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ESTIMATES OF ONE-DIMENSIONAL OSCILLATORY INTEGRALS

by Detlef MÜLLER

1. Introduction.

If \( U \) is an open domain in \( \mathbb{R}^k \) and if \( f \) is a smooth, real valued function on \( U \), one may define the associated oscillatory integral as

\[
E_f(\theta) = \int_U \theta(x)e^{2\pi i f(x)} \, dx,
\]

where \( \theta \) belongs to \( \mathcal{D}(U) \), the space of test functions on \( U \).

When \( f \) has the form \( f = \sum_{j=1}^n \eta_j \psi_j \), where the \( \psi_j \in C^\infty(U) \) are real-valued functions and \( \eta_j \) are real parameters, one is interested in the asymptotic behaviour of \( E_{\Sigma\eta_j\psi_j}(\theta) \) as \( (\eta_1, \ldots, \eta_n) \) tends to infinity, for several reasons.

For example, if \( \mu \) is a smooth measure on a smooth submanifold of \( \mathbb{R}^m \), and if the support of \( \mu \) is sufficiently small, then the Fourier-Stieltjes transform \( \hat{\mu}(\eta_1, \ldots, \eta_m) \) may always be written as \( E_{\Sigma\eta_j\psi_j}(\theta) \) for certain functions \( \psi_j \) and \( \theta \).

Good information about the asymptotic behaviour of such Fourier-Stieltjes transforms is needed to solve the synthesis problem for smooth submanifolds of \( \mathbb{R}^m \) (see e.g. [7]). And, as Professor Y. Domar has pointed out to me, such knowledge would also yield information about the decay at infinity of solutions of partial differential equations (see e.g. [5]).

As far as I know, satisfactory answers to the above problem have only been given for oscillatory integrals \( E_{\Sigma\eta_j\psi_j}(\theta) \) with

\[
\Sigma\eta_j\psi_j(x_1, \ldots, x_k) = \sum_{j=1}^k \eta_j x_j + \eta_{k+1} \psi_{k+1}(x_1, \ldots, x_k),
\]
which correspond to surface carried measures (see [2], [4], [6]). In some sense, the other extreme is the case where $\sum \eta_j \psi_j$ is a function of only one real variable, which corresponds to measures on curves. For this case, we will prove some quite general results.

2.

Let $\psi \in C^\infty(I, \mathbb{R}^n)$, $\psi = (\psi_1, \ldots, \psi_n)$, where $I \neq \emptyset$ is some bounded open interval in $\mathbb{R}$. For $\xi$, $\eta \in \mathbb{R}^n$ let $\xi \cdot \eta$ denote the Euclidean inner product on $\mathbb{R}^n$, and correspondingly let

$$\eta \cdot \psi(x) = \sum_{j=1}^{n} \eta_j \psi_j(x).$$

Further let

$$|\eta| : = \max_j |\eta_j| \quad \text{for} \quad \eta \in \mathbb{R}^n.$$

Define the torsion $\tau$ of $\psi$ by

$$\tau(x) = \det (\psi^{(i+1)}(x))_{i,j=1,\ldots,n} = \det (\psi''(x)\psi'''(x) \ldots \psi^{(n+1)}(x)),$$

where $\psi$ is regarded as a column vector and $\psi^{(k)}$ denotes the $k$-th derivative of $\psi$. At least for $n = 2$ we have $\tau(x) = k(x)\psi''(x)^2$, where $k$ is the torsion of the curve $\gamma = \{(x, \psi(x)) : x \in I\}$ in $\mathbb{R}^{n+1}$. Let

$$e(t) = e^{2\pi i t} \quad \text{for} \quad t \in \mathbb{R}, \quad \text{and} \quad e(g) = e \circ g$$

for $g \in C^\infty(I, \mathbb{R})$. If $\psi_0(x) = x$ for $x \in \mathbb{R}$, then for $\theta \in \mathcal{D}(I)$, $\eta_0 \in \mathbb{R}$ and $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$, we have

$$E_{\eta} \sum_0 \eta_j \psi_j (\theta) = (\theta e(\eta \cdot \psi)) \hat{\tau}(-\eta_0).$$

