimensional vector subspace all of whose non-zero elements are not extendable in x_0 .*

Proof. — Obviously, we may suppose $\mathfrak{F} = \mathcal{O}_X$. Since the problem is local we also may suppose that X is a closed analytic subset of some open set $G \subset \mathbb{C}^N$ and that

$$Y = \{x \in X \mid \varphi(x) > \varphi(x_0)\},$$

where $\varphi \in C^\infty(G, \mathbb{R})$ is a strongly plurisubharmonic function.

Writing the Taylor expansion of φ at x_0 we get :

$$\varphi(x) = \varphi(x_0) + 2 \operatorname{Re} f(x) + \mathcal{L}(\varphi)(x) + O(\|x - x_0\|^3)$$

where f is a polynomial of degree two in x and $\mathcal{L}(\varphi)$ is the Levi form. Let $g = f|_X$ and $Z_0 = \{x \in X \mid g(x) = 0\}$.

Replacing G by a smaller subset we may suppose that $Z_0 \setminus \{x_0\} \subset Y$. Moreover, using the perturbation argument in [7, pp. 357-358], we may suppose that the image of g in $\mathcal{O}_{X,x}$ is

not a zero-divisor for any $x \in X$. Consider the space (Z_0, \mathcal{O}_{Z_0}) where $\mathcal{O}_{Z_0} = \mathcal{O}_X/g \mathcal{O}_X$. Since X is a perfect space and the image of g in $\mathcal{O}_{X,x}$ is not a zero-divisor for any $x \in X$ it follows that (Z_0, \mathcal{O}_{Z_0}) is also perfect.

Put $n_0 = \dim \mathcal{O}_{X,x_0}$, hence $n_0 - 1 = \dim \mathcal{O}_{Z_0,x_0}$, and let $L_{x_0} = \varinjlim_{U' \in \mathcal{V}'_{x_0}} H_{x_0}^{n_0-1}(U', \mathcal{O}_{Z_0})$ where \mathcal{V}'_{x_0} = the set of all open neighbourhoods U' of x_0 in Z_0 .

Consider the exact sequence of sheaves on Y

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\cdot g} \mathcal{O}_X \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0. \quad (1)$$

If $U \subset X$ is an open neighbourhood of x_0 , then (1) together with the long exact sequence of cohomology provide the exact sequence

$$H^q(Y \cap U, \mathcal{O}_X) \longrightarrow H^q(U' \setminus \{x_0\}, \mathcal{O}_{Z_0}) \longrightarrow H^{q+1}(Y \cap U, \mathcal{O}_X) \quad (2)$$

where $U' = U \cap Z_0$ (recall that by choice of Z_0 we have $Y \cap U' = U' \setminus \{x_0\}$).

Consider first the case $n_0 \geq 3$. Making $q = n_0 - 2$ in (2) and taking inductive limit we get the exact sequence

$$\begin{aligned} H^{n_0-2}(Y, x_0, \mathcal{O}_X) &\longrightarrow H^{n_0-2}(Z_0 \setminus \{x_0\}, x_0, \mathcal{O}_{Z_0}) \\ &\longrightarrow H^{n_0-1}(Y, x_0, \mathcal{O}_X). \end{aligned} \quad (3)$$

By [2, Théorème 9] we get $H^{n_0-2}(Y, x_0, \mathcal{O}_X) = 0$. Since $H^{n_0-2}(Z_0 \setminus \{x_0\}, x_0, \mathcal{O}_{Z_0}) \cong L_{x_0}$, Proposition 1 implies that $\dim_{\mathbb{C}} H^{n_0-1}(Y, x_0, \mathcal{O}_X) = \infty$ hence the theorem is proved for $n_0 \geq 3$.

For $n_0 = 1$ the theorem is obvious, hence to conclude the proof we only have to deal with the case $n_0 = 2$. If $U \subset X$ is an open neighbourhood of x_0 , then by (1) and the long exact sequence of cohomology we get the exact sequence

$$H^0(Y \cap U, \mathcal{O}_X) \longrightarrow H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0}) \longrightarrow H^1(Y \cap U, \mathcal{O}_X) \quad (4)$$

where $U' = U \cap Z_0$.

