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MIHNEA COLTOIU

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## THE LEVI PROBLEM FOR COHOMOLOGY CLASSES

by Mihnea COLTOIU

### Introduction.

The aim of this paper is to extend some of the results of Andreotti and Norguet from [4] to complex spaces.

The paper is divided into two paragraphs :

- 1) The local problem
- 2) The global problem

In the first paragraph we prove the following

**THEOREM 1.** — *Let  $X$  be a perfect complex space,  $Y \subset X$  an open subset,  $x_0 \in \partial Y$  and  $\mathcal{F}$  a sheaf which is locally free in a neighbourhood of  $x_0$ . Suppose  $Y$  is strongly pseudoconcave in  $x_0$  and let  $n_0 = \dim \mathcal{O}_{X, x_0} > 0$ . Then  $H^{n_0-1}(Y, x_0, \mathcal{F})$  contains an infinitely dimensional vector subspace all of whose non-zero elements are not extendable in  $x_0$ .*

When  $X$  is a complex manifold this result was proved in [4] using a generalization of an integral formula of E. Martinelli. In the proof of Theorem 1 we use elementary results of local cohomology (one needs only supports consisting of a point) and the local structure theorems of a strongly pseudoconcave domain from [2].

The second paragraph is devoted to the generalization of Theorem 3 from [4]. More precisely we prove

**THEOREM 2.** — *Let  $X$  be a complex space and  $Y \subset \subset X$  an open subset which is strongly  $q$ -pseudoconvex. Suppose  $Y$  is strictly  $q$ -pseudoconvex in every point of  $\partial Y \cap \text{Reg}(X)$  and let  $\mathcal{F} \in \text{Coh}(X)$  such that  $\partial Y \subset \text{supp}(\mathcal{F})$ . Then there exists an element in  $H^q(Y, \mathcal{F})$  which is not extendable in any point of  $\partial Y$ .*

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### 1. The local problem.

Let us briefly recall some definitions from [4] which will be used throughout this paper.

Let  $\mathcal{F}$  be a sheaf of vector spaces on a topological space  $X$ ,  $Y \subset X$  an open subset and  $x_0$  a point in  $\partial Y$ . Put :

$$\begin{aligned} H^r(Y, x_0, \mathcal{F}) &= \varinjlim_{U \in \mathcal{V}_{x_0}} H^r(Y \cap U, \mathcal{F}) \\ H_+^r(Y \cup \{x_0\}, \mathcal{F}) &= \varinjlim_{U \in \mathcal{V}_{x_0}} H^r(Y \cup U, \mathcal{F}) \\ H^r(x_0, \mathcal{F}) &= \varinjlim_{U \in \mathcal{V}_{x_0}} H^r(U, \mathcal{F}) \end{aligned}$$

where  $\mathcal{V}_{x_0}$  = the set of all open neighbourhoods  $U$  of  $x_0$  in  $X$ .

We have  $H^0(x_0, \mathcal{F}) = \mathcal{F}_{x_0}$  and  $H^r(x_0, \mathcal{F}) = \{0\}$  for  $r \geq 1$  (cf. [6, pp. 192-193]). Consider the natural restriction maps :

$$\begin{aligned} r_1 : H^r(x_0, \mathcal{F}) &\longrightarrow H^r(Y, x_0, \mathcal{F}) \\ r_2 : H_+^r(Y \cup \{x_0\}, \mathcal{F}) &\longrightarrow H^r(Y, \mathcal{F}). \end{aligned}$$

An element in  $H^r(Y, x_0, \mathcal{F})$  (in  $H^r(Y, \mathcal{F})$ ) will be called extendable in  $x_0 \in \partial Y$  if it belongs to the image of the map  $r_1$  ( $r_2$  respectively).

Suppose now that  $X$  is a complex space. We say that  $Y$  is strongly pseudoconcave in  $x_0$  if there exist an open neighbourhood  $U$  of  $x_0$  in  $X$  and  $\varphi \in C^\infty(U, \mathbb{R})$  a strongly plurisubharmonic function such that  $U \cap Y = \{x \in U \mid \varphi(x) > \varphi(x_0)\}$ .

If  $x_0 \in \text{Reg}(X)$  we say that  $Y$  is strictly  $q$ -pseudoconvex in  $x_0$  if there exist an open neighbourhood  $U$  of  $x_0$  and  $\varphi \in C^\infty(U, \mathbb{R})$  such that :

- i)  $(d\varphi)_{x_0} \neq 0$
- ii)  $U \cap Y = \{x \in U \mid \varphi(x) < \varphi(x_0)\}$

iii) the restriction of the Levi form  $\mathcal{L}(\varphi)$  to the analytic tangent hyperplane to  $\partial Y$  at  $x_0$  is nondegenerate and admits precisely  $q$  strictly negative eigenvalues.

Let us also recall that a complex space  $X$  is called perfect if  $\mathcal{O}_{X,x}$  is Cohen-Macaulay for any  $x \in X$ . We denote by  $H_{x_0}^i(X, \cdot)$  the cohomology groups with support in  $\{x_0\}$ . In order to prove Theorem 1 we shall need the following statement

PROPOSITION 1. — *Let  $X$  be a perfect complex space (not necessarily reduced),  $x_0 \in X$  and  $n_0 = \dim \mathcal{O}_{X,x_0} > 0$ . put  $L_{x_0} = \varinjlim_{U \in \mathfrak{V}_{x_0}} H_{x_0}^{n_0}(U, \mathcal{O}_X)$ . Then  $\dim_{\mathbb{C}} L_{x_0} = \infty$ .*

The above proposition is an immediate consequence of [5, pp. 86, Corollaire 4.5.].