So it will be slightly more general to study the behaviour of $|\theta e(\eta \cdot \psi)|_{PM}$ as $|\eta| \to \infty$, where

$$|\varphi|_{PM} = \sup_{t \in \mathbb{R}} |\varphi(t)|$$

for every $\varphi \in \mathcal{D}(\mathbb{R})$. 
For certain reasons (see [3], [7], Th. 4.1), we will also study $|\varphi(\eta \cdot \psi)|_{\lambda}$, where
\[ |\varphi|_{\lambda} = \int |\varphi(t)| \, dt \]
for every $\varphi \in \mathcal{D}(\mathbb{R})$.

We will first state our main results and prove some corollaries:

**Theorem 1.** Let $\mathcal{S} \in \mathcal{D}(I)$. Then

(i) $|\varphi(\eta \cdot \psi)|_{\lambda} = O(|\eta|^{1/2})$, as $|\eta| \to \infty$.

(ii) If for some subinterval $J$ of $I$ and some $\sigma > 0$
\[ |\mathcal{S}(x)| \geq \sigma \quad \text{and} \quad |\mathcal{S}(x) - \mathcal{S}(y)| < \sigma/2 \quad \text{for all} \quad x, y \in J, \]
and if $\psi_1, \ldots, \psi_n$ are linearly independent modulo affine linear functions, then there is a constant $C > 0$, such that
\[ |\varphi(\eta \cdot \psi)|_{\lambda} \geq C(1 + |\eta|)^{1/2} \]
for all $\eta \in \mathbb{R}^n$.

**Corollary 1.** The following two conditions are equivalent:

(i) For each $\mathcal{S} \in \mathcal{D}(\mathbb{R})$, $\mathcal{S} \neq 0$, there are constants $c > 0$, $C > 0$, such that for all $\eta \in \mathbb{R}^n$
\[ c(1 + |\eta|)^{1/2} \leq |\varphi(\eta \cdot \psi)|_{\lambda} \leq C(1 + |\eta|)^{1/2}. \]

(ii) $\psi_1, \ldots, \psi_n$ are linearly independent modulo affine linear functions on every non empty open subinterval of $I$.

**Proof of Corollary 1.** (i) follows directly from (ii) by Theorem 1. Now suppose that there exists a vector $v \in \mathbb{R}^n$, $v \neq 0$, such that $v \cdot \psi$ is affine linear on some open subinterval $J \neq \emptyset$ of $I$. Then we have for any non-trivial $\mathcal{S} \in \mathcal{D}(J)$
\[ |\varphi(sv \cdot \psi)|_{\lambda} = |\mathcal{S}|_{\lambda} \neq 0 \quad \text{for all} \quad s \in \mathbb{R}, \]
since $e(sv \cdot \psi)$ is the product of a unimodular complex number and a unitary character of $\mathbb{R}$.

Thus (i) is not fulfilled, q.e.d.
Remark. — Condition (ii) of Corollary 1 is clearly satisfied if \( \tau^{-1}(\{0\}) \) has empty interior. As will be shown later (Lemma 3), this is always the case if \( \psi_1, \ldots, \psi_n \) are real analytic and linearly independent modulo affine mappings. However one should notice that global linear independence does not in general imply local linear independence.

**Theorem 2.** — (i) If \( \tau^{-1}(\{0\}) = \emptyset \), then for \( \xi \in \mathcal{D}(I) \)

\[
|\xi(e \cdot \psi)|_{PM} = 0(|\eta|^{-1/(n+1)}) \quad \text{as} \quad |\eta| \to \infty.
\]

(ii) If \( \xi \in \mathcal{D}(I) \), and if there exists an \( x_0 \in I \) with \( \xi(x_0) \neq 0 \) and \( \tau(x_0) \neq 0 \), then there exists an \( \varepsilon > 0 \) and a function \( \xi \in C^\infty((-\varepsilon, \varepsilon), \mathbb{R}^n) \) with

\[
\det (\xi(y)\xi'(y) \cdots \xi^{(n-1)}(y)) \neq 0 \quad \text{for all} \quad y \in (-\varepsilon, \varepsilon),
\]

such that, for some \( C > 0 \),

\[
|\xi(e \cdot \psi)|_{PM} \geq C(1 + |s|)^{-1/(n+1)}
\]

for all \( s \in \mathbb{R} \) and \( y \in (-\varepsilon, \varepsilon) \).