By [2, Théorème 10] there exists a fundamental system of Stein neighbourhoods U of x_0 in X such that the restriction map $H^0(U, \mathcal{O}_X) \longrightarrow H^0(Y \cap U, \mathcal{O}_X)$ is bijective. The commutative diagram

$$\begin{array}{ccc} H^0(U, \mathcal{O}_X) & \xrightarrow{\sim} & H^0(Y \cap U, \mathcal{O}_X) \\ \downarrow & & \downarrow \\ H^0(U', \mathcal{O}_{Z_0}) & \longrightarrow & H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0}) \end{array}$$

and the surjectivity of the map $H^0(U, \mathcal{O}_X) \longrightarrow H^0(U', \mathcal{O}_{Z_0})$ imply that

$$\begin{aligned} \text{Im}(H^0(Y \cap U, \mathcal{O}_X) \longrightarrow H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0})) \\ = \text{Im}(H^0(U', \mathcal{O}_{Z_0}) \longrightarrow H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0})), \end{aligned}$$

hence there is a natural injection $H^1_{x_0}(U', \mathcal{O}_{Z_0}) \longrightarrow H^1(Y \cap U, \mathcal{O}_X)$. Taking inductive limit it follows that the map $L_{x_0} \longrightarrow H^1(Y, x_0, \mathcal{O}_X)$ is injective, hence by Proposition 1 we get $\dim_{\mathbb{C}} H^1(Y, x_0, \mathcal{O}_X) = \infty$, and we are done.

COROLLARY 1 [4, Proposition 6]. — *Let Y be an open subset of a complex manifold X , $x_0 \in \partial Y$ and suppose Y is strictly q -pseudoconvex in x_0 . Let \mathcal{F} be a sheaf which is locally free in a neighbourhood of x_0 . Then $H^q(Y, x_0, \mathcal{F})$ contains an infinitely dimensional vector subspace all of whose non-zero elements are not extendable in x_0 .*

Proof. — We may suppose $\mathcal{F} = \mathcal{O}_X$ and $q > 0$ (the case $q = 0$ is obvious).

By definition of strictly q -pseudoconvexity it immediately follows that :

- i) Y is strongly q -pseudoconvex in a neighbourhood of x_0 .
- ii) In some neighbourhood of x_0 there exists an analytic submanifold B containing x_0 such that $\dim B = q + 1$ and $B \cap Y$ is strongly pseudoconcave in x_0 . By [2, Théorème 5] we deduce that the map

$$H^q(Y, x_0, \mathcal{O}_X) \longrightarrow H^q(B \cap Y, x_0, \mathcal{O}_B)$$

is surjective and using Theorem 1 we get $\dim_{\mathbb{C}} H^q(Y, x_0, \mathcal{O}_X) = \infty$.

Remark 2. — Let φ be a strongly plurisubharmonic function in some neighbourhood U of the origin in \mathbb{C}^n ($n \geq 2$), $(d\varphi)_0 \neq 0$ and put $Y = \{z \in U \mid \varphi(z) > \varphi(0)\}$. In suitable coordinates the Taylor expansion of φ at 0 has the form

$$\varphi(z) = \varphi(0) + 2 \operatorname{Re} z_1 + \sum_{1 \leq j, k \leq n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} (0) z_j \bar{z}_k + O(\|z\|^3).$$

Put exactly as in [4]

$$\psi_\alpha = \left(\sum_{1 \leq j \leq n} z_j^{\alpha_j} \bar{z}_j^{\alpha_j} \right)^{-n} \sum_{1 \leq j \leq n} (-1)^{j-1} \bar{z}_j^{\alpha_j} \wedge_{\substack{1 \leq k \leq n \\ k \neq j}} d(\bar{z}_k^{\alpha_k}).$$

By [4, Proposition 5] it follows that the images of the differential forms $\psi_{\alpha+1}$ ($\alpha \in \mathbb{N}^n$) in $H^{n-1}(Y \cap U, \Theta)$ are linearly independent. Let M be the linear span of the above images.

