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David Spring

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CONVEX INTEGRATION OF NON-LINEAR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

by David SPRING

1. Introduction.

In this paper existence theorems are proved for C^r solutions to underdetermined systems of non-linear r-th order partial differential equations which are subject to certain convex conditions, $r \in \{1,2,3,\dots\}$. Furthermore, the solutions satisfy given approximations on lower order derivatives.

The main geometrical construction, Convex Integration, was first introduced by M. Gromov [1] in his study of 1st order systems. Our main results generalize Gromov's theorems on 1st order systems to r-th order systems of partial differential equations, $r \in \{1,2,3,\dots\}$. Despite the analytic character of the main results (§ 2, § 3), the proofs are mainly geometrical in conception.

From the point of view of differential topology, an r-th order system of partial differential equations is a closed differential relation in the space of r-jets of C^r maps between smooth manifolds. There are very few general theorems in topology which yield solutions to closed differential relations. One goal of this paper is to provide a self-contained, detailed exposition of the Convex Integration technique as it applies to a system of partial differential equations defined on an open set in Euclidean space. The principal local result, The Local Extension Theorem (Appendix 6), is also the main step in all the applications of the Convex Integration technique to solving open and closed differential relations in jet spaces.

In particular, in the case of open differential relations, some of the classical results reprovable by Convex Integration are the Immersion Theorem of Hirsch [5], and the principal theorems of Gromov and Eliasberg on the removal of singularities [2] (cf. Gromov [4], for a recent summary of results).

While the systems of partial differential equations studied in this paper are in some respects quite general, the convex conditions *exclude* linear or quasi-linear systems of partial differential equations.

To understand the scope of the method, let $\mathfrak{S}$ be an r-th order system of m partial differential equations in q unknown real-valued C^r-functions $f_1, f_2, \ldots, f_q$ defined on an open set U in $\mathbf{R}^n$, $n \geqslant 1$.

$\mathfrak{S}$ may be written as follows:

$\mathfrak{S} \equiv F_i(x, D^\beta f_j(x)) = 0, \ 1 \leqslant i \leqslant m; \ 1 \leqslant j \leqslant q; \ x \in U$, where

1) D^β runs over all local differential operators (on C^r functions of n-variables) which have the following form:

Let $(u_1, u_2, \ldots, u_n)$ be coordinates in $\mathbf{R}^n$.

Then $D^\beta = \dfrac{\partial^{|\beta|}}{\partial u_1^{a_1} \, \partial u_2^{a_2} \ldots \partial u_n^{a_n}}$; $|\beta| = a_1 + a_2 \ldots + a_n \leqslant r$.

Note. – Throughout this paper, a differential operator D^β on functions of n-variables will mean a differential operator as defined above.

2) F_i is a real-valued continuous function on the space of r-jets of maps from U to $\mathbf{R}^q$, $i = 1, 2, 3, \ldots, m$.

The analysis of the system provided by the method of Convex Integration may be described as follows:

Coordinates in $\mathbf{R}^n$ are $(u_1, u_2, \ldots, u_{n-1}, t)$.

For each $x \in U$, the m-equations of the system $\mathfrak{S}$ define a (singular) surface $\Omega_f(x) \equiv \Omega(x)$ in $\mathbf{R}^q$ according to the following prescription:

A point $(x_1, x_2, \ldots, x_q) \in \mathbf{R}^q$ lies in $\Omega(x)$ if and only if, $F_i(x, D^\beta f_j(x), x_1, x_2, \ldots, x_q) = 0, \ 1 \leqslant i \leqslant m; \ 1 \leqslant j \leqslant q$, where in the above equations for the system $\mathfrak{S}$, one replaces the derivative $\dfrac{\partial^r f_j}{\partial t^r}(x)$ by the independent variable x_j, $1 \leqslant j \leqslant q$, and, D^β runs over all differential operators on functions of n-variables such that $|\beta| \leqslant r$ and $D^\beta \neq \dfrac{\partial^r}{\partial t^r}$. Thus $(D^\beta f_j(x))_{j,\beta}$, β as above, is a sequence of constants in these equations for the surface $\Omega(x) \subset \mathbf{R}^q$.