Remark 1. — If  $U \in \mathfrak{V}_{x_0}$  we have the exact sequence

$$H^{n_0-1}(U, \mathcal{O}_X) \longrightarrow H^{n_0-1}(U \setminus \{x_0\}, \mathcal{O}_X) \longrightarrow H_{x_0}^{n_0}(U, \mathcal{O}_X) \longrightarrow H^{n_0}(U, \mathcal{O}_X).$$

Taking inductive limit we get

$$L_{x_0} \cong H^{n_0-1}(X \setminus \{x_0\}, \mathcal{O}_X) \text{ for } n_0 \geq 2.$$

THEOREM 1. — *Let  $X$  be a perfect complex space,  $Y \subset X$  an open subset,  $x_0 \in \partial Y$  and  $\mathfrak{F}$  a sheaf which is locally free in a neighbourhood of  $x_0$ . Suppose  $Y$  is strongly pseudoconcave in  $x_0$  and let  $n_0 = \dim \mathcal{O}_{X,x_0} > 0$ . Then  $H^{n_0-1}(Y, x_0, \mathfrak{F})$  contains an infinitely dimensional vector subspace all of whose non-zero elements are not extendable in  $x_0$ .*

Proof. — Obviously, we may suppose  $\mathfrak{F} = \mathcal{O}_X$ . Since the problem is local we also may suppose that  $X$  is a closed analytic subset of some open set  $G \subset \mathbb{C}^N$  and that

$$Y = \{x \in X \mid \varphi(x) > \varphi(x_0)\},$$

where  $\varphi \in C^\infty(G, \mathbb{R})$  is a strongly plurisubharmonic function.

Writing the Taylor expansion of  $\varphi$  at  $x_0$  we get :

$$\varphi(x) = \varphi(x_0) + 2 \operatorname{Re} f(x) + \mathcal{L}(\varphi)(x) + O(\|x - x_0\|^3)$$

where  $f$  is a polynomial of degree two in  $x$  and  $\mathcal{L}(\varphi)$  is the Levi form. Let  $g = f|_X$  and  $Z_0 = \{x \in X \mid g(x) = 0\}$ .

Replacing  $G$  by a smaller subset we may suppose that  $Z_0 \setminus \{x_0\} \subset Y$ . Moreover, using the perturbation argument in [7, pp. 357-358], we may suppose that the image of  $g$  in  $\mathcal{O}_{X,x}$  is

not a zero-divisor for any  $x \in X$ . Consider the space  $(Z_0, \mathcal{O}_{Z_0})$  where  $\mathcal{O}_{Z_0} = \mathcal{O}_X/g \cdot \mathcal{O}_X$ . Since  $X$  is a perfect space and the image of  $g$  in  $\mathcal{O}_{X,x}$  is not a zero-divisor for any  $x \in X$  it follows that  $(Z_0, \mathcal{O}_{Z_0})$  is also perfect.

Put  $n_0 = \dim \mathcal{O}_{X,x_0}$ , hence  $n_0 - 1 = \dim \mathcal{O}_{Z_0,x_0}$ , and let  $L_{x_0} = \varinjlim_{U' \in \mathcal{V}'_{x_0}} H_{x_0}^{n_0-1}(U', \mathcal{O}_{Z_0})$  where  $\mathcal{V}'_{x_0}$  = the set of all open neighbourhoods  $U'$  of  $x_0$  in  $Z_0$ .

Consider the exact sequence of sheaves on  $Y$

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\cdot g} \mathcal{O}_X \longrightarrow \mathcal{O}_{Z_0} \longrightarrow 0. \quad (1)$$

If  $U \subset X$  is an open neighbourhood of  $x_0$ , then (1) together with the long exact sequence of cohomology provide the exact sequence

$$H^q(Y \cap U, \mathcal{O}_X) \longrightarrow H^q(U' \setminus \{x_0\}, \mathcal{O}_{Z_0}) \longrightarrow H^{q+1}(Y \cap U, \mathcal{O}_X) \quad (2)$$

where  $U' = U \cap Z_0$  (recall that by choice of  $Z_0$  we have  $Y \cap U' = U' \setminus \{x_0\}$ ).

Consider first the case  $n_0 \geq 3$ . Making  $q = n_0 - 2$  in (2) and taking inductive limit we get the exact sequence

$$\begin{aligned} H^{n_0-2}(Y, x_0, \mathcal{O}_X) &\longrightarrow H^{n_0-2}(Z_0 \setminus \{x_0\}, x_0, \mathcal{O}_{Z_0}) \\ &\longrightarrow H^{n_0-1}(Y, x_0, \mathcal{O}_X). \end{aligned} \quad (3)$$

By [2, Théorème 9] we get  $H^{n_0-2}(Y, x_0, \mathcal{O}_X) = 0$ . Since  $H^{n_0-2}(Z_0 \setminus \{x_0\}, x_0, \mathcal{O}_{Z_0}) \cong L_{x_0}$ , Proposition 1 implies that  $\dim_{\mathbb{C}} H^{n_0-1}(Y, x_0, \mathcal{O}_X) = \infty$  hence the theorem is proved for  $n_0 \geq 3$ .