We shall now investigate the relation between M and the vector space considered in the proof of Theorem 1 (which we denote now by L_1). Recall that L_1 is the kernel of the map $\alpha_1 =$ multiplication by z_1 ,

$$\alpha_1 : H^{n-1}(Y \cap U, \Theta) \longrightarrow H^{n-1}(Y \cap U, \Theta).$$

In the same way we define $\alpha_k =$ multiplication by z_k^k ,

$$\alpha_k : H^{n-1}(Y \cap U, \Theta) \longrightarrow H^{n-1}(Y \cap U, \Theta)$$

and put $L_k = \ker \alpha_k$, $L = \bigcap_{k=1}^{\infty} L_k$. We claim that $M \subset L$. To

prove this inclusion we use the relation $z_1^{\alpha_1} \psi_\alpha = \bar{\partial} \mu_\alpha$ where

$$\mu_\alpha = \frac{1}{n-1} \left(\sum_{1 \leq j \leq n} z_j^{\alpha_j} \bar{z}_j^{\alpha_j} \right)^{1-n} \wedge \sum_{2 \leq j \leq n} (-1)^j \bar{z}_j^{\alpha_j} d(\bar{z}_k^{\alpha_k}).$$

This equality shows that the image of $\psi_{\alpha+1}$ in $H^{n-1}(Y \cap U, \Theta)$ is contained in L_{α_1+1} , hence $M \subset L$.

2. The global problem.

$\alpha)$ Let U be an open subset of \mathbb{C}^n and $\varphi \in C^\infty(U, \mathbb{R})$. Recall that φ is called strongly q -pseudoconvex ($0 \leq q \leq n-1$) if the Levi form $\mathcal{L}(\varphi)$ has at least $(n-q)$ strictly positive eigenvalues at any point in U . Using local embeddings in the Zarisky tangent space one easily extends the notion of strongly q -pseudoconvex function in the case of complex spaces (for details see [1, pp. 12-13]).

Remark 3. — Let X be a complex space and $\varphi : X \rightarrow \mathbb{R}$ a strongly q -pseudoconvex function. For any $x \in X$ put $\mu(x) = \min \dim X_x^i$ where X_x^i are the irreducible components

of X_x (X_x denotes the germ of X in x). From the above definitions it immediately follows that $q < \min_{x \in X} \mu(x)$.

To state our theorem recall the following definition : an open subset $Y \subset X$ is called strongly q -pseudoconvex if there exist an open neighbourhood V of ∂Y and $\varphi \in C^\infty(V, \mathbf{R})$ a strongly q -pseudoconvex function such that $V \cap Y = \{x \in V \mid \varphi(x) < 0\}$.

If $\mathfrak{F} \in \text{Coh}(X)$ and $Y \subset X$ is strongly q -pseudoconvex we have [2, Théorème 11] $\dim_{\mathbf{C}} H^r(Y, \mathfrak{F}) < \infty$ if $r \geq q + 1$.

As we already announced in the introduction the aim of this paragraph is to prove the following

THEOREM 2. — *Let X be a complex space and $Y \subset X$ an open subset which is strongly q -pseudoconvex. Suppose Y is strictly q -pseudoconvex in every point of $\partial Y \cap \text{Reg}(X)$ and let $\mathfrak{F} \in \text{Coh}(X)$ such that $\partial Y \subset \text{supp}(\mathfrak{F})$. Then there exists an element in $H^q(Y, \mathfrak{F})$ which is not extendable in any point of ∂Y .*

β) **LEMMA 1.** — *Let $Y \subset X$ be an open subset such that Y is strongly q -pseudoconvex and let $A \subset X$ be an analytic closed subset such that $\dim_x A < \dim_x X$ for any $x \in A$. Then $\partial Y \setminus A$ is dense in ∂Y .*

Proof. — Let V be an open neighbourhood of ∂Y and $\varphi \in C^\infty(V, \mathbf{R})$ a strongly q -pseudoconvex function such that $V \cap Y = \{x \in V \mid \varphi(x) < 0\}$. Let's make a couple of remarks :

1) For any point $x \in A$ with X_x irreducible there exists a fundamental system of open neighbourhoods $(U_i)_{i \in \mathbf{N}}$ of x such that $U_i \setminus A$ is connected.

2) For any point $x \in \partial Y$ there exists a germ of analytic set Q_x passing through x , $\dim_x Q_x \geq 1$ and $\varphi|_{Q_x}$ is strongly plurisubharmonic.