Assume that \( \tau^{-1}(\{0\}) \) has empty interior. Then we have

**Corollary 2.** — There exists a \( \xi \in \mathcal{D}(I) \), \( \xi \neq 0 \), such that for all positive \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) with \( \sum_1^n \alpha_j \leq (n+1)^{-1} \), there exists a constant \( C = C(\alpha_1, \ldots, \alpha_n) > 0 \) such that

\[
(2.1) \quad |\xi(e \cdot \psi)|_{PM} \leq C \prod_{j=1}^n |\eta_j|^{-\alpha_j}.
\]

Conversely, if \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) are positive, and if there exists a \( \xi \in \mathcal{D}(I) \), \( \xi \neq 0 \), and a \( C > 0 \) such that (2.1) holds, then

\[
\sum_1^n \alpha_j \leq (n+1)^{-1}.
\]

**Proof of Corollary 2.** — If \( \tau^{-1}(\{0\}) \) has empty interior, then there is of course an \( x_0 \in I \) with \( \tau(x_0) \neq 0 \), and so, for \( \xi \in \mathcal{D}(I) \) with sufficiently small support near \( x_0 \),

\[
|\xi(e \cdot \psi)|_{PM} \leq C(1 + |\eta|)^{-1/(n+1)}
\]

by Theorem 2, (i).
If \( \alpha_1, \ldots, \alpha_n \) are positive and \( \Sigma \alpha_j \leq (n+1)^{-1} \), then
\[
\prod_j |\eta_j|^{\alpha_j} \leq |\eta|^{1/(n+1)} \quad \text{for} \quad |\eta| \geq 1,
\]
hence
\[
|\mathfrak{g}(\eta \cdot \Psi)|_{PM} \leq C \prod_j |\eta_j|^{-\alpha_j} \quad \text{for} \quad |\eta| \geq 1,
\]
and the same estimate holds for all \( \eta \) if one replaces \( C \) by \( C + |\mathfrak{g}|_{L^1} \).

Conversely, let now \( \mathfrak{g} \in \mathscr{D}(I) \), \( \mathfrak{g} \neq 0 \), such that (2.1) holds for some \( \alpha_j \geq 0 \), and assume
\[
\Sigma \alpha_j = (n+1)^{-1} + \delta, \quad \delta > 0.
\]
Since \( \tau^{-1}(\{0\}) \) has empty interior, there is an \( x_0 \in I \) with \( \mathfrak{g}(x_0) \neq 0 \) and \( \tau(x_0) \neq 0 \). Choose \( \varepsilon > 0 \) and \( \xi \in C^\infty((-\varepsilon, \varepsilon), \mathbb{R}^n) \) as in Theorem 2 (ii).
Since \( \det (\xi_j(y)) \xi_j^e(y) \ldots \xi^{(n-1)}(y)) \neq 0 \) for all \( y \in (-\varepsilon, \varepsilon) \), there exists a \( y_0 \in (-\varepsilon, \varepsilon) \) with
\[
\xi_j(y_0) \neq 0 \quad \text{for} \quad j = 1, \ldots, n.
\]
It follows
\[
|\mathfrak{g}(\xi(y_0) \cdot \Psi)|_{PM} \geq C'(1 + |s|)^{-1/(n+1)}.
\]
On the other hand, (2.1) yields
\[
|\mathfrak{g}(s\xi(y_0) \cdot \Psi)|_{PM} \leq C \prod_j |s\xi_j(y_0)|^{-\alpha_j}
= \left(C \prod_j |\xi_j(y_0)|^{-\alpha_j}\right)|s|^{-1/(n+1)}|s|^{-\delta}.
\]
For \( |s| \) sufficiently large this leads to a contradiction to (2.2), q.e.d.