Evidently, a C^r map $f = (f_1, f_2, \ldots, f_q) : U \longrightarrow \mathbf{R}^q$ is a solution to the system $\mathfrak{S}$ on $U \subset \mathbf{R}^q$, if and only if:

For each $x \in U$,

$$\frac{\partial^r f}{\partial t^r}(x) = \left(\frac{\partial^r f_1}{\partial t^r}, \frac{\partial^r f_2}{\partial t^r}, \ldots, \frac{\partial^r f_q}{\partial t^r}(x)\right) \in \Omega(x).$$

Example. – Let $\mathcal{S}$ be the 2nd order system consisting of one equation in 3 unknown functions on an open set $U \subset \mathbf{R}^2$ (coordinates in $\mathbf{R}^2$ are (y, t)):

$$\frac{\partial^2 f_1}{\partial t^2} - \left[\frac{\partial^2 f_2}{\partial t^2}\right]^2 + \left[\frac{\partial^2 f_3}{\partial t^2}\right]^2 = \mathrm{F}\left(y, t, f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial t\, \partial y}\right),$$

where $f = (f_1, f_2, f_3)\colon\ U \longrightarrow \mathbf{R}^3$, and F is a continuous map.

Evidently for each $(y, t) \in U$, $\Omega(y, t) \subset \mathbf{R}^3$ is the surface (a hyperbolic paraboloid),

$$\Omega(y, t) = \left\{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1 - x_2^2 + x_3^2 = \mathrm{F}\left(y, t, f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial t\, \partial y}\right)\right\}.$$

Remark. – Generically, a system $\mathcal{S}$ of m equations in q unknown functions $(q > m)$ defines a submanifold $\Omega(x) \subset \mathbf{R}^q$ of dimension $q - m$ for each $x \in U$.

The Convex Integration technique for solving the system $\mathcal{S}$ depends on simple geometric properties of the surfaces $\Omega(x)$.

Specifically, for each C^r map $h\colon\ U \longrightarrow \mathbf{R}^q$ and $x \in U$, one requires the following properties which are here informally stated:

P(1) : $\Omega_h(x) \equiv \Omega(x)$ is closed in $\mathbf{R}^q$ and is locally arc-wise connected.

P(2) : For each $z \in \Omega(x)$ the convex hull of every neighbourhood of z in $\Omega(x)$ has non empty interior in $\mathbf{R}^q$.

P(3) : There is a neighbourhood $N(\Omega)(x)$ of $\Omega(x)$ in $\mathbf{R}^q$ and a retraction $R\colon\ N(\Omega)(x) \longrightarrow \Omega(x)$.

P(4) : (Coherence properties):

The (closed) set $\Omega = \bigcup_{x,h} \Omega_h(x)$ is a topological fiberbundle, and Ω is a retract of a neighbourhood of Ω in the space of r-jets of C^r maps from U to $\mathbf{R}^q$.

Precise statements of the above properties, especially P(4), are provided in Appendix 1 and in § 3.

Formal solution to $\mathcal{S}$.

Assume now that the system $\mathcal{S}$ satisfies properties P(1) to P(4) above.

Let $h = (h_1, h_2, \ldots, h_q) : \mathrm{U} \longrightarrow \mathbf{R}^q$ be a C^r map and $\Phi = (\Phi_1, \ldots, \Phi_q) : \mathrm{U} \longrightarrow \mathbf{R}^q$ a continuous map such that the following conditions obtain:

F(1): For each $x \in \mathrm{U}$,

$$\mathrm{F}_i(x, \mathrm{D}^\beta h_j(x), \Phi_1(x), \Phi_2(x), \ldots, \Phi_q(x)) = 0,$$