Assertion 1) is well known and 2) may be deduced from [8, pp. 46, Corollary 4] using the condition $q < \min_{x \in \partial Y} \dim \mathcal{O}_{X,x}$ (which is a consequence of Remark 3). Let's show now that $\partial Y \setminus A$ is dense in ∂Y .

a) Take first $x_0 \in \partial Y \cap A$ such that X_{x_0} is irreducible and

let $(U_i)_{i \in \mathbb{N}}$ be a fundamental system of open neighbourhoods of x_0 such that $U_i \setminus A$ is connected and $U_i \subset V$. We must prove that for any i $\partial Y \cap U_i \not\subset A \cap U_i$. If there existed an i_0 such that $\partial Y \cap U_{i_0} \subset A \cap U_{i_0}$ we would get

$$U_{i_0} \setminus A = [(U_{i_0} \cap Y) \setminus A] \cup [(U_{i_0} \cap \mathbb{C} \bar{Y}) \setminus A]$$

and since $U_{i_0} \setminus A$ is connected we would get $(U_{i_0} \cap \mathbb{C} \bar{Y}) \setminus A = \emptyset$, hence $U_{i_0} \subset \bar{Y}$. In particular we would have $\varphi \leq 0$ on U_{i_0} .

Since $\varphi(x_0) = 0$ and $\varphi|_{Q_{x_0}}$ is strongly plurisubharmonic the maximum principle yields a contradiction and we are done.

b) Take now $x_0 \in \partial Y \cap A$ and suppose that X_{x_0} is not irreducible. Let $X_{x_0} = \bigcup_{i=1}^{k_0} X_{x_0}^i$ be the decomposition of X_{x_0} into irreducible components. One may easily deduce that there exist $i_0 \in \{1, \dots, k_0\}$ and an open neighbourhood $U = U(x_0)$ of x_0 such that $X_{x_0}^{i_0}$ is induced in U by an irreducible subspace $Z = Z(x_0)$ with $x_0 \in \partial(Y \cap Z)$. On the other hand by Remark 3 we get that $q < \dim Z$. If we put $A' = A \cap Z$ and $\varphi' = \varphi|_Z$ it follows that $\dim A' < \dim Z$ and φ' is strongly q -pseudoconvex. Hence there exists a germ of analytic set Q'_{x_0} passing through x_0 with $\dim_{x_0} Q'_{x_0} \geq 1$, $Q'_{x_0} \subset Z$ and $\varphi'|_{Q'_{x_0}}$ is strongly plurisubharmonic. Since Z_{x_0} is irreducible the same reasoning as in a) shows that we may find a sequence $(x_n)_{n \in \mathbb{N}}$, $x_n \rightarrow x_0$ and $x_n \in \partial(Y \cap Z) \setminus A'$. Lemma 1 is completely proved.

COROLLARY 2. — *Let $Y \subset\subset X$ be an open subset such that Y is strongly q -pseudoconvex and let $\mathfrak{F} \in \text{Coh}(X)$ such that $\partial Y \subset \text{supp}(\mathfrak{F})$. Then there exists an open subset $D \subset X$ such that :*

- a) $D \subset \text{Reg}(X)$
- b) $\mathfrak{F}|_D$ is locally free of rank ≥ 1 (the rank not being necessarily constant)
- c) $\partial Y \cap D$ is dense in ∂Y .

Proof. — Put $A_1 = \{x \in X \mid \mathfrak{F}_x \text{ is not a free } \mathcal{O}_{x,x}\text{-module}\}$. It is well known that A_1 is an analytic closed subset of X and $\dim_x A_1 < \dim_x X$ for any $x \in A_1$. Put $D_1 = X \setminus (A_1 \cup \text{Sing}(X))$

and $D = D_1 \cap \text{supp}(\mathcal{F})$. By Lemma 1 we immediately deduce that D satisfies conditions a), b), c) and we are done.

