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ON THE BOUNDARY LIMITS OF HARMONIC FUNCTIONS WITH GRADIENT IN L^p

by Yoshihiro MIZUTA

1. Introduction.

Let u be a function harmonic in the half space

$$\mathbf{R}_+^n = \{x = (x_1, \dots, x_n); x_n > 0\}$$

and satisfying the condition :

$$\int_G |\text{grad } u(x)|^p x_n^\alpha dx < \infty \quad (1)$$

for any bounded open set $G \subset \mathbf{R}_+^n$, where $p > 1$ and $\alpha < p - 1$.
For $\xi \in \partial \mathbf{R}_+^n$, $\gamma \geq 1$ and $a > 1$, set

$$T_\gamma(\xi, a) = \{x \in \mathbf{R}_+^n; |x - \xi| < ax_n^{1/\gamma}\}.$$

The existence of nontangential limits of u , that is, the limit of $u(x)$ as $x \rightarrow \xi$, $x \in T_1(\xi, a)$, was studied by Carleson [1] ($n = p = 2$ and $0 \leq \alpha < 1$), Wallin [10] ($p = 2$ and $0 \leq \alpha < 1$) and Mizuta [6] in the present situation.

Recently Cruzeiro [2] proved the existence of $\lim u(x)$ as $x \rightarrow \xi$, $x \in T_\gamma(\xi, a)$, for a harmonic function u satisfying (1) with $p = n$ and $\alpha = 0$. The existence of such limits for Green potentials in $\mathbf{R}_+^n$ was obtained by Wu [11]. Taking these results into account, we give the following theorem :

THEOREM. — *Let u be a function harmonic in $\mathbf{R}_+^n$ and satisfying (1) with $p > 1$ and $\alpha < p - 1$.*

(i) *If $n - p + \alpha > 0$, then for each $\gamma > 1$ there exists a set $E_\gamma \subset \partial \mathbf{R}_+^n$ such that $H_{\gamma(n-p+\alpha)}(E_\gamma) = 0$ and*

$$\lim_{x \rightarrow \xi, x \in T_\gamma(\xi, a)} u(x) \quad (2)$$

exists and is finite for any $\xi \in \partial \mathbf{R}_+^n - E_\gamma$ and any $a > 1$.

(ii) If $n - p + \alpha = 0$, then there exists a set $E \subset \partial \mathbf{R}_+^n$ such that $B_{n/p, p}(E) = 0$ and (2) exists and is finite for any $\xi \in \partial \mathbf{R}_+^n - E$, any $\gamma \geq 1$ and any $a > 1$.

(iii) If $n - p + \alpha < 0$, then $\lim_{x \rightarrow \xi, x \in \mathbf{R}_+^n} u(x)$ exists and is finite for any $\xi \in \partial \mathbf{R}_+^n$.

Here H_d denotes the d -dimensional Hausdorff measure, and $B_{\beta, p}$ the Bessel capacity of index (β, p) (cf. [3]). In view of [3; Theorems 21 and 22], one notes the following results:

a) If $q > 1$ and $n - d \geq \beta q$, then $H_d(E) < \infty$ implies $B_{\beta, q}(E) = 0$;

b) If $q > 1$ and $n - d < \beta q$, then $B_{\beta, q}(E) = 0$ implies $H_d(E) = 0$.

Recently Nagel, Rudin and Shapiro [7] studied tangential behaviors of Poisson integrals of potential type functions (see Sec. 4, Remark 2). Their results are not applicable to our case unless $p = 2$.

Remark. — The same result as in the theorem is also valid for a domain Ω for which any function v satisfying

$$\int_{\Omega} |\text{grad } v(x)|^p \delta(x)^\alpha dx < \infty, \quad p > 1, \alpha < p - 1, \quad (3)$$

can be extended to a function satisfying (3) with Ω replaced by $\mathbf{R}^n$, where $\delta(x)$ denotes the distance from x to $\partial\Omega$. The special Lipschitz domains in [9; Chap. VI] are typical examples of Ω .