For  $n_0 = 1$  the theorem is obvious, hence to conclude the proof we only have to deal with the case  $n_0 = 2$ . If  $U \subset X$  is an open neighbourhood of  $x_0$ , then by (1) and the long exact sequence of cohomology we get the exact sequence

$$H^0(Y \cap U, \mathcal{O}_X) \longrightarrow H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0}) \longrightarrow H^1(Y \cap U, \mathcal{O}_X) \quad (4)$$

where  $U' = U \cap Z_0$ .

By [2, Théorème 10] there exists a fundamental system of Stein neighbourhoods  $U$  of  $x_0$  in  $X$  such that the restriction map  $H^0(U, \mathcal{O}_X) \longrightarrow H^0(Y \cap U, \mathcal{O}_X)$  is bijective. The commutative diagram

$$\begin{array}{ccc} H^0(U, \mathcal{O}_X) & \xrightarrow{\sim} & H^0(Y \cap U, \mathcal{O}_X) \\ \downarrow & & \downarrow \\ H^0(U', \mathcal{O}_{Z_0}) & \longrightarrow & H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0}) \end{array}$$

and the surjectivity of the map  $H^0(U, \mathcal{O}_X) \longrightarrow H^0(U', \mathcal{O}_{Z_0})$  imply that

$$\begin{aligned} \text{Im}(H^0(Y \cap U, \mathcal{O}_X) \longrightarrow H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0})) \\ = \text{Im}(H^0(U', \mathcal{O}_{Z_0}) \longrightarrow H^0(U' \setminus \{x_0\}, \mathcal{O}_{Z_0})), \end{aligned}$$

hence there is a natural injection  $H^1_{x_0}(U', \mathcal{O}_{Z_0}) \longrightarrow H^1(Y \cap U, \mathcal{O}_X)$ . Taking inductive limit it follows that the map  $L_{x_0} \longrightarrow H^1(Y, x_0, \mathcal{O}_X)$  is injective, hence by Proposition 1 we get  $\dim_{\mathbb{C}} H^1(Y, x_0, \mathcal{O}_X) = \infty$ , and we are done.

**COROLLARY 1** [4, Proposition 6]. — *Let  $Y$  be an open subset of a complex manifold  $X$ ,  $x_0 \in \partial Y$  and suppose  $Y$  is strictly  $q$ -pseudoconvex in  $x_0$ . Let  $\mathcal{F}$  be a sheaf which is locally free in a neighbourhood of  $x_0$ . Then  $H^q(Y, x_0, \mathcal{F})$  contains an infinitely dimensional vector subspace all of whose non-zero elements are not extendable in  $x_0$ .*

*Proof.* — We may suppose  $\mathcal{F} = \mathcal{O}_X$  and  $q > 0$  (the case  $q = 0$  is obvious).

By definition of strictly  $q$ -pseudoconvexity it immediately follows that :

i)  $Y$  is strongly  $q$ -pseudoconvex in a neighbourhood of  $x_0$ .

ii) In some neighbourhood of  $x_0$  there exists an analytic submanifold  $B$  containing  $x_0$  such that  $\dim B = q + 1$  and  $B \cap Y$  is strongly pseudoconcave in  $x_0$ . By [2, Théorème 5] we deduce that the map

$$H^q(Y, x_0, \mathcal{O}_X) \longrightarrow H^q(B \cap Y, x_0, \mathcal{O}_B)$$

is surjective and using Theorem 1 we get  $\dim_{\mathbb{C}} H^q(Y, x_0, \mathcal{O}_X) = \infty$ .

**Remark 2.** — Let  $\varphi$  be a strongly plurisubharmonic function in some neighbourhood  $U$  of the origin in  $\mathbb{C}^n$  ( $n \geq 2$ ),  $(d\varphi)_0 \neq 0$  and put  $Y = \{z \in U \mid \varphi(z) > \varphi(0)\}$ . In suitable coordinates the Taylor expansion of  $\varphi$  at 0 has the form

$$\varphi(z) = \varphi(0) + 2 \operatorname{Re} z_1 + \sum_{1 \leq j, k \leq n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} (0) z_j \bar{z}_k + O(\|z\|^3).$$

Put exactly as in [4]

$$\psi_\alpha = \left( \sum_{1 \leq j \leq n} z_j^{\alpha_j} \bar{z}_j^{\alpha_j} \right)^{-n} \sum_{1 \leq j \leq n} (-1)^{j-1} \bar{z}_j^{\alpha_j} \bigwedge_{\substack{1 \leq k \leq n \\ k \neq j}} d(\bar{z}_k^{\alpha_k}).$$

By [4, Proposition 5] it follows that the images of the differential forms  $\psi_{\alpha+1}$  ( $\alpha \in \mathbb{N}^n$ ) in  $H^{n-1}(Y \cap U, \Theta)$  are linearly independent. Let  $M$  be the linear span of the above images.