Corollary 2 demonstrates that the result in Theorem 2 is in some sense best possible.

3.

Before we start to prove the theorems above we will state some lemmas. The first one is due to J.-E. Björk and is cited in [3], Lemma 1.6:

**Lemma 1.** - Let \( I \neq \emptyset \) be a bounded, open interval in \( \mathbb{R} \), and let \( \varphi \in \mathscr{D}(I) \), \( g \in C^p(I) \) with
\[
0 < C_1 \leq |g'(x)| + |g''(x)| + \cdots + |g^{(p)}(x)| \leq C_2
\]
if \( x \in I \), where \( C_1 \) and \( C_2 \) are constants and \( p \) is a positive integer. Then there exists a constant \( C \) not depending on \( g \), such that

\[
\left| \int \phi(x) e^{2 \pi i g(x)} \, dx \right| \leq C (1 + |t|)^{-1/p}
\]

for every \( t \in \mathbb{R} \).

The second lemma will be used to prove the remark following Corollary 1. I would like to thank Professor H. Leptin for pointing out to me a shorter proof than my original one. By \( \wedge \) we denote the exterior product in the Grassmann algebra \( \wedge (\mathbb{R}^n) \).

**Lemma 2.** — Let \( \psi \in C^\infty(I, \mathbb{R}^n) \). Then

\[
\psi(x) \wedge \psi'(x) \ldots \wedge \psi^{(n-1)}(x) = 0
\]

for all \( x \in I \) implies

\[
\psi^{(k_1)}(x) \wedge \psi^{(k_2)}(x) \wedge \ldots \wedge \psi^{(k_n)}(x) = 0
\]

for all \( x \in I \) and \( k_1, \ldots, k_n \in \mathbb{N}_0 \).

**Proof.** — Fix \( x_0 \in I \), and assume first \( \psi(x_0) \neq 0 \). If \( u \in C^\infty(I, \mathbb{R}) \), then

\[
(u\psi)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} u^{(k-j)} \psi^{(j)},
\]

so \( \psi \wedge \psi' \wedge \ldots \wedge \psi^{(n-1)} \equiv 0 \) implies

\[
(u\psi) \wedge (u\psi)' \wedge \ldots \wedge (u\psi)^{(n-1)} \equiv 0.
\]

So, it is no loss of generality to assume

\[
\psi_n(x) = 1 \quad \text{for} \quad x \in I.
\]

If \( \{e_j\}_{j=1}^{n-1} \) denotes the canonical basis of \( \mathbb{R}^n \), we may thus write

\[
\psi(x) = \sum_{j=1}^{n-1} \psi_j(x) e_j + e_n = \rho(x) + e_n, \quad \text{where} \quad \rho(x) \in \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n.
\]

This yields

\[
0 = \psi(x) \wedge \psi'(x) \wedge \ldots \wedge \psi^{(n-1)}(x) = \rho(x) \wedge \rho'(x) \wedge \ldots \wedge \rho^{(n-1)}(x) + e_n \wedge \rho'(x) \wedge \ldots \wedge \rho^{(n-1)}(x),
\]
and since $p(x)$, $p'(x)$, ..., $p^{(n-1)}(x)$ are clearly linearly dependent, we get
\[ 0 = p'(x) \wedge p''(x) \wedge \ldots \wedge p^{(n-1)}(x). \]
By induction over $n$, we now may assume
\[ 0 = p^{(k_2)}(x) \wedge p^{(k_3)}(x) \wedge \ldots \wedge p^{(k_n)}(x) \]
for $x \in I$ and $k_j \geq 1$.