2. Lemmas.

First we note the following result, which follows readily from the fact in [4; p. 165].

LEMMA 1. — Let f be a locally integrable function on $\mathbf{R}^n$. For $\beta > 0$, we set

$$A_\beta = \{ \xi \in \partial \mathbf{R}_+^n ; \limsup_{r \downarrow 0} r^{-\beta} \int_{B(\xi, r)} |f(y)| dy > 0 \},$$

where $B(x, r)$ denotes the open ball with center at x and radius r . Then $H_\beta(A_\beta) = 0$.

By [3 ; Theorem 21] and the result of [4 ; p. 165], we have

LEMMA 2. — Let f be a locally integrable function on $\mathbf{R}^n$. For $p > 1$, we set

$$B_p = \{ x ; \limsup_{r \downarrow 0} (\log r^{-1})^{p-1} \int_{B(x, r)} |f(y)| dy > 0 \}.$$

Then $B_{n/p, p}(B_p) = 0$.

Next we prove the following technical result.

LEMMA 3. — Let $c_1 > 0$, $c_2 > 0$, $\gamma \geq 1$, $p > 1$ and $p - n \leq \alpha < p - 1$. Then

$$\left\{ \int_{\{y : c_1 x_n < |x-y| < c_2 x_n^{1/\gamma}\}} |x-y|^{p'(1-n)} |y_n|^{-\alpha p'/p} dy \right\}^{1/p'} \\ \leq C \begin{cases} x_n^{(p-n-\alpha)/p} & \text{if } n-p+\alpha > 0, \\ (\log x_n^{-1})^{1/p'} & \text{if } n-p+\alpha = 0, \end{cases}$$

where $1/p + 1/p' = 1$ and C is a positive constant independent of $x = (x_1, \dots, x_n)$ with $0 < x_n < 1/2$.

Proof. — Let $x^* = (0, \dots, 0, 1)$. By change of variables we see that the left hand side is equal to

$$x_n^{1-n-\alpha/p+n/p'} \left\{ \int_{E_1} |x^* - z|^{p'(1-n)} |z_n|^{-\alpha p'/p} dz \right\}^{1/p'} \\ \leq C x_n^{(p-n-\alpha)/p} \left\{ \int_{E_2} (1 + |z|)^{p'(1-n)} |z_n|^{-\alpha p'/p} dz \right\}^{1/p'},$$

where $E_1 = \{z, c_1 < |x^* - z| < c_2 x_n^{1/\gamma-1}\}$,

$$E_2 = B(0, (c_2 + 1) x_n^{1/\gamma-1}),$$

C is a positive constant independent of x with $0 < x_n < 1$. The required inequalities are established by estimating the last integral.

In the same manner we can prove

LEMMA 4. — Let $p > 1$ and $\alpha < p - n$. Then

$$\left\{ \int_{\{y: x_n/2 < |x-y| < |x|/2\}} |x-y|^{p'(1-n)} |y_n|^{-\alpha p'/p} dy \right\}^{1/p'} \leq C |x|^{(p-n-\alpha)/p}$$

for any $x \in \mathbf{R}_+^n$, where C is a positive constant independent of x .

Finally we borrow a result from [6 ; Lemma 4].

LEMMA 5. — Let $p > 1$, $\alpha < p - 1$ and f be a measurable function on $\mathbf{R}^n$ such that $\int_G |f(y)|^p |y_n|^\alpha dy < \infty$ for any bounded open set $G \subset \mathbf{R}^n$. If we set

$$E' = \left\{ \xi \in \partial \mathbf{R}_+^n ; \int_{B(\xi, 1)} |\xi - y|^{1-n} |f(y)| dy = \infty \right\},$$

then $B_{1-\alpha/p, p}(E') = \emptyset$.

Remark. — If $p - \alpha > n$, then one sees that E' is empty.