γ) Let X be a complex space, $\mathcal{F} \in \text{Coh}(X)$, $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ a locally finite open covering of X . Put :

$Z^p(\mathcal{U}, \mathcal{F}) =$ the group of p -cocycles with values in \mathcal{F} , with its natural topology of Fréchet space

$H^p(\mathcal{U}, \mathcal{F}) =$ the p -th group of Čech cohomology of \mathcal{F} with respect to \mathcal{U}

$H^p(X, \mathcal{F}) =$ the p -th cohomology group of \mathcal{F} computed using the canonical resolution of Godement

$\Theta_{\mathcal{U}} : H^p(\mathcal{U}, \mathcal{F}) \rightarrow H^p(X, \mathcal{F})$ the natural maps between the above groups.

If U_i is Stein for any i then $\Theta_{\mathcal{U}}$ are isomorphisms. Let now $X' \subset X$ be an open subset and $\mathcal{U}' = (U'_i)_{i \in \mathbb{N}}$ the covering defined by $U'_i = U_i \cap X'$. We have a commutative diagram :

$$\begin{array}{ccc} H^p(\mathcal{U}, \mathcal{F}) & \xrightarrow{\Theta_{\mathcal{U}}} & H^p(X, \mathcal{F}) \\ \downarrow & & \downarrow \\ H^p(\mathcal{U}', \mathcal{F}) & \xrightarrow{\Theta_{\mathcal{U}'}} & H^p(X', \mathcal{F}) \end{array}$$

Suppose now X is a complex manifold and E is a holomorphic vector bundle over X . Put $\mathcal{F} = \mathcal{O}(E)$ which is a locally free sheaf on X . Let $\mathcal{G}^{p,q}(E)$ be the sheaf of germs of C^∞ E -valued forms of type (p,q) . Consider the Dolbeault resolution

$$0 \rightarrow \mathcal{O}(E) \rightarrow \mathcal{G}^{0,0}(E) \xrightarrow{\bar{\partial}} \mathcal{G}^{0,1}(E) \xrightarrow{\bar{\partial}} \dots$$

Put :

$$Z^p(X, E) = \ker \{ \Gamma(X, \mathcal{G}^{0,p}(E)) \xrightarrow{\bar{\partial}} \Gamma(X, \mathcal{G}^{0,p+1}(E)) \}$$

with its natural topology of Fréchet space

$$B^p(X, E) = \text{Im} \{ \Gamma(X, \mathcal{G}^{0,p-1}(E)) \xrightarrow{\bar{\partial}} \Gamma(X, \mathcal{G}^{0,p}(E)) \}$$

$$H^p_{\bar{\partial}}(X, E) = Z^p(X, E) / B^p(X, E).$$

Let $\psi = (\psi_i)_{i \in \mathbb{N}}$ be a partition of unity with respect to $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$. Define $T_{\mathcal{U}, \psi} : Z^p(\mathcal{U}, \mathcal{O}(E)) \rightarrow Z^p(X, E)$ by

$$T_{\mathcal{U}, \psi}(\xi) = \sum_{i_0 \dots i_p} \xi_{i_0 \dots i_p} \psi_{i_0} \bar{\partial} \psi_{i_1} \wedge \dots \wedge \bar{\partial} \psi_{i_p}$$

$T_{\mathcal{U}, \psi}$ is a continuous linear operator. The operator

$$T_{\mathcal{U}} : H^p(\mathcal{U}, \mathcal{O}(E)) \longrightarrow H^p_{\mathfrak{S}}(X, E),$$

induced by $T_{\mathcal{U}, \psi}$, does not depend on ψ . Furthermore if $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ is a Stein covering then $T_{\mathcal{U}}$ is an algebraic and topological isomorphism (cf. [3, pp. 225-227]).

Let now $X' \subset X$ be an open subset and $\mathcal{U}' = (U'_i)_{i \in \mathbb{N}}$ the covering defined by $U'_i = U_i \cap X'$. Since $T_{\mathcal{U}}$ does not depend on ψ we get the following commutative diagram :

$$\begin{array}{ccc} H^p(\mathcal{U}, \mathcal{O}(E)) & \xrightarrow{T_{\mathcal{U}}} & H^p_{\mathfrak{S}}(X, E) \\ \downarrow & & \downarrow \\ H^p(\mathcal{U}', \mathcal{O}(E)) & \xrightarrow{T_{\mathcal{U}'}} & H^p_{\mathfrak{S}}(X', E) \end{array}$$