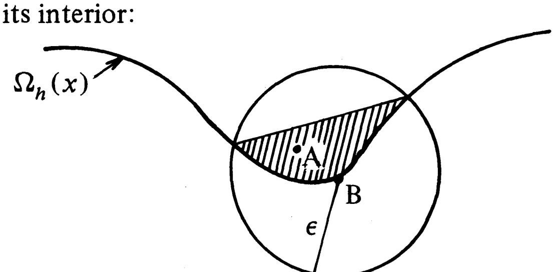
$1 \leqslant i \leqslant m$; $1 \leqslant j \leqslant q$; $|\beta| \leqslant r$, and $\mathrm{D}^\beta \neq \dfrac{\partial^r}{\partial t^r}$.

In other words, for each $x \in \mathrm{U}$, $\Phi(x)$ lies on the surface $\Omega_h(x) \subset \mathbf{R}^q$. There is, of course, no a priori relationship between the continuous maps $\Phi : \mathrm{U} \longrightarrow \mathbf{R}^q$, $\dfrac{\partial^r h}{\partial t^r} : \mathrm{U} \longrightarrow \mathbf{R}^q$.

Evidently, if $f : \mathrm{U} \longrightarrow \mathbf{R}^q$ is a C^r map which solves the system $\mathcal{S}$, then the maps f, $\Phi = \dfrac{\partial^r f}{\partial t^r}$ satisfy F(1). Thus F(1) is a (weak) necessary condition for the existence of a solution to the system $\mathcal{S}$.

F(2): For each $x \in \mathrm{U}$, the convex hull of the arc-component of $\Phi(x)$ in $\Omega_h(x)$ contains $\dfrac{\partial^r h}{\partial t^r}(x)$ in its interior. The following refined form of F(2) is also used.

F(2)(ϵ): Fix $\epsilon > 0$. For each $x \in \mathrm{U}$, the convex hull of the arc-component of $\Phi(x)$ in $\Omega_h(x) \cap \mathrm{D}(\Phi(x); \epsilon)$ contains $\dfrac{\partial^r h}{\partial t^r}(x)$ in its interior:



$$\mathrm{A} = \frac{\partial^r h}{\partial t^r}(x); \quad \mathrm{B} = \Phi(x).$$

The shaded area is the convex hull of the arc-component of $\Phi(x)$ in $\Omega_h(x) \cap D(\Phi(x);\epsilon)$, where $D(y;\delta)$ is the open ball in $\mathbf{R}^q$, centre y, radius $\delta > 0$.

DEFINITION. – *A formal solution to* $\mathcal{S}$ *is a pair of maps* (h, Φ) *which satisfies condition* F(1), F(2) *above.*

Example. – In the above example of a *2nd* order partial differential equation, properties P(1) to P(4) are satisfied. Note also that the convex hull of $\Omega(y,t)$ is $\mathbf{R}^3$, for all $(y,t) \in U$.

For any C^2 map $h : U \longrightarrow \mathbf{R}^3$, one easily constructs a continuous map $\Phi : U \longrightarrow \mathbf{R}^3$ such that (h, Φ) is a formal solution to the equation $\mathcal{S}$.

Statement of the Main Theorem.

The main result of this paper, theorem 2 (§ 3), states that if (h, Φ) is a formal solution to an r-th order system $\mathcal{S}$ which satisfies the geometrical conditions P(1) to P(4), then there is a C^r map $f : U \longrightarrow \mathbf{R}^q$ which solves $\mathcal{S}$, and is such that the maps $D^\beta f : U \longrightarrow \mathbf{R}^q$, $D^\beta h : U \longrightarrow \mathbf{R}^q$ satisfy given approximations for all β such that $|\beta| \leqslant r$, $D^\beta \neq \dfrac{\partial^r}{\partial t^r}$.

In particular, $f : U \longrightarrow \mathbf{R}^q$ is a C^{r-1} approximation to the map h in the Whitney fine C^{r-1} topology. Thus, in the above example, any C^2 map $h : U \longrightarrow \mathbf{R}^3$ can be approximated, in the fine C^1-topology, by a C^2 solution to the equation $\mathcal{S}$.