3. Proof of the theorem.

Take q such that $q = p$ if $\alpha \leq 0$ and $1 < q < p/(\alpha + 1)$ if $\alpha > 0$. Then Hölder's inequality implies that

$$\int_G |\text{grad } u(x)|^q dx < \infty$$

for any bounded open set $G \subset \mathbf{R}_+^n$. Hence the function

$$v(x) = \begin{cases} u(x_1, \dots, x_{n-1}, x_n) & \text{for } x_n > 0, \\ u(x_1, \dots, x_{n-1}, -x_n) & \text{for } x_n < 0, \end{cases}$$

can be extended to a locally q -precise function w on $\mathbf{R}^n$ in view of [8 ; Theorem 5.6] (for q -precise functions, see Ohtsuka [8 ; Chap. IV] and Ziemer [12]). Define

$$E' = \left\{ \xi \in \partial \mathbf{R}_+^n ; \int_{B(\xi, 1)} |\xi - y|^{1-n} |\operatorname{grad} w(y)| dy = \infty \right\}.$$

Then we have $B_{1-\alpha/p, p}(E') = 0$ on account of Lemma 5. We also define A_β and B_p with $f(y) = |\operatorname{grad} w(y)|^p |y_n|^\alpha$. By Lemmas 1 and 2, we see that $H_\beta(A_\beta) = 0$ and $B_{n/p, p}(B_p) = 0$.

First suppose $n - p + \alpha > 0$. Let $\gamma > 1$ be given. We shall show that (2) exists and is finite for any

$$\xi \in \partial \mathbf{R}_+^n - (E' \cup A_{\gamma(n-p+\alpha)})$$

and any $a > 1$. Let $\xi \in \partial \mathbf{R}_+^n - (E' \cup A_{\gamma(n-p+\alpha)})$. Take $N > 1$ such that $\xi \in B(0, N)$, and let ϕ_N be a function in $C_0^\infty(\mathbf{R}^n)$ such that $\phi_N = 1$ on $B(0, 2N)$ and $\phi_N = 0$ on $\mathbf{R}^n - B(0, 3N)$. Set $w_N = \phi_N w$. Then w_N is q -precise on $\mathbf{R}^n$ and satisfies

$$w_N(x) = c \sum_{i=1}^n \int \left(\frac{\partial}{\partial x_i} R_2 \right) (x - y) \frac{\partial w_N}{\partial y_i} dy \quad \text{for } x \in \mathbf{R}_+^n,$$

where c is a constant, $R_2(x) = \log(1/|x|)$ if $n = 2$ and $R_2(x) = |x|^{2-n}$ if $n \geq 3$. In fact, since w_N is continuously differentiable on $\mathbf{R}_+^n$, the right hand side is continuous on $\mathbf{R}_+^n$ and the required equality holds for any $x \in \mathbf{R}_+^n$ on account of Ohtsuka [8; Theorem 9.11].

For $x \in \mathbf{R}_+^n$, we write

$$w_N(x) = w_{N,1}(x) + w_{N,2}(x) + w_{N,3}(x),$$

where

$$w_{N,1}(x) = c \sum_{i=1}^n \int_{B(x, x_n/2)} \left(\frac{\partial}{\partial x_i} R_2 \right) (x - y) \frac{\partial w_N}{\partial y_i} dy,$$

$$w_{N,2}(x) = c \sum_{i=1}^n \int_{B(x, |\xi - x|/2) - B(x, x_n/2)} \left(\frac{\partial}{\partial x_i} R_2 \right) (x - y) \frac{\partial w_N}{\partial y_i} dy,$$

$$w_{N,3}(x) = c \sum_{i=1}^n \int_{\mathbf{R}^n - B(x, |\xi - x|/2)} \left(\frac{\partial}{\partial x_i} R_2 \right) (x - y) \frac{\partial w_N}{\partial y_i} dy.$$