We shall now investigate the relation between  $M$  and the vector space considered in the proof of Theorem 1 (which we denote now by  $L_1$ ). Recall that  $L_1$  is the kernel of the map  $\alpha_1 =$  multiplication by  $z_1$ ,

$$\alpha_1 : H^{n-1}(Y \cap U, \Theta) \longrightarrow H^{n-1}(Y \cap U, \Theta).$$

In the same way we define  $\alpha_k =$  multiplication by  $z_k$ ,

$$\alpha_k : H^{n-1}(Y \cap U, \Theta) \longrightarrow H^{n-1}(Y \cap U, \Theta)$$

and put  $L_k = \ker \alpha_k$ ,  $L = \bigcap_{k=1}^{\infty} L_k$ . We claim that  $M \subset L$ . To

prove this inclusion we use the relation  $z_1^{\alpha_1} \psi_\alpha = \bar{\partial} \mu_\alpha$  where

$$\mu_\alpha = \frac{1}{n-1} \left( \sum_{1 \leq j \leq n} z_j^{\alpha_j} \bar{z}_j^{\alpha_j} \right)^{1-n} \wedge \sum_{2 \leq j \leq n} (-1)^j \bar{z}_j^{\alpha_j} d(\bar{z}_k^{\alpha_k}).$$

This equality shows that the image of  $\psi_{\alpha+1}$  in  $H^{n-1}(Y \cap U, \Theta)$  is contained in  $L_{\alpha_1+1}$ , hence  $M \subset L$ .

## 2. The global problem.

$\alpha)$  Let  $U$  be an open subset of  $\mathbb{C}^n$  and  $\varphi \in C^\infty(U, \mathbb{R})$ . Recall that  $\varphi$  is called strongly  $q$ -pseudoconvex ( $0 \leq q \leq n-1$ ) if the Levi form  $\mathcal{L}(\varphi)$  has at least  $(n-q)$  strictly positive eigenvalues at any point in  $U$ . Using local embeddings in the Zarisky tangent space one easily extends the notion of strongly  $q$ -pseudoconvex function in the case of complex spaces (for details see [1, pp. 12-13]).

*Remark 3.* — Let  $X$  be a complex space and  $\varphi : X \longrightarrow \mathbb{R}$  a strongly  $q$ -pseudoconvex function. For any  $x \in X$  put  $\mu(x) = \min \dim X_x^i$  where  $X_x^i$  are the irreducible components

of  $X_x$  ( $X_x$  denotes the germ of  $X$  in  $x$ ). From the above definitions it immediately follows that  $q < \min_{x \in X} \mu(x)$ .

To state our theorem recall the following definition : an open subset  $Y \subset\subset X$  is called strongly  $q$ -pseudoconvex if there exist an open neighbourhood  $V$  of  $\partial Y$  and  $\varphi \in C^\infty(V, \mathbb{R})$  a strongly  $q$ -pseudoconvex function such that  $V \cap Y = \{x \in V \mid \varphi(x) < 0\}$ .

If  $\mathcal{F} \in \text{Coh}(X)$  and  $Y \subset\subset X$  is strongly  $q$ -pseudoconvex we have [2, Théorème 11]  $\dim_{\mathbb{C}} H^r(Y, \mathcal{F}) < \infty$  if  $r \geq q + 1$ .

As we already announced in the introduction the aim of this paragraph is to prove the following

**THEOREM 2.** — *Let  $X$  be a complex space and  $Y \subset\subset X$  an open subset which is strongly  $q$ -pseudoconvex. Suppose  $Y$  is strictly  $q$ -pseudoconvex in every point of  $\partial Y \cap \text{Reg}(X)$  and let  $\mathcal{F} \in \text{Coh}(X)$  such that  $\partial Y \subset \text{supp}(\mathcal{F})$ . Then there exists an element in  $H^q(Y, \mathcal{F})$  which is not extendable in any point of  $\partial Y$ .*

**$\beta$ ) LEMMA 1.** — *Let  $Y \subset\subset X$  be an open subset such that  $Y$  is strongly  $q$ -pseudoconvex and let  $A \subset X$  be an analytic closed subset such that  $\dim_x A < \dim_x X$  for any  $x \in A$ . Then  $\partial Y \setminus A$  is dense in  $\partial Y$ .*

*Proof.* — Let  $V$  be an open neighbourhood of  $\partial Y$  and  $\varphi \in C^\infty(V, \mathbb{R})$  a strongly  $q$ -pseudoconvex function such that  $V \cap Y = \{x \in V \mid \varphi(x) < 0\}$ . Let's make a couple of remarks :

1) For any point  $x \in A$  with  $X_x$  irreducible there exists a fundamental system of open neighbourhoods  $(U_i)_{i \in \mathbb{N}}$  of  $x$  such that  $U_i \setminus A$  is connected.

2) For any point  $x \in \partial Y$  there exists a germ of analytic set  $Q_x$  passing through  $x$ ,  $\dim_x Q_x \geq 1$  and  $\varphi|_{Q_x}$  is strongly plurisubharmonic.

Assertion 1) is well known and 2) may be deduced from [8, pp. 46, Corollary 4] using the condition  $q < \min_{x \in \partial Y} \dim \mathcal{O}_{X,x}$  (which is a consequence of Remark 3). Let's show now that  $\partial Y \setminus A$  is dense in  $\partial Y$ .

a) Take first  $x_0 \in \partial Y \cap A$  such that  $X_{x_0}$  is irreducible and



let  $(U_i)_{i \in \mathbb{N}}$  be a fundamental system of open neighbourhoods of  $x_0$  such that  $U_i \setminus A$  is connected and  $U_i \subset V$ . We must prove that for any  $i$   $\partial Y \cap U_i \not\subset A \cap U_i$ . If there existed an  $i_0$  such that  $\partial Y \cap U_{i_0} \subset A \cap U_{i_0}$  we would get

$$U_{i_0} \setminus A = [(U_{i_0} \cap Y) \setminus A] \cup [(U_{i_0} \cap \mathbb{C} \bar{Y}) \setminus A]$$

and since  $U_{i_0} \setminus A$  is connected we would get  $(U_{i_0} \cap \mathbb{C} \bar{Y}) \setminus A = \emptyset$ , hence  $U_{i_0} \subset \bar{Y}$ . In particular we would have  $\varphi \leq 0$  on  $U_{i_0}$ .