The proof of the theorem depends ultimately on the solution of the following C^r-approximation problem which is of independent interest.

A sketch of the proof of Theorem 2 follows the discussion of this approximation result.

The C^r*-Cube-Lemma* (cf. Appendix 5).

Let $[0,1]^n$ be the n-cube in $\mathbf{R}^n$. Coordinates in $\mathbf{R}^n$ are $(s_1, s_2, \ldots, s_{n-1}, t)$. Let $Q \subset [0,1]^n \times \mathbf{R}^q$ be an open subset. Let $Q(x) = Q \cap (x \times \mathbf{R}^q)$, $x \in [0,1]^n$; thus $Q(x) \subset x \times \mathbf{R}^q \equiv \mathbf{R}^q$. Let $\Phi : [0,1]^n \longrightarrow \mathbf{R}^q$ be a continuous map such that the graph of Φ is contained in Q.

Fix an integer $r \geqslant 1$.

Suppose $G : [0,1]^n \longrightarrow \mathbf{R}^q$ is a C^r map and $K_0 \subset [0,1]^n$ a closed set such that :

a) For each $x \in [0,1]^n$, the convex hull of the arc-component of $\Phi(x)$ in $Q(x)$ contains $\frac{\partial^r G}{\partial t^r}(x)$ in its interior.

b) There is a neighbourhood U of K_0 in K such that, for each $x \in U$, $\frac{\partial^r G}{\partial t^r} = \Phi(x)$.

The approximation problem, informally stated, is to find a C^r map $F : [0,1]^n \longrightarrow \mathbf{R}^q$ such that the graph of $\frac{\partial^r F}{\partial t^r} : [0,1]^n \longrightarrow \mathbf{R}^q$ is contained in Q and such that the maps F, G satisfy given approximations on all lower order derivatives. This problem was first formulated and solved by Gromov [1] in the case $r = 1$ (the one-dimensional lemma). In Appendix 5, we solve the approximation problem in a uniform manner to obtain the following precise result:

Fix $\epsilon > 0$.

There is a C^r map $F : [0,1]^n \longrightarrow \mathbf{R}^q$ and a neighbourhood V of K_0 in K, $\overline{V} \subset U$, such that:

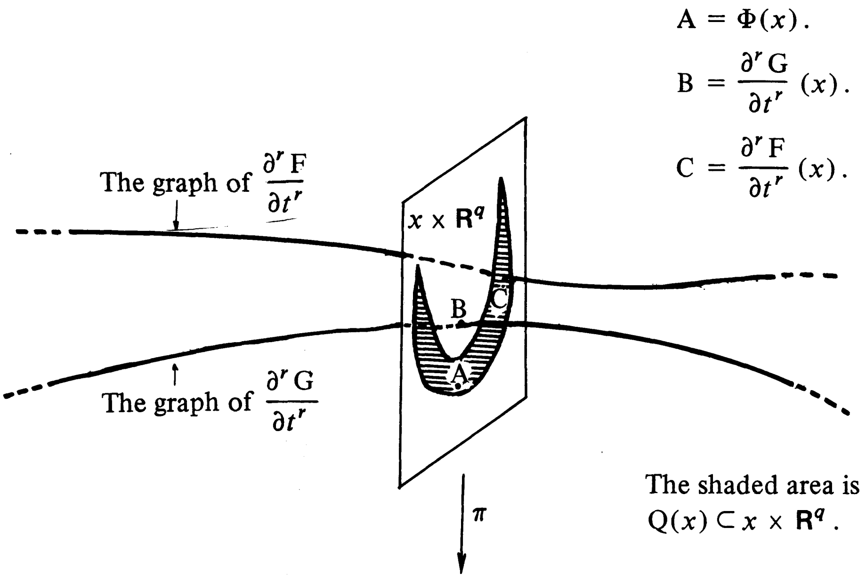
(i) the graph of $\frac{\partial^r F}{\partial t^r} : [0,1]^n \longrightarrow \mathbf{R}^q$ is contained in Q. More precisely, for each $x \in [0,1]^n$, $\frac{\partial^r F}{\partial t^r}(x)$ is contained in the arc-component of $\Phi(x)$ in $Q(x)$.