This implies
\[ \psi^{(k_1)}(x) \wedge \ldots \wedge \psi^{(k_n)}(x) = e^{(k_1)}_n(x) \wedge p^{(k_2)}(x) \wedge \ldots \wedge p^{(k_n)}(x) = 0 \]
for $0 \leq k_1 < k_2 < \ldots < k_n$, where we considered $e_n$ as the function $e_n(x) = e_n$.

Thus we have proved
\[ \psi^{(k_1)}(x_0) \wedge \psi^{(k_2)}(x_0) \wedge \ldots \wedge \psi^{(k_n)}(x_0) = 0 \]
for all $x_0 \in I_0 = \{x \in I : \psi(x) \neq 0\}$ and $k_j \geq 0$. By continuity, the same holds true for $x_0 \in I_0 \cap I$, hence for all $x_0 \in I$, since for $y \in I \setminus I_0$ clearly $\psi^{(k)}(y) = 0$ for every $k \in \mathbb{N}_0$.

**Lemma 3.** – If $\psi = (\psi_1, \ldots, \psi_n) \in \mathcal{C}^\infty(I, \mathbb{R}^n)$ is real analytic, and if $\psi_1$, $\ldots$, $\psi_n$ are linearly independent modulo affine mappings, then $\tau^{-1}(\{0\})$ has empty interior, where $\tau$ denotes the torsion of $\psi$.

**Proof.** – Assume $\psi(x) = 0$ for every $x$ in some nonempty open interval $J \subset I$. Fix $x_0 \in J$. Then, passing to a possibly smaller interval, we may assume that $\psi_j$ has an absolute convergent series expansion
\[ \psi_j(x) = \sum_{k=0}^{\infty} a_k j^k(x-x_0)^k, \quad j = 1, \ldots, n, \quad x \in J. \]
Define vectors
\[ a_k = (a_k^j)_{j=1,\ldots,n} \in \mathbb{R}^n \]
and
\[ a^j = (a_j^k)_{k=2,\ldots,\infty} \in \mathbb{R}^{n_1}, \quad N_1 = N\setminus\{0,1\}. \]
By Lemma 2, $\psi^{(k_1)}(x_0), \ldots, \psi^{(k_n)}(x_0)$ are linearly dependent for any $k_j \in \mathbb{N}$ with $2 \leq k_1 < \ldots < k_n$, i.e. $a_{k_1}, \ldots, a_{k_n}$ are linearly dependent for $2 \leq k_1 < \ldots < k_n$. But this implies that $a^1, \ldots, a^n$ are linearly
dependent, i.e. there exist \( v_1, \ldots, v_n \in \mathbb{R} \), not all zero, with

\[
0 = \sum_j v_j a^j, \quad \text{i.e.}
\]

\[
\sum_j v_j \psi_j(x) = \sum_j v_j a^j_0 + v_j a^j_1(x-x_0) \quad \text{for } x \in J.
\]

But, since \( \psi \) is real analytic, this equation holds for all \( x \in I \), i.e. \( \sum_j v_j \psi_j \) is affine linear.

4.

Proof of Theorem 1. — It is well-known (see e.g. [1], [7]) that for \( \varphi \in \mathcal{D}(\mathbb{R}) \) one has the estimate

\[
|\varphi|_\Lambda \leq \{2 |\text{supp } \varphi||\varphi|_\infty|\varphi'|_\infty\}^{1/2},
\]

where \( |\text{supp } \varphi| \) denotes the Lebesgue measure of the support of \( \varphi \). From (4.1) one immediately gets (i) of Theorem 1.

Now, suppose there exists a subinterval \( J \) in \( I \) and a \( \sigma > 0 \) such that \( |\theta(x)| \geq \sigma \) and \( |\theta(x) - \theta(y)| < \sigma/2 \) for \( x, y \in J \), and such that \( \psi_1, \ldots, \psi_n \) are linearly independent modulo affine mappings on \( J \).