Since  $\varphi(x_0) = 0$  and  $\varphi|_{Q_{x_0}}$  is strongly plurisubharmonic the maximum principle yields a contradiction and we are done.

b) Take now  $x_0 \in \partial Y \cap A$  and suppose that  $X_{x_0}$  is not irreducible. Let  $X_{x_0} = \bigcup_{i=1}^{k_0} X_{x_0}^i$  be the decomposition of  $X_{x_0}$  into irreducible components. One may easily deduce that there exist  $i_0 \in \{1, \dots, k_0\}$  and an open neighbourhood  $U = U(x_0)$  of  $x_0$  such that  $X_{x_0}^{i_0}$  is induced in  $U$  by an irreducible subspace  $Z = Z(x_0)$  with  $x_0 \in \partial(Y \cap Z)$ . On the other hand by Remark 3 we get that  $q < \dim Z$ . If we put  $A' = A \cap Z$  and  $\varphi' = \varphi|_Z$  it follows that  $\dim A' < \dim Z$  and  $\varphi'$  is strongly  $q$ -pseudoconvex. Hence there exists a germ of analytic set  $Q'_{x_0}$  passing through  $x_0$  with  $\dim_{x_0} Q'_{x_0} \geq 1$ ,  $Q'_{x_0} \subset Z$  and  $\varphi'|_{Q'_{x_0}}$  is strongly plurisubharmonic. Since  $Z_{x_0}$  is irreducible the same reasoning as in a) shows that we may find a sequence  $(x_n)_{n \in \mathbb{N}}$ ,  $x_n \rightarrow x_0$  and  $x_n \in \partial(Y \cap Z) \setminus A'$ . Lemma 1 is completely proved.

**COROLLARY 2.** — *Let  $Y \subset \subset X$  be an open subset such that  $Y$  is strongly  $q$ -pseudoconvex and let  $\mathfrak{F} \in \text{Coh}(X)$  such that  $\partial Y \subset \text{supp}(\mathfrak{F})$ . Then there exists an open subset  $D \subset X$  such that :*

- a)  $D \subset \text{Reg}(X)$
- b)  $\mathfrak{F}|_D$  is locally free of rank  $\geq 1$  (the rank not being necessarily constant)
- c)  $\partial Y \cap D$  is dense in  $\partial Y$ .

*Proof.* — Put  $A_1 = \{x \in X \mid \mathfrak{F}_x \text{ is not a free } \mathcal{O}_{X,x}\text{-module}\}$ . It is well known that  $A_1$  is an analytic closed subset of  $X$  and  $\dim_x A_1 < \dim_x X$  for any  $x \in A_1$ . Put  $D_1 = X \setminus (A_1 \cup \text{Sing}(X))$

and  $D = D_1 \cap \text{supp}(\mathcal{F})$ . By Lemma 1 we immediately deduce that  $D$  satisfies conditions a), b), c) and we are done.

γ) Let  $X$  be a complex space,  $\mathcal{F} \in \text{Coh}(X)$ ,  $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$  a locally finite open covering of  $X$ . Put :

$Z^p(\mathcal{U}, \mathcal{F})$  = the group of  $p$ -cocycles with values in  $\mathcal{F}$ , with its natural topology of Fréchet space

$H^p(\mathcal{U}, \mathcal{F})$  = the  $p$ -th group of Čech cohomology of  $\mathcal{F}$  with respect to  $\mathcal{U}$

$H^p(X, \mathcal{F})$  = the  $p$ -th cohomology group of  $\mathcal{F}$  computed using the canonical resolution of Godement

$\Theta_{\mathcal{U}} : H^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$  the natural maps between the above groups.

If  $U_i$  is Stein for any  $i$  then  $\Theta_{\mathcal{U}}$  are isomorphisms. Let now  $X' \subset X$  be an open subset and  $\mathcal{U}' = (U'_i)_{i \in \mathbb{N}}$  the covering defined by  $U'_i = U_i \cap X'$ . We have a commutative diagram :

$$\begin{array}{ccc} H^p(\mathcal{U}, \mathcal{F}) & \xrightarrow{\Theta_{\mathcal{U}}} & H^p(X, \mathcal{F}) \\ \downarrow & & \downarrow \\ H^p(\mathcal{U}', \mathcal{F}) & \xrightarrow{\Theta_{\mathcal{U}'}} & H^p(X', \mathcal{F}) \end{array}$$

Suppose now  $X$  is a complex manifold and  $E$  is a holomorphic vector bundle over  $X$ . Put  $\mathcal{F} = \mathcal{O}(E)$  which is a locally free sheaf on  $X$ . Let  $\mathcal{G}^{p,q}(E)$  be the sheaf of germs of  $C^\infty$   $E$ -valued forms of type  $(p, q)$ . Consider the Dolbeault resolution

$$0 \longrightarrow \mathcal{O}(E) \longrightarrow \mathcal{G}^{0,0}(E) \xrightarrow{\bar{\partial}} \mathcal{G}^{0,1}(E) \xrightarrow{\bar{\partial}} \dots$$