(ii) For each $(y, t) \in [0,1]^{n-1} \times [0,1] = [0,1]^n$, ($\| \ \|$ is the Euclidean norm on $\mathbf{R}^q$).

$\| D^\beta (F - G)(y, t) \| \leqslant \epsilon$, where D^β runs over all differential operators on functions of n-variables such that $|\beta| \leqslant r$ and D^β involves at most $(r - 1)$ differentiations in the t-coordinate.

(iii) For each $x \in V$, $F(x) = G(x)$.

This is represented schematically as follows.



Sketch of the Proof of Theorem 2.

One constructs a suitable sequence of metaneighbourhoods $(\mathrm{Met}_m\, \Omega)_{m \geqslant -1}$ of Ω in the space of r-jets such that $\bigcap_m \overline{\mathrm{Met}_m\, \Omega} = \Omega$.

A metaneighbourhood of Ω is an open set, whose closure contains Ω, and which satisfies certain convexity properties (precise definitions and properties of metaneighbourhoods are provided in Appendix 1). Let (h, Φ) be a formal solution to the closed condition Ω. One constructs a sequence of formal solutions $(h_m, \Phi_m)_{m \geqslant 0}$ to the closed condition Ω such that:

1) $j^r h_m(\mathrm{U}) \subset \mathrm{Met}_m\, \Omega$, $m = 0, 1, 2, 3, \ldots$.

That is, $h_m : \mathrm{U} \longrightarrow \mathbf{R}^q$ is a C^r map which solves the *open* condition $\mathrm{Met}_m\, \Omega$, $m \geqslant 0$.

2) For large m, $\Phi_m : \mathrm{U} \longrightarrow \mathbf{R}^q$ is a close approximation to the derivative map $\dfrac{\partial^r h_m}{\partial t^r} : \mathrm{U} \longrightarrow \mathbf{R}^q$.

3) The C^r maps h, $h_m : \mathrm{U} \longrightarrow \mathbf{R}^q$ satisfy given approximations on lower order derivatives, $m = 0, 1, 2, 3, \ldots$.

The limit map $f = \lim_{m \to \infty} h_m : \mathrm{U} \longrightarrow \mathbf{R}^q$ exists, is of class C^r and solves the closed condition Ω.

In this way, the closed condition Ω is solved by passing to a suitable sequence of solutions to "nearby" open conditions. The corresponding problem of existence of solutions to open conditions in jet spaces is solved in Theorem 1, § 2.

Brief Sketch of the proof of Theorem 1.

Suppose (h, Φ) is a formal solution to an open condition Y in the space of r-jets.

The construction of a C^r solution $f: \mathrm{U} \longrightarrow \mathbf{R}^q$ to Y proceeds as follows: One covers U with a suitable locally-finite cover by n-cubes. The construction of the map f proceeds by an infinite series of modification of maps (beginning with the map $h: \mathrm{U} \longrightarrow \mathbf{R}^q$), each modification taking place within one of the n-cubes of the cover.

Within each n-cube, the required modifications are made by applying the Local Extension Theorem (Appendix 6) which is a closely related refinement of the Cube lemma discussed above.

Remarks. – 1) Generically, for underdetermined systems (i.e. $q > m$), the geometric properties P(1) to P(4) associated to the system $\mathcal{S}$ are always satisfied, and one may construct formal solutions, at least locally in U. One therefore concludes that, generically, all underdetermined r-th order systems of partial differential equations are solvable, at least locally, $r \in \{1, 2, 3, \ldots\}$.