Then a simple compactness argument yields:

There are constants \( \varepsilon > 0, \delta > 0 \), such that for every \( \eta \in \mathbb{R}^n \) with \( |\eta| = 1 \) there is an interval \( J_\eta \) of length \( 2\varepsilon \) in \( J \) with

\[
|\eta \cdot \psi''(x)| \geq \delta \quad \text{for all } x \in J_\eta.
\]

Now choose \( \varphi \in \mathcal{D}(-\varepsilon, \varepsilon), \varphi \geq 0 \), with \( \int \varphi(x) \, dx = 1 \). For fixed \( \eta \in \mathbb{R}^n, \eta \neq 0 \), set \( \eta' = |\eta|^{-1} \eta \), and choose \( J_{\eta'} \) as in (4.2). Let \( \tilde{\varphi} \) be a suitable translate of \( \varphi \) such that \( \text{supp } \tilde{\varphi} \subset J_{\eta'} \). Then we get

\[
0 < \sigma/2 \leq \left| \int \theta(x) \tilde{\varphi}(x) \, dx \right|
\]

\[
= \left| \int \theta(x)e(\eta \cdot \psi)(x)\tilde{\varphi}(x)e(-\eta \cdot \psi)(x) \, dx \right|
\]

\[
\leq |\theta e(\eta \cdot \psi)|_\Lambda |\tilde{\varphi} e(-\eta \cdot \psi)|_{PM},
\]

since \( J_{\eta'} \subset J \).
For $\xi \in \mathbb{R}$ one has
\[
\{\phi e(\eta \cdot \psi)\}^* (-\xi) = \int \phi(x) e(-\xi x - \eta \cdot \psi(x)) \, dx = \int \varphi(x) e(-|\eta|g(x)) \, dx,
\]
where $g$ is a function on $[-\varepsilon, \varepsilon]$ which is a certain translate of the function
\[
x \mapsto \xi' x + \eta' \cdot \psi(x)
\]
on $J_\eta'$,
where $\xi' = |\eta|^{-1} \xi$.

But (4.2) implies
\[
\delta \leq |g''(x)| \quad \text{for every} \quad x \in [-\varepsilon, \varepsilon].
\]
Moreover, if we set $A = 2 \sup_{x \in J} |\psi'(x)|$, $B = \sup_{x \in J} |\psi''(x)|$, then for $|\xi| \leq A|\eta|$:
\[
|g'(x)| + |g''(x)| \leq |\xi'| + |\eta'| (A + B) \leq 2A + B
\]
for every $x \in [-\varepsilon, \varepsilon]$.

Thus, by Lemma 1, there exists a $C > 0$, such that for $|\xi| \leq A|\eta|$:
\[
(4.4) \quad \left| \int \phi(x) e(-\xi x - \eta \cdot \psi(x)) \, dx \right| \leq C (1 + |\eta|)^{-1/2}.
\]
And, if $|\xi| > A|\eta|$, then integration by parts yields
\[
(4.5) \quad \left| \int \phi(x) e(-\xi x - \eta \cdot \psi(x)) \, dx \right| \\
= \left| \int \phi' (-|\eta|g(x)) \left( \frac{\phi}{2\pi|\eta|g'} \right)'(x) \, dx \right| \\
\leq (2\pi|\eta|)^{-1} \int \left\{ \left| \frac{\phi'(x)}{|g'(x)|} \right| + \frac{|\phi'(x)||g''(x)|}{|g'(x)|^2} \right\} \, dx \\
\leq C'|\eta|^{-1},
\]
where $C'$ is some constant depending on $\phi$, $\psi$ and $A$ only, since for $x \in [-\varepsilon, \varepsilon]$ we have $|g''(x)| \leq B$ and $|g'(x)| = |\xi'| + |\eta' \psi'(y)| \geq A - A/2$ for some $y \in J$. 
Now, by (4.4), (4.5),

$$|\tilde{\phi}(e^{-\eta \cdot \psi})|_{PM} \leq (C + C')|\eta|^{-1/2}$$

if \(|\eta| \geq 1\),

which together with (4.3) proves Theorem 1 (ii).

**Proof of Theorem 2.** — Assume \(\tau(x) \neq 0\) for every \(x \in I\), and let \(g \in \mathcal{G}(I)\), \(g \neq 0\). Passing to a smaller interval, we may even assume that \(I\) is closed.