Put :

$$Z^p(X, E) = \ker \{ \Gamma(X, \mathcal{G}^{0,p}(E)) \xrightarrow{\bar{\partial}} \Gamma(X, \mathcal{G}^{0,p+1}(E)) \}$$

with its natural topology of Fréchet space

$$B^p(X, E) = \text{Im} \{ \Gamma(X, \mathcal{G}^{0,p-1}(E)) \xrightarrow{\bar{\partial}} \Gamma(X, \mathcal{G}^{0,p}(E)) \}$$

$$H^p_{\bar{\partial}}(X, E) = Z^p(X, E) / B^p(X, E).$$

Let  $\psi = (\psi_i)_{i \in \mathbb{N}}$  be a partition of unity with respect to  $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ . Define  $T_{\mathcal{U}, \psi} : Z^p(\mathcal{U}, \mathcal{O}(E)) \longrightarrow Z^p(X, E)$  by

$$T_{\mathcal{U}, \psi}(\xi) = \sum_{i_0 \dots i_p} \xi_{i_0 \dots i_p} \psi_{i_0} \bar{\partial} \psi_{i_1} \wedge \dots \wedge \bar{\partial} \psi_{i_p}$$

$T_{\mathfrak{U}, \psi}$  is a continuous linear operator. The operator

$$T_{\mathfrak{U}} : H^p(\mathfrak{U}, \mathcal{O}(E)) \longrightarrow H^p_{\mathfrak{g}}(X, E),$$

induced by  $T_{\mathfrak{U}, \psi}$ , does not depend on  $\psi$ . Furthermore if  $\mathfrak{U} = (U_i)_{i \in \mathbb{N}}$  is a Stein covering then  $T_{\mathfrak{U}}$  is an algebraic and topological isomorphism (cf. [3, pp. 225-227]).

Let now  $X' \subset X$  be an open subset and  $\mathfrak{U}' = (U'_i)_{i \in \mathbb{N}}$  the covering defined by  $U'_i = U_i \cap X'$ . Since  $T_{\mathfrak{U}}$  does not depend on  $\psi$  we get the following commutative diagram :

$$\begin{array}{ccc} H^p(\mathfrak{U}, \mathcal{O}(E)) & \xrightarrow{T_{\mathfrak{U}}} & H^p_{\mathfrak{g}}(X, E) \\ \downarrow & & \downarrow \\ H^p(\mathfrak{U}', \mathcal{O}(E)) & \xrightarrow{T_{\mathfrak{U}'}} & H^p_{\mathfrak{g}}(X', E) \end{array}$$

If  $\mathfrak{U} = (U_i)_{i \in \mathbb{N}}$  is a Stein covering of  $X$  we may define the isomorphism  $H^p_{\mathfrak{g}}(X, E) \longrightarrow H^p(X, \mathcal{O}(E))$  as the composed map  $H^p_{\mathfrak{g}}(X, E) \xrightarrow{T_{\mathfrak{U}}^{-1}} H^p(\mathfrak{U}, \mathcal{O}(E)) \xrightarrow{\Theta_{\mathfrak{U}}} H^p(X, \mathcal{O}(E))$ . One verifies immediately that the above isomorphism does not depend on  $\mathfrak{U}$  and denote this isomorphism by  $L_X$ . For any open subset  $X' \subset X$  we have a commutative diagram :

$$\begin{array}{ccc} H^p_{\mathfrak{g}}(X, E) & \xrightarrow{L_X} & H^p(X, \mathcal{O}(E)) \\ \downarrow & & \downarrow \\ H^p_{\mathfrak{g}}(X', E) & \xrightarrow{L_{X'}} & H^p(X', \mathcal{O}(E)) \end{array}$$

#### δ) Proof of Theorem 2

We shall suppose  $q > 0$  since the case  $q = 0$  is well known. Let  $\mathfrak{U} = (U_i)_{i \in \mathbb{N}}$  be a locally finite Stein covering of  $Y$  and  $D \subset X$  having properties a), b), c) from Corollary 2. Put  $D' = D \cap Y$ ,  $U'_i = U_i \cap D$ ,  $\mathfrak{U}' = (U'_i)_{i \in \mathbb{N}}$  = a locally finite open covering of  $D'$ . Let  $\psi = (\psi_i)_{i \in \mathbb{N}}$  be a partition of unity with respect to  $\mathfrak{U}'$  and let  $E$  be a holomorphic vector bundle over  $D$  such that  $\mathcal{F}|_D \xrightarrow{\sigma} \mathcal{O}(E)$ .

Consider the linear continuous map

$$R : Z^q(\mathfrak{U}, \mathcal{F}) \longrightarrow Z^q(D', E)$$

obtained by composition of the maps

$$Z^q(\mathfrak{U}, \mathcal{F}) \longrightarrow Z^q(\mathfrak{U}', \mathcal{F}) \xrightarrow{\sigma} Z^q(\mathfrak{U}', \mathcal{O}(E)) \xrightarrow{T_{\mathfrak{U}'}, \psi} Z^q(D', E).$$

Let  $V$  be an open neighbourhood of  $\partial Y$  and let  $\varphi \in C^\infty(V, \mathbf{R})$  be a strongly  $q$ -pseudoconvex function such that

$$V \cap Y = \{x \in V \mid \varphi(x) < 0\}.$$

Let  $(p_j)_{j \in \mathbf{N}} \subset \partial Y \cap D$  be a dense subset of points of  $\partial Y \cap D$ ,  $p_i \neq p_j$  for  $i \neq j$ .