2) Suppose now the maps F_i are C^∞ (that is the system $\mathcal{S}$ is defined by C^∞ data). In this paper we are unable to obtain results about C^∞ solutions to the r-th order system $\mathcal{S}$. The question of C^∞ solutions in the presence of C^∞ data for the r-th order system $\mathcal{S}$ remains an outstanding problem.

3) Our results are existence theorems only. In general, for systems $\mathcal{S}$ which satisfy the geometrical conditions P(1) to P(4) above, there are infinitely many solutions to $\mathcal{S}$ which satisfy the given approximations on lower order derivatives.

Historical Remarks.

The method of Convex Integration is close in spirit to the work of J. Nash on the C^1 isometric embedding problem (Nash [8], and also Kuiper [7]). In order to realize a given continuous metric by a C^1 embedding in Euclidean space, Nash begins with a "short" embedding, and, working locally via an inductive procedure, proceeds to lengthen the embedding to obtain a new short embedding for which the induced metric is closer to the given metric. In the limit, a C^1 embedding is obtained which realizes the given metric.

The key step of "stretching" a given short embedding in a small neighbourhood is achieved by a one-dimensional lemma analogous to the one-dimensional Lemma (Appendix 3). This one-dimensional lemma was clarified by Kuiper who obtained also codimension $\geqslant 1$ results.

Indeed, the method of Convex Integration, detailed in this paper, may be applied to recover and extend Nash's C^1 isometric embedding theorem (cf. Gromov [3] for an announcement of his results on this problem).

Organization of the Paper.

In § 2, complete details are provided for the solution of open conditions in jet spaces which admit a formal solution. In § 3, the results on open conditions in the space of r-jets are applied to solve closed conditions in jet spaces which satisfy the geometrical conditions P(1) to P(4) above. In § 4, related results on mixed derivatives are discussed.

In order not to overburden the proofs in each section, the following steps have been taken:

1) Additional refinements to Theorems 1, 2 (§ 2, § 3) have been placed at the end of the sections, in a series of Complements.

2) The main technical results concerning metaneighbourhoods, Convex Hulls, the Cube Lemma, the Local Extension Theorem, etc., have been relegated to a series of Appendices in order to minimize the numerous digressions into technical details that would otherwise become necessary to discuss in § 2, § 3.

Finally I should like to thank M. Gromov for the many useful discussions we had together about Immersion Theory and Convex Integration Theory. In particular, I thank him for introducing me to his paper on C^1 systems [1], and for his encouragement to write

a complete account of the results in this paper, which were first announced in the preprint [9]. Independently, using a different approach to solving the Approximation Theorem (Appendix 3), he sketched a proof (unpublished) of the principal results of this paper. His solution to open and (generic) closed differential relations in jet spaces, as well as a wealth of other remarkable results, will appear in somewhat condensed form in a forthcoming book.

2. Solving Open Conditions in Jet Spaces.

In this section, it is proved that a formal solution to an open condition in the space of r-jets may be transformed into an exact solution (i.e., which does satisfy the given open condition), while controlling in a precise way all lower order derivatives.

Briefly, one performs an infinite series of transformations of a given formal solution, each transformation taking place within an n-cube (of a locally finite cover by n-cubes). Within each n-cube, the transformation is performed by employing the Local Extension theorem (Appendix 6); thus, the success of the process depends on a local construction designed to solve the approximation problem discussed in the introduction.

Let $\mathrm{U} \subset \mathbf{R}^n$ be open, $n \geqslant 1$.

$s\colon \mathrm{J}^r(\mathrm{U}, \mathbf{R}^q) \longrightarrow \mathrm{U}$ is the product bundle of r-jets of C^r maps from U to $\mathbf{R}^q$ (s is the source map), $r \in \{1, 2, 3, \ldots\}$. Thus if $f\colon \mathrm{U} \longrightarrow \mathbf{R}^q$ is a C^r map, $j^r f\colon \mathrm{U} \longrightarrow \mathrm{J}^r(\mathrm{U}, \mathbf{R}^q)$ is the continuous section, $j^r f(x) = (x, f(x) \ \ldots, \mathrm{D}^\alpha f(x), \ldots)$, where D^α runs over all differential operators on functions of n-variables such that $|\alpha| \leqslant r$.