If $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ is a Stein covering of X we may define the isomorphism $H^p_{\mathfrak{S}}(X, E) \longrightarrow H^p(X, \mathcal{O}(E))$ as the composed map $H^p_{\mathfrak{S}}(X, E) \xrightarrow{T_{\mathcal{U}}^{-1}} H^p(\mathcal{U}, \mathcal{O}(E)) \xrightarrow{\Theta_{\mathcal{U}}} H^p(X, \mathcal{O}(E))$. One verifies immediately that the above isomorphism does not depend on \mathcal{U} and denote this isomorphism by L_X . For any open subset $X' \subset X$ we have a commutative diagram :

$$\begin{array}{ccc} H^p_{\mathfrak{S}}(X, E) & \xrightarrow{L_X} & H^p(X, \mathcal{O}(E)) \\ \downarrow & & \downarrow \\ H^p_{\mathfrak{S}}(X', E) & \xrightarrow{L_{X'}} & H^p(X', \mathcal{O}(E)) \end{array}$$

δ) *Proof of Theorem 2*

We shall suppose $q > 0$ since the case $q = 0$ is well known. Let $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ be a locally finite Stein covering of Y and $D \subset X$ having properties a), b), c) from Corollary 2. Put $D' = D \cap Y$, $U'_i = U_i \cap D$, $\mathcal{U}' = (U'_i)_{i \in \mathbb{N}}$ = a locally finite open covering of D' . Let $\psi = (\psi_i)_{i \in \mathbb{N}}$ be a partition of unity with respect to \mathcal{U}' and let E be a holomorphic vector bundle over D such that $\mathfrak{F}|_D \xrightarrow{\sigma} \mathcal{O}(E)$.

Consider the linear continuous map

$$R : Z^q(\mathcal{U}, \mathfrak{F}) \longrightarrow Z^q(D', E)$$

obtained by composition of the maps

$$Z^q(\mathcal{U}, \mathfrak{F}) \longrightarrow Z^q(\mathcal{U}', \mathfrak{F}) \xrightarrow{\sigma} Z^q(\mathcal{U}', \mathcal{O}(E)) \xrightarrow{T_{\mathcal{U}'}, \psi} Z^q(D', E).$$

Let V be an open neighbourhood of ∂Y and let $\varphi \in C^\infty(V, \mathbf{R})$ be a strongly q -pseudoconvex function such that

$$V \cap Y = \{x \in V \mid \varphi(x) < 0\}.$$

Let $(p_j)_{j \in \mathbf{N}} \subset \partial Y \cap D$ be a dense subset of points of $\partial Y \cap D$, $p_i \neq p_j$ for $i \neq j$.

For each $j \in \mathbf{N}$ we may find a neighbourhood $V_j \subset\subset V \cap D$ of p_j and we may find in V_j :

– q -discs $D_{\nu, j}(r)$ $0 < r \leq r_j$ $\nu \in \mathbf{N}^*$ having the properties from the proof of [4, Théorème 3]

– $L_j \subset V_j$ closed submanifolds such that $L_j \cap \bar{Y} = \{p_j\}$ (here L_j corresponds to the set A in the proof of [4, Proposition 6])

– differential forms $t_\alpha^j \in Z^q(V_j \setminus L_j, E)$ ($\alpha \in \mathbf{N}^{q+1}$) such that the following holds :

for any element of the form $t_j = \sum_\alpha c_\alpha t_{\alpha+1}^j$ $c_\alpha \in \mathbf{C}$ (the sum being finite and not all of the c_α 's being zero) there exists an E^* -valued $(q, 0)$ holomorphic form γ_j on V_j (E^* is the dual of E) such

$$\text{that } \lim_{\nu \rightarrow \infty} \left| \int_{D_{\nu, j}(r_j)} \gamma_j \wedge t_j \right| = \infty.$$

Let $\rho_j \in C_0^\infty(V, \mathbf{R})$, $\rho_j \geq 0$, $\rho_j|_{L_j} = 0$, $\rho_j > 0$ on $\partial Y \setminus \{p_j\}$ and choose $\epsilon_j > 0$ such that $\varphi - \epsilon_j \rho_j$ is strongly q -pseudoconvex on V . Putting $Y_j = Y \cup \{x \in V \mid \varphi(x) - \epsilon_j \rho_j(x) < 0\}$ we get $\bar{Y} \setminus \{p_j\} \subset Y_j$, $p_j \in \partial Y \cap \partial Y_j$ and $Y_j \cap L_j = \emptyset$.