Set \(A = 2 \sup_{x \in I} |\psi'(x)|\), and for \(\xi' \in \mathbb{R}, \ |\xi'| \leq A, \ \eta' \in \mathbb{R}^n, \ |\eta'| = 1, \ x \in I\) let

$$Q_{\xi',\eta'}(x) = \sum_{j=1}^{n+1} |(\xi' x + \eta' \cdot \psi(x))^{(j)}(x)|.$$

Since \(\tau^{-1}([0]) = \emptyset\), we have \(Q_{\xi',\eta'}(x) \neq 0\) for every \(x \in I\), and since \(Q_{\xi',\eta'}(x)\) is continuous in \(\xi', \eta'\) and \(x\) on the compact space \([-A,A] \times \{\eta' \in \mathbb{R}^n: \ |\eta'| = 1\} \times I\), there exist constants \(C_1 > 0, C_2 > 0\), such that

$$C_1 \leq Q_{\xi',\eta'}(x) \leq C_2$$

for all \(x \in I, \ \xi', \eta'\) with \(\ |\xi'| \leq A, \ |\eta'| = 1\).

So, using quite the same arguments as in the proof of Theorem 1 (ii), we can deduce from (4.6) by Lemma 1:

$$|\tilde{\phi}(\eta \cdot \psi)|_{PM} \leq C(1 + |\eta|)^{-1/(n+1)}$$

for some constant \(C > 0\), which proves (i).

To prove (ii), we will assume, for convenience, \(x_0 = 0\), i.e. \(0 \in I\), and \(g(0) \neq 0, \ \tau(0) \neq 0\).

Let \(\varepsilon > 0\) such that \(\tau(x) \neq 0\) for \(x \in [-\varepsilon, \varepsilon]\).

Since \(\psi''(x), \ \psi'''(x), \ldots, \psi^{(n+1)}(x)\) are linearly independent for \(x \in [-\varepsilon, \varepsilon]\), there exists a function \(\xi \in C^\infty([-\varepsilon, \varepsilon], \mathbb{R}^n)\), such that for every \(x \in [-\varepsilon, \varepsilon]\)

$$\xi(x) \cdot \psi^{(j)}(x) = 0, \quad j = 2, \ldots, n,$$

and

$$\xi(x) \cdot \psi^{(n+1)}(x) = 1.$$
Differentiating (4.7) and inserting (4.8), we get
\[ \xi'(x) \cdot \psi^{(j)}(x) = 0 \quad \text{for} \quad j = 2, \ldots, n - 1, \]
and
\[ \xi'(x) \psi^{(n)}(x) = -1. \]

Repeating this process, one inductively obtains for \( k = 0, \ldots, n - 1 \)
\begin{equation}
(4.9) \quad \begin{cases}
\xi^{(k)}(x) \cdot \psi^{(j)}(x) = 0 & \text{for} \quad j = 2, \ldots, n - k, \\
\xi^{(k)}(x) \cdot \psi^{(n+1-k)}(x) = (-1)^k.
\end{cases}
\end{equation}

So, if we define matrices
\[ S(x) = (\xi^{(n-1)}(x))_{i,j=1,\ldots,n}, \quad T(x) = (\psi^{(j+1)}(x))_{i,j=1,\ldots,n}, \]
then (4.9) means that \( S(x)T(x) \) is an upper triangular matrix with
diagonal elements 1 or \(-1\), which yields
\begin{equation}
(4.10) \quad |\det (\xi(x)\xi'(x) \ldots \xi^{(n-1)}(x))| = |\det S(x)| = |\tau(x)|^{-1} \neq 0
\end{equation}
for all \( x \in [-\varepsilon, \varepsilon] \).