In particular, $x \in \mathrm{U} \Longrightarrow s \circ j^r f(x) = x$.

Coordinates in $\mathbf{R}^n$ are denoted by $(s_1, s_2, \ldots, s_{n-1}, t)$. $\mathrm{J}^r(\mathrm{U}, \mathbf{R}^q) = \mathrm{J}^\perp(\mathrm{U}, \mathbf{R}^q) \times \mathbf{R}^q$ where the $\mathbf{R}^q$ factor in this product decomposition corresponds to the "pure" r-th order derivative $\dfrac{\partial^r}{\partial t^r}$.

One employs the notation, $j^r f(x) = \left(j^\perp f(x), \dfrac{\partial^r f}{\partial t^r}(x)\right)$, $x \in \mathrm{U}$, where $j^\perp f$ is the corresponding continuous section of the bundle,

$s: J^{\perp}(U, \mathbf{R}^q) \longrightarrow U$. Note that $\pi: J^r(U, \mathbf{R}^q) \longrightarrow J^{\perp}(U, \mathbf{R}^q)$ is a trivial Euclidean vector bundle, fiber $\mathbf{R}^q$, where π is the projection map onto the first factor.

Statement of the Problem.

Let $Y \subset J^r(U, \mathbf{R}^q)$ be an open set.

We seek reasonable conditions which ensure the existence of a C^r map $f: U \longrightarrow \mathbf{R}^q$ such that: $x \in U \Longrightarrow j^r f(x) \in Y$.

These conditions are stated in terms of the Euclidean vector bundle $\pi: J^r(U, \mathbf{R}^q) \longrightarrow J^{\perp}(U, \mathbf{R}^q)$.

The following notation is convenient (cf. also Appendix 1).

1) For each $z \in J^{\perp}(U, \mathbf{R}^q)$, $Y(z) = Y \cap \pi^{-1}(z) \subset z \times \mathbf{R}^q \equiv \mathbf{R}^q$.

2) Let $w \in Y(z)$.

(i) $\mathrm{Arc}\,(Y(z); w)$ is the arc component of w in $Y(z)$.

(ii) $\mathrm{Env}\,(Y(z); w)$ is the interior (in $\mathbf{R}^q$) of the convex hull of $\mathrm{Arc}\,(Y(z); w)$.

(iii) Let $\epsilon > 0$.

$\mathrm{Arc}_\epsilon(Y(z); w) = \mathrm{Arc}\,(Y(z) \cap D(w; \epsilon); w)$.

$\mathrm{Env}_\epsilon(Y(z); w) = \mathrm{Env}\,(Y(z) \cap D(w; \epsilon); w)$, where $D(w; \epsilon)$ is the open q-ball in $\mathbf{R}^q$ with center w, radius ϵ.

Formal Solution to Y.

A formal solution to the open condition $Y \subset J^r(U, \mathbf{R}^q)$ is a pair of maps (h, g) where $h: U \longrightarrow \mathbf{R}^q$ is a C^r map and $g: U \longrightarrow \mathbf{R}^q$ is a continuous map, such that:

F(1): For each $x \in U$, $(j^{\perp} h(x), g(x)) \in Y$.

F(2): For each $x \in U$, $\dfrac{\partial^r h}{\partial t^r}(x) \in \mathrm{Env}\,(Y(j^{\perp} h(x)); g(x))$.

Remarks. – Suppose $h: U \longrightarrow \mathbf{R}^q$ is a C^r map which solves the condition Y; i.e., $j^r h(U) \subset Y$. Then $\left(h, \dfrac{\partial^r h}{\partial t^r}\right)$ is a formal solution to Y. Thus the existence of a formal solution is a (weak) necessary condition for the existence of a C^r solution to Y.