Since w_N is harmonic on $B(0, 2N) \cap \mathbf{R}_+^n$, $w_{N,1}(x) = 0$ for $x \in B(0, N) \cap \mathbf{R}_+^n$. It follows from our assumption $\xi \notin E'$ that $\int |\xi - y|^{1-n} |\text{grad } w_N(y)| dy < \infty$. Hence Lebesgue's dominated convergence theorem gives

$$\lim_{x \rightarrow \xi, x \in \mathbf{R}_+^n} w_{N,3}(x) = c \sum_{i=1}^n \int \left(\frac{\partial}{\partial x_i} R_2 \right) (\xi - y) \frac{\partial w_N}{\partial y_i} dy.$$

For $w_{N,2}$ we apply Hölder's inequality to obtain by Lemma 3,

$$\begin{aligned} & |w_{N,2}(x)| \\ & \leq \text{const.} \left\{ \int_{\{y; x_n/2 < |x-y| < |\xi-x|/2\}} |x-y|^{p'(1-n)} |y_n|^{-\alpha p'/p} dy \right\}^{1/p'} \\ & \times \left\{ \int_{B(x, |\xi-x|/2)} |\text{grad } w_N(y)|^p |y_n|^\alpha dy \right\}^{1/p} \\ & \leq \text{const.} \left\{ x_n^{p-\alpha-n} \int_{B(\xi, 2ax_n^{1/\gamma})} |\text{grad } w(y)|^p |y_n|^\alpha dy \right\}^{1/p} \end{aligned}$$

for $x \in B(0, N) \cap T_\gamma(\xi, a)$. Since $\xi \notin A_{\gamma(n-p+\alpha)}$,

$$\lim_{x \rightarrow \xi, x \in T_\gamma(\xi, a)} w_{N,2}(x) = 0.$$

$$\begin{aligned} \text{Thus } \lim_{x \rightarrow \xi, x \in T_\gamma(\xi, a)} u(x) &= \lim_{x \rightarrow \xi, x \in T_\gamma(\xi, a)} w_N(x) \\ &= \lim_{x \rightarrow \xi, x \in T_\gamma(\xi, a)} w_{N,3}(x), \end{aligned}$$

which is finite.

If $n - p + \alpha = 0$, then we can prove that (2) exists and is finite for any $\xi \in \partial \mathbf{R}_+^n - (E' \cup B_p)$, any $\gamma \geq 1$ and any $a > 1$.

If $n - p + \alpha < 0$, then E' is empty, so that similar arguments yield the required assertion with the aid of Lemma 4.

4. Remarks.

Remark 1. — If $p > n$, $p - \alpha - n > 0$ and u is a locally p -precise function on $\mathbf{R}_+^n$ satisfying (1), then $u(x)$ has a finite limit as $x \rightarrow \xi$, $x \in \mathbf{R}_+^n$, for any $\xi \in \partial \mathbf{R}_+^n$.

Remark 2. — Nagel, Rudin and Shapiro [7] studied tangential behaviors of functions of the form

$$(P_{x_n} * (K * g))(x'), \quad x = (x', x_n) \in \mathbf{R}_+^n,$$

where P is the Poisson kernel in $\mathbf{R}_+^n$, K is a nonnegative kernel, which is radial and decreasing, and g is a function in $L^p(\mathbf{R}^{n-1})$. The function in our theorem has a boundary value in the Lipschitz space $\Lambda_{\beta}^{p,p}(\mathbf{R}^{n-1})$ with $\beta = 1 - (\alpha + 1)/p$ locally, provided $-1 < \alpha < p + 1$ (cf. [9; Chap. VI, §§ 4.3, 4.5]). We do not know whether functions f in $\Lambda_{\beta}^{p,p}(\mathbf{R}^{n-1})$ can be written as $f = K * g$, where K is an appropriate kernel function which is determined independently of f and $g \in L^p(\mathbf{R}^{n-1})$.