We now claim:

There is a constant \( C > 0 \), such that for all \( y \in (-\varepsilon, \varepsilon) \) and \( s \in \mathbb{R} \)
\begin{equation}
(4.11) \quad |\delta s(\xi(y) \cdot \psi)|_{PM} \geq C(1 + |s|)^{-1/(n+1)}.
\end{equation}

Choose \( y \in (-\varepsilon, \varepsilon) \). Then by (4.7), \( (\xi(y) \cdot \psi)^{(j)}(y) = \delta_{j,n+1} \) for
\( j = 2, \ldots, n + 1 \), and so a Taylor expansion of \( \xi(y) \cdot \psi \) yields (for \( \varepsilon \) small enough)
\begin{equation}
(4.12) \quad (\xi(y) \cdot \psi)(x) = \alpha + \beta x + (x-y)^{n+1}g(x) \quad \text{for} \quad x \in (-2\varepsilon, 2\varepsilon),
\end{equation}
where \( g \) is some smooth function on \((-2\varepsilon, 2\varepsilon)\) which depends on \( y \), and
where \( \alpha \) and \( \beta \) are some real numbers.

Let us remark here that although \( g = g_y \) depends on \( y \), \( \sup_{|s| < 2\varepsilon} |g'_y(x)| \) is
uniformly bounded for \( y \in (-\varepsilon, \varepsilon) \).

Now take \( \rho \in \mathcal{D}(\mathbb{R}) \) with \( \text{supp} \, \rho \subset (-\varepsilon, \varepsilon) \), \( \rho \geq 0 \) and
\[ \int \rho(x) \, dx = 1, \quad \text{and set} \quad \tilde{\rho}(x) = \rho(|s|^{1/(n+1)}(x-y)).\]
If we choose $\varepsilon$ small enough such that

$$|\Theta(0) - \Theta(x)| < \frac{1}{2} |\Theta(0)|$$

for $x \in (-2\varepsilon, 2\varepsilon)$, then we get

$$\left| \int \Theta(x) \tilde{p}(x) \, dx \right| = \left| \int \Theta(|s|^{-1/(n+1)}x + y) \rho(x) \, dx \right| |s|^{-1/(n+1)}$$

$$\geq \frac{1}{2} |\Theta(0)| |s|^{-1/(n+1)}, \quad \text{if } |s| \geq 1;$$

and since

$$\left| \int \Theta(x) \tilde{p}(x) \, dx \right| = \left| \int \Theta(x) e(s\xi(y) \cdot \psi) \tilde{p}(x) e(-s\xi(y) \cdot \psi) \, dx \right|$$

$$\leq |\Theta e(s\xi(y) \cdot \psi)|_{PM} |\tilde{p} e(-s\xi(y) \cdot \psi)|_A,$$

(4.11) will follow if we can show that $|\tilde{p} e(-s\xi(y) \cdot \psi)|_A$ is uniformly bounded for $y \in (-\varepsilon, \varepsilon)$ and $|s| \geq 1$.

Now, regular affine mappings of $\mathbb{R}$ induce isometries of the Fourier algebra $A = A(\mathbb{R})$, thus

$$|\tilde{p} e(-s\xi(y) \cdot \psi)|_A = |\rho e(-s\xi(y) \cdot \tilde{\psi})|_A,$$

where $\tilde{\psi}(x) = \psi(|s|^{-1/(n+1)}x + y)$.

Since for $x \in \text{supp} \rho$ and $|s| \geq 1$,

$$|s|^{-1/(n+1)}x + y \in (-2\varepsilon, 2\varepsilon),$$

(4.12) yields

$$\xi(y) \cdot \tilde{\psi}(x) = \alpha + \beta y + \beta|s|^{-1/(n+1)}x + |s|^{-1}x^{n+1}g(|s|^{-1/(n+1)}x + y).$$

Thus

$$|\tilde{p} e(-s\xi(y) \cdot \psi)|_A = |\rho e(h)|_A,$$

where $h(x) = -s|s|^{-1}x^{n+1}g(|s|^{-1/(n+1)}x + y)$. If we again apply estimate (4.1), we easily see that $|\rho e(h)|_A$ is uniformly bounded for $y \in (-\varepsilon, \varepsilon)$ and $|s| \geq 1$, q.e.d.
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