Take now $h_j \in C_0^\infty(V_j, \mathbf{R})$, $h_j \geq 0$, $h_j(p_j) > 0$ and $\epsilon'_j > 0$ such that $\varphi - \epsilon_j \rho_j - \epsilon'_j h_j$ is strongly q -pseudoconvex on V and put $V'_j = \{x \in V_j \mid \varphi(x) - \epsilon_j \rho_j(x) - \epsilon'_j h_j(x) < 0\}$ and $Y'_j = Y_j \cup V'_j$. Then V'_j is an open neighbourhood of p_j , $Y_j \cap V'_j = Y_j \cap V_j$ and Y'_j is strongly q -pseudoconvex, hence $\dim_{\mathbf{C}} H^{q+1}(Y'_j, \mathfrak{F}) < \infty$.

Let $S_j \subset Z^q(Y_j \cap V_j, E)$ be the linear span of the elements of the form $t_{\alpha+1}^j$ ($\alpha \in \mathbf{N}^{q+1}$) and let $K_j \subset H^q(Y_j \cap V_j, \mathfrak{F})$ be the image of S_j by the map

$$\delta_j : Z^q(Y_j \cap V_j, E) \longrightarrow H^q(Y_j \cap V_j, \mathfrak{F})$$

obtained by composing the maps

$$Z^q(Y_j \cap V_j, E) \longrightarrow H^q_0(Y_j \cap V_j, E)$$

$$\xrightarrow{L_{Y_j \cap V_j}} H^q(Y_j \cap V_j, \mathcal{O}(E)) \xrightarrow{\sigma} H^q(Y_j \cap V_j, \mathcal{F}).$$

By [4, Proposition 6] we have $\dim_{\mathbf{C}} K_j = \infty$. By Mayer-Vietoris exact sequence

$$H^q(Y_j, \mathcal{F}) \oplus H^q(V'_j, \mathcal{F}) \xrightarrow{\alpha_j} H^q(Y_j \cap V_j, \mathcal{F}) \xrightarrow{\beta_j} H^{q+1}(Y'_j, \mathcal{F})$$

and by the conditions $\dim_{\mathbf{C}} K_j = \infty, \dim_{\mathbf{C}} H^{q+1}(Y'_j, \mathcal{F}) < \infty$ there exists $d_j \in K_j \setminus \{0\}$ such that $\beta_j(d_j) = 0$. Let $t_j \in S_j$ such that $\delta_j(t_j) = d_j$ and let $\xi_j \in H^q(Y_j, \mathcal{F}), v_j \in H^q(V'_j, \mathcal{F})$ such that $\xi_j|_{Y_j \cap V_j} - v_j|_{Y_j \cap V_j} = d_j$.

If $V''_j \subset V'_j$ is a Stein neighbourhood of p_j we have $\xi_j|_{Y_j \cap V''_j} = d_j$. Put $\xi'_j = \xi_j|_Y$ and let $\tau_j \in Z^q(\mathcal{U}, \mathcal{F})$ be such that ξ'_j is the image of τ_j by the map

$$Z^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\Theta_{\mathcal{U}}} H^q(Y, \mathcal{F}).$$

Let η_j be the restriction of τ_j on D' , i.e. $\eta_j = R(\tau_j)$.

We claim that for any point p_s and for any $j \in \mathbf{N}$ there exist a Stein neighbourhood U^j_s of p_s , $U^j_s \subset D$, and an E -valued C^∞ form λ^j_s of type $(0, q-1)$ on $V^j_s = Y \cap U^j_s$ such that

- a) $\eta_j|_{V^j_s} = \bar{\partial} \lambda^j_s$ for $j \neq s$
- b) $\eta_j|_{V^j_j} = t_j + \bar{\partial} \lambda^j_j$ for $j = s$.

The claim can be proved like this : for any $s \neq j$ take U^j_s a Stein neighbourhood of p_s contained in $Y_j \cap D$ and for $s = j$ take $U^j_j = V''_j$.