If $g \in L^p(\mathbf{R}^n)$, $\beta = 1 - (\alpha + 1)/p$ and $-1 < \alpha < p - 1$, then

$$F(x') = \int g_{1-\alpha/p}((x', 0) - y) g(y) dy$$

belongs to $\Lambda_{\beta}^{p,p}(\mathbf{R}^{n-1})$, where g_{ℓ} denotes the Bessel kernel of order ℓ (cf. [9; Chap. VI, § 4.3]). Hence $u(x) = P_{x_n} * F(x')$ satisfies

$$\int_0^\infty \left[x_n^{k-\beta} \left\{ \int_{\mathbf{R}^{n-1}} \left| \left(\frac{\partial}{\partial x_n} \right)^k u(x', x_n) \right|^p dx' \right\}^{1/p} \right]^p x_n^{-1} dx_n < \infty,$$

where k is an integer greater than β , in view of [9; p. 152]. This implies, by the observation given after Lemma 4' in [9; Chap. V], that u satisfies

$$\int_{\mathbf{R}_+^n} |\text{grad } u(x)|^p x_n^\alpha dx < \infty. \quad (1)'$$

Thus our theorem is applicable to this function u .

Remark 3. — In case $p - \alpha - n = 0$, our theorem gives the best possible result as to the size of the exceptional set as the next proposition shows.

PROPOSITION. — Let E be a compact set in $\partial \mathbf{R}_+^n$ with $B_{1-\alpha/p,p}(E) = 0$, where $p > 1$ and $-1 < \alpha < p - 1$. Then there exists a function u which is harmonic in $\mathbf{R}_+^n$ and satisfies (1)' such that $\lim_{x_n \downarrow 0} u(x', x_n)$ does not exist for any $(x', 0) \in E$.

Proof. — Since $B_{1-\alpha/p,p}(E) = 0$, we can find a nonnegative function $f \in L^p$ such that

$$F(x') \equiv \int g_{1-\alpha/p}((x', 0) - y) f(y) dy = \infty$$

for any $(x', 0) \in E$. As seen above, $P_{x_n} * F(x')$ satisfies (1)'.

Take a_1, b_1 and c_1 such that

$$0 < a_1 < 1, \quad 0 < b_1 < c_1 < 1$$

and $P_{a_1} * F_1(x') > 1$ for $(x', 0) \in E$, where

$$F_1(x') = \int_{\{b_1 < |y_n| < c_1\}} g_{1-\alpha/p}((x', 0) - y) f(y) dy.$$

We proceed inductively and obtain $\{a_j\}$, $\{b_j\}$ and $\{c_j\}$ such that $0 < a_{j+1} < a_j$, $0 < c_{j+1} < b_j < c_j$, $P_{a_k} * F_j(x') < 2^{-j}$

if $k < j$ and $(x', 0) \in E$ and $P_{a_j} * F_j(x') > j + \sum_{k=1}^{j-1} M_k$ if $(x', 0) \in E$, where

$$F_j(x') = \int_{\{b_j < |y_n| < c_j\}} g_{1-\alpha/p}((x', 0) - y) f(y) dy$$

and $M_k = \max \{F_k(x'); (x', 0) \in \partial R_+^n\}$.

Define

$$u(x', x_n) = \sum_{j=1}^{\infty} (-1)^j P_{x_n} * F_j(x').$$

Then one sees easily that $\{u(x', a_j)\}$ does not converge as $j \rightarrow \infty$ for any $(x', 0) \in E$. Since u satisfies (1)' and is harmonic in R_+^n , u is the required function.

Remark 4. — Let $p > 1$, $p - n < \alpha < p - 1$ and $\alpha > -1$. Then we can find a function u harmonic in R_+^n and satisfying the following conditions :

- (i) u has a finite nontangential limit at 0 ;
- (ii) $\limsup_{x \rightarrow 0, x \in T_\gamma(0, a)} u(x) = \infty$ for any $\gamma > 1$ and any $a > 1$;
- (iii) $\int_{R_+^n} |\text{grad } u(x)|^p x_n^\alpha dx < \infty$.