For each  $j \in \mathbf{N}$  we may find a neighbourhood  $V_j \subset V \cap D$  of  $p_j$  and we may find in  $V_j$ :

–  $q$ -discs  $D_{\nu, j}(r)$   $0 < r \leq r_j$   $\nu \in \mathbf{N}^*$  having the properties from the proof of [4, Théorème 3]

–  $L_j \subset V_j$  closed submanifolds such that  $L_j \cap \bar{Y} = \{p_j\}$  (here  $L_j$  corresponds to the set  $A$  in the proof of [4, Proposition 6])

– differential forms  $t_\alpha^j \in Z^q(V_j \setminus L_j, E)$  ( $\alpha \in \mathbf{N}^{q+1}$ ) such that the following holds:

for any element of the form  $t_j = \sum_\alpha c_\alpha t_{\alpha+1}^j$   $c_\alpha \in \mathbf{C}$  (the sum being finite and not all of the  $c_\alpha$ 's being zero) there exists an  $E^*$ -valued  $(q, 0)$  holomorphic form  $\gamma_j$  on  $V_j$  ( $E^*$  is the dual of  $E$ ) such

that  $\lim_{\nu \rightarrow \infty} \left| \int_{D_{\nu, j}(r_j)} \gamma_j \wedge t_j \right| = \infty$ .

Let  $\rho_j \in C_0^\infty(V, \mathbf{R})$ ,  $\rho_j \geq 0$ ,  $\rho_j|_{L_j} = 0$ ,  $\rho_j > 0$  on  $\partial Y \setminus \{p_j\}$  and choose  $\epsilon_j > 0$  such that  $\varphi - \epsilon_j \rho_j$  is strongly  $q$ -pseudoconvex on  $V$ . Putting  $Y_j = Y \cup \{x \in V \mid \varphi(x) - \epsilon_j \rho_j(x) < 0\}$  we get  $\bar{Y} \setminus \{p_j\} \subset Y_j$ ,  $p_j \in \partial Y \cap \partial Y_j$  and  $Y_j \cap L_j = \emptyset$ .

Take now  $h_j \in C_0^\infty(V_j, \mathbf{R})$ ,  $h_j \geq 0$ ,  $h_j(p_j) > 0$  and  $\epsilon'_j > 0$  such that  $\varphi - \epsilon_j \rho_j - \epsilon'_j h_j$  is strongly  $q$ -pseudoconvex on  $V$  and put  $V'_j = \{x \in V_j \mid \varphi(x) - \epsilon_j \rho_j(x) - \epsilon'_j h_j(x) < 0\}$  and  $Y'_j = Y_j \cup V'_j$ . Then  $V'_j$  is an open neighbourhood of  $p_j$ ,  $Y_j \cap V'_j = Y_j \cap V_j$  and  $Y'_j$  is strongly  $q$ -pseudoconvex, hence  $\dim_{\mathbf{C}} H^{q+1}(Y'_j, \mathfrak{H}) < \infty$ .

Let  $S_j \subset Z^q(Y_j \cap V_j, E)$  be the linear span of the elements of the form  $t_{\alpha+1}^j$  ( $\alpha \in \mathbf{N}^{q+1}$ ) and let  $K_j \subset H^q(Y_j \cap V_j, \mathfrak{H})$  be the image of  $S_j$  by the map

$$\delta_j: Z^q(Y_j \cap V_j, E) \longrightarrow H^q(Y_j \cap V_j, \mathfrak{H})$$

obtained by composing the maps

$$Z^q(Y_j \cap V_j, E) \longrightarrow H^q_0(Y_j \cap V_j, E)$$

$$\xrightarrow{L_{Y_j \cap V_j}} H^q(Y_j \cap V_j, \mathcal{O}(E)) \xrightarrow{\sigma} H^q(Y_j \cap V_j, \mathcal{F}).$$

By [4, Proposition 6] we have  $\dim_{\mathbb{C}} K_j = \infty$ . By Mayer-Vietoris exact sequence

$$H^q(Y_j, \mathcal{F}) \oplus H^q(V'_j, \mathcal{F}) \xrightarrow{\alpha_j} H^q(Y_j \cap V_j, \mathcal{F}) \xrightarrow{\beta_j} H^{q+1}(Y'_j, \mathcal{F})$$

and by the conditions  $\dim_{\mathbb{C}} K_j = \infty$ ,  $\dim_{\mathbb{C}} H^{q+1}(Y'_j, \mathcal{F}) < \infty$  there exists  $d_j \in K_j \setminus \{0\}$  such that  $\beta_j(d_j) = 0$ . Let  $t_j \in S_j$  such that  $\delta_j(t_j) = d_j$  and let  $\xi_j \in H^q(Y_j, \mathcal{F})$ ,  $v_j \in H^q(V'_j, \mathcal{F})$  such that  $\xi_j|_{Y_j \cap V_j} - v_j|_{Y_j \cap V_j} = d_j$ .

If  $V''_j \subset V'_j$  is a Stein neighbourhood of  $p_j$  we have  $\xi_j|_{Y_j \cap V''_j} = d_j$ . Put  $\xi'_j = \xi_j|_Y$  and let  $\tau_j \in Z^q(\mathcal{U}, \mathcal{F})$  be such that  $\xi'_j$  is the image of  $\tau_j$  by the map

$$Z^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\Theta_{\mathcal{U}}} H^q(Y, \mathcal{F}).$$

Let  $\eta_j$  be the restriction of  $\tau_j$  on  $D'$ , i.e.  $\eta_j = R(\tau_j)$ .