Let \mathcal{U}^j_s be the Stein covering of V^j_s given by $\{U_i \cap V^j_s | i \in \mathbf{N}\}$.

We have a commutative diagram

$$\begin{CD} H^q(\mathcal{U}, \mathcal{F}) @>>> H^q(\mathcal{U}^j_s, \mathcal{F}) @>\sigma>> H^q(\mathcal{U}^j_s, \mathcal{O}(E)) @>\tau_{\mathcal{U}^j_s}>> H^q_0(V^j_s, E) \\ @V\downarrow\Theta_{\mathcal{U}}VV @V\downarrow\Theta_{\mathcal{U}^j_s}VV @V\downarrow\Theta_{\mathcal{U}^j_s}VV @V\downarrow L_{V^j_s}VV \\ H^q(Y, \mathcal{F}) @>>> H^q(V^j_s, \mathcal{F}) @>\sigma>> H^q(V^j_s, \mathcal{O}(E)) @>\text{id}>> H^q(V^j_s, \mathcal{O}(E)) \end{CD}$$

which gives us a). Property b) can be deduced from the following diagram

$$\begin{array}{ccccccc}
 H^q(\mathcal{U}, \mathfrak{F}) & \longrightarrow & H^q(\mathcal{U}'_j, \mathfrak{F}) & \xrightarrow{\sigma} & H^q(\mathcal{U}'_j, \mathcal{O}(E)) & \xrightarrow{T_{\sigma_j}} & H^q(V'_j, E) \longrightarrow H^q(Y_j \cap V_j, E) \\
 \downarrow \Theta_{\mathcal{U}} & & \downarrow \Theta_{\mathcal{U}'_j} & & \downarrow \Theta_{\mathcal{U}'_j} & & \downarrow L_{V'_j} \\
 H^q(Y, \mathfrak{F}) & \longrightarrow & H^q(V'_j, \mathfrak{F}) & \xrightarrow{\sigma} & H^q(V'_j, \mathcal{O}(E)) & \xrightarrow{id} & H^q(V'_j, \mathcal{O}(E)) \longleftarrow H^q(Y_j \cap V_j, \mathcal{O}(E)) \\
 & & & & & & \downarrow \downarrow \\
 & & & & & & H^q(Y_j \cap V_j, \mathfrak{F})
 \end{array}$$

Let now γ_j be an E^* -valued holomorphic $(q, 0)$ form on V_j such that

$$1) \lim_{\nu \rightarrow \infty} \int_{D_{\nu,j}(r_j)} \gamma_j \wedge t_j = \infty.$$

Using 1), relations a), b), Stokes' theorem and the fact that for any $0 < r \leq r_j$ we have $\bigcup_{\nu=1}^{\infty} [D_{\nu,j}(r_j) \setminus D_{\nu,j}(r)] \subset\subset D'$ it follows that

$$2) \lim_{\nu \rightarrow \infty} \int_{D_{\nu,j}(r_j)} \gamma_j \wedge \eta_j = \infty$$

and

$$3) \int_{D_{\nu,j}(r_j)} \gamma_j \wedge \eta_s \leq p_s^j \text{ if } j \neq s$$

where $0 < p_s^j < \infty$.

Let $k_j > 0$ be sufficiently small real numbers such that for $|c_j| < k_j, c_j \in \mathbf{C}$, the series $\sum_j c_j \tau_j$ converges in $Z^q(\mathcal{U}, \mathfrak{F})$ and put $\eta = R \left(\sum_j c_j \tau_j \right) \in Z^q(D', E)$. If $c_j \neq 0$ are chosen sufficiently small then we get by 2) and 3) that

$$4) \lim_{\nu \rightarrow \infty} \int_{D_{\nu,j}(r_j)} \gamma_j \wedge \eta = \infty.$$

Since $\bigcup_{\nu=1}^{\infty} [D_{\nu,j}(r_j) \setminus D_{\nu,j}(r)] \subset\subset D'$ we get that 4) holds for any $0 < r \leq r_j$ from which we immediately deduce, via Stokes' theorem, that $\sum_j c_j \tau_j$ defines an element in $H^q(Y, \mathfrak{F})$ not extendable in any point of ∂Y . Theorem 2 is completely proved.

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