To see this, let $x^{(j)} = (2^{-j}, 0, \dots, 0)$ and define

$$f_j(y) = \begin{cases} a_j |x^{(j)} - y|^{-1} & \text{if } y \in B(x^{(j)}, 2^{-j-2}) - \mathbf{R}_+^n, \\ 0 & \text{otherwise,} \end{cases}$$

where a_j is a positive number determined later. Setting

$$u(x) = \int (x_n - y_n) |x - y|^{-n} f(y) dy,$$

where $f = \sum_{j=1}^{\infty} f_j$, we note the following facts :

$$a) u(0) \leq \int |y|^{1-n} f(y) dy \leq C_1 \sum_{j=1}^{\infty} a_j;$$

b) If $x = (2^{-j}, 0, \dots, 0, x_n)$ and $0 < x_n < 2^{-j-1}$, then $u(x) \geq C_2 a_j \log(2^{-j}/x_n)$;

$$c) \int f(y)^p |y_n|^\alpha dy \leq C_3 \sum_{j=1}^{\infty} a_j^p 2^{-j(n-p+\alpha)},$$

where C_1, C_2 and C_3 are positive constants independent of x and j .

Now we choose $\{a_j\}$ so that (a) and (c) are finite but $\limsup_{j \rightarrow \infty} ja_j = \infty$. Then (a) implies that u has a nontangential limit at 0, and (iii) follows from (c) and [5; Lemma 6](*). By (b) and the construction of $\{a_j\}$, (ii) is fulfilled. Thus u satisfies (i), (ii) and (iii).

Remark 5. — If $p > 1$ and $\alpha = p - n > -1$, then for each $\gamma > 1$ there exists a function u harmonic in $\mathbf{R}_+^n$ such that :

(i) u has a finite nontangential limit at 0;

(ii) $\limsup_{x \rightarrow 0, x \in T_{\gamma'}(\xi, a)} u(x) = \infty$ for any $\gamma' \geq \gamma$ and any $a > 1$;

$$(iii) \int_{\mathbf{R}_+^n} |\text{grad } u(x)|^p x_n^\alpha dx < \infty.$$

In fact we modify above f_j by setting

$$f_j(y) = \begin{cases} a_j |x^{(j)} - y|^{-1} & \text{if } y \in B(x^{(j)}, 2^{-j-2}) - B(x^{(j)}, 2^{-\gamma j}) - \mathbf{R}_+^n, \\ 0 & \text{otherwise,} \end{cases}$$

(*) One notes that the conclusion of [5; Lemma 6] is true in case $p > 1$ and $-1 < \alpha < p - 1$ if g in the lemma has compact support.

and consider $u(x) = \sum_{j=1}^{\infty} \int (x_n - y_n) |x - y|^{-n} f_j(y) dy$. Then as in Remark 3 we can choose $\{a_j\}$ such that (i), (ii) and (iii) hold.

Remark 6. — Let $p > 1$ and h be a positive function on the interval $(0, \infty)$ such that $\lim_{r \downarrow 0} h(r) (\log r^{-1})^{p-1} = 0$. Define for $\xi \in \partial \mathbf{R}_+^n$,

$$T_h(\xi) = \left\{ x \in \mathbf{R}_+^n ; \log \frac{|x - \xi|}{x_n} \leq h(|x - \xi|)^{-1/(p-1)} \right\}.$$

If u is a function harmonic in $\mathbf{R}_+^n$ and satisfying (1) with $\alpha = p - n > -1$, then $\lim_{x \rightarrow \xi, x \in T_h(\xi)} u(x)$ exists and is finite for any $\xi \in \partial \mathbf{R}_+^n - (E' \cup B_h)$, where

$$B_h = \left\{ \xi \in \partial \mathbf{R}_+^n ; \limsup_{r \downarrow 0} h(r)^{-1} \int_{B(\xi, r) \cap \mathbf{R}_+^n} |\text{grad } u(x)|^p x_n^\alpha dx > 0 \right\}.$$

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