We claim that for any point  $p_s$  and for any  $j \in \mathbb{N}$  there exist a Stein neighbourhood  $U^j_s$  of  $p_s$ ,  $U^j_s \subset D$ , and an  $E$ -valued  $C^\infty$  form  $\lambda^j_s$  of type  $(0, q-1)$  on  $V^j_s = Y \cap U^j_s$  such that

- a)  $\eta_j|_{V^j_s} = \bar{\partial} \lambda^j_s$  for  $j \neq s$
- b)  $\eta_j|_{V^j_j} = t_j + \bar{\partial} \lambda^j_j$  for  $j = s$ .

The claim can be proved like this : for any  $s \neq j$  take  $U^j_s$  a Stein neighbourhood of  $p_s$  contained in  $Y_j \cap D$  and for  $s = j$  take  $U^j_j = V^j_j$ .

Let  $\mathcal{V}^j_s$  be the Stein covering of  $V^j_s$  given by  $\{U_i \cap V^j_s | i \in \mathbb{N}\}$ .

We have a commutative diagram

$$\begin{array}{ccccccc} H^q(\mathcal{U}, \mathcal{F}) & \longrightarrow & H^q(\mathcal{V}^j_s, \mathcal{F}) & \xrightarrow{\sigma} & H^q(\mathcal{V}^j_s, \mathcal{O}(E)) & \xrightarrow{\tau_{\mathcal{V}^j_s}} & H^q_0(V^j_s, E) \\ \downarrow \Theta_{\mathcal{U}} & & \downarrow \Theta_{\mathcal{V}^j_s} & & \downarrow \Theta_{\mathcal{V}^j_s} & & \downarrow L_{V^j_s} \\ H^q(Y, \mathcal{F}) & \longrightarrow & H^q(V^j_s, \mathcal{F}) & \xrightarrow{\sigma} & H^q(V^j_s, \mathcal{O}(E)) & \xrightarrow{\text{id}} & H^q(V^j_s, \mathcal{O}(E)) \end{array}$$

which gives us a). Property b) can be deduced from the following diagram

$$\begin{array}{ccccccc}
H^q(\mathcal{U}, \mathfrak{F}) & \longrightarrow & H^q(\mathcal{U}_j^j, \mathfrak{F}) & \xrightarrow{\sigma} & H^q(\mathcal{U}_j^j, \mathcal{O}(E)) & \xrightarrow{\tau_{\pi_j^j}} & H^q(V_j^j, E) \longrightarrow H^q(Y_j \cap V_j, E) \\
\downarrow \circlearrowleft_{\mathcal{U}} & & \downarrow \circlearrowleft_{\mathcal{U}_j^j} & & \downarrow \circlearrowleft_{\mathcal{U}_j^j} & & \downarrow L_{V_j^j} \\
H^q(Y, \mathfrak{F}) & \longrightarrow & H^q(V_j^j, \mathfrak{F}) & \xrightarrow{\sigma} & H^q(V_j^j, \mathcal{O}(E)) & \xrightarrow{id} & H^q(V_j^j, \mathcal{O}(E)) \longleftarrow H^q(Y_j \cap V_j, \mathcal{O}(E)) \\
& & & & & & \downarrow L_{Y_j \cap V_j} \\
& & & & & & H^q(Y_j \cap V_j, \mathfrak{F})
\end{array}$$

Let now  $\gamma_j$  be an  $E^*$ -valued holomorphic  $(q, 0)$  form on  $V_j$  such that

$$1) \lim_{\nu \rightarrow \infty} \left| \int_{D_{\nu,j}(r_j)} \gamma_j \wedge t_j \right| = \infty.$$

Using 1), relations a), b), Stokes' theorem and the fact that for any  $0 < r \leq r_j$  we have  $\bigcup_{\nu=1}^{\infty} [D_{\nu,j}(r_j) \setminus D_{\nu,j}(r)] \subset\subset D'$  it follows that

$$2) \lim_{\nu \rightarrow \infty} \left| \int_{D_{\nu,j}(r_j)} \gamma_j \wedge \eta_j \right| = \infty$$

and

$$3) \left| \int_{D_{\nu,j}(r_j)} \gamma_j \wedge \eta_s \right| \leq p_s^j \quad \text{if } j \neq s$$

where  $0 < p_s^j < \infty$ .

Let  $k_j > 0$  be sufficiently small real numbers such that for  $|c_j| < k_j, c_j \in \mathbf{C}$ , the series  $\sum_j c_j \tau_j$  converges in  $Z^q(\mathcal{U}, \mathfrak{F})$  and put  $\eta = R \left( \sum_j c_j \tau_j \right) \in Z^q(D', E)$ . If  $c_j \neq 0$  are chosen sufficiently small then we get by 2) and 3) that

$$4) \lim_{\nu \rightarrow \infty} \left| \int_{D_{\nu,j}(r_j)} \gamma_j \wedge \eta \right| = \infty.$$

Since  $\bigcup_{\nu=1}^{\infty} [D_{\nu,j}(r_j) \setminus D_{\nu,j}(r)] \subset\subset D'$  we get that 4) holds for any  $0 < r \leq r_j$  from which we immediately deduce, via Stokes' theorem, that  $\sum_j c_j \tau_j$  defines an element in  $H^q(Y, \mathfrak{F})$  not extendable in any point of  $\partial Y$ . Theorem 2 is completely proved.

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Mihnea COLTOIU,  
National Institute for Scientific  
and Technical Creation  
Bd. Păcii 220  
77538 Bucharest (Romania).