Let $\sigma : U \longrightarrow (0, \infty)$ be a continuous map. The pair (h, g) is a formal solution to within tolerance σ if, in addition, $F(\sigma)$: for each $x \in U$, $\frac{\partial^r h}{\partial t^r}(x) \in \mathrm{Env}_{\sigma(x)}(Y(j^\perp h(x)); g(x))$.

The principal result of this section, Theorem 1, may be informally stated as follows (the precise statement of Theorem 1 is provided below):

Let (h, g) be a formal solution to an open condition $Y \subset J^r(U, \mathbf{R}^q)$. Then there is a C^r map $f: U \longrightarrow \mathbf{R}^q$ such that $j^r f(U) \subset Y$. The proof of Theorem 1 reduces to solving the following extension problem.

The Extension Problem.

Let A, B be closed sets in U, $A \subset B$. Let (h, g) be a formal solution to $Y \subset J^r(U, \mathbf{R}^q)$ such that $\frac{\partial^r h}{\partial t^r} = g$ in a neighbourhood of A in U (i.e., the C^r map $h: U \longrightarrow \mathbf{R}^q$ solves the open condition Y in a neighbourhood of A in U).

The formal solution (h, g) extends to a formal solution $(\bar{h}, \bar{g})$ to Y with respect to the closed sets $A \subset B$, if $\frac{\partial^r \bar{h}}{\partial t^r} = \bar{g}$ in a neighbourhood of B in U and if there is a neighbourhood of A in U on which $h = \bar{h}$; $g = \bar{g}$.

In particular, there is a neighbourhood N(A) of A in U such that: For each $x \in N(A)$, $\frac{\partial^r h}{\partial t^r}(x) = \frac{\partial^r \bar{h}}{\partial t^r}(x) = \bar{g}(x) = g(x)$.

The proof of Theorem 1 reduces to proving the extension problem in case $\overline{B = A}$ is contained in the interior of an n-cube (this is the main inductive step in the proof of Theorem 1). This step in the proof is solved by the Local Extension Theorem (Appendix 6). By employing a suitable locally finite cover of U by n-cubes, a proof of Theorem 1 is obtained.

Remarks. – Let $(\bar{h}, \bar{g})$ be a formal solution to Y which extends (h, g) with respect to the closed sets $A \subset B$.

One constructs $(\bar{h}, \bar{g})$ to satisfy the additional properties:

a) h, $\bar{h}$ satisfy given approximations on lower order derivatives.

b) There is a homotopy of formal solutions to Y, rel a neighbourhood of A in U, connecting (h, g) to $\bar{h}, \bar{g})$. (cf. Complement 2, below).

THEOREM 1. – *Let* $Y \subset J^r(U, \mathbf{R}^q)$ *be an open set.*

Let (h, g) *be a formal solution to* Y.

Let $T = \left\{ x \in U \;\middle|\; \dfrac{\partial^r h}{\partial t^r}(x) = g(x) \right\}$. *In particular, the* C^r *map* $h : U \longrightarrow \mathbf{R}^q$ *solves the open condition* Y *on the closed set* T.

Let $W(h)$ *be a neighbourhood of* $j^\perp h(U)$ *in* $J^\perp(U, \mathbf{R}^q)$.

There is a C^r *map* $f : U \longrightarrow \mathbf{R}^q$ *such that:*

a) $j^\perp f(U) \subset W(h)$.

b) *The r-jets* $j^r f$, $j^r h : U \longrightarrow J^r(U, \mathbf{R}^q)$ *coïncide in a neighbourhood of* T *in* U.

c) *For each* $x \in U$, $\dfrac{\partial^r f}{\partial t^r}(x) \in \mathrm{Arc}\,(Y(j^\perp f(x)) ; g(x))$.

(In particular $j^r f(U) \subset Y$).

Remarks. – It follows from conclusion a) that in particular, f is a C^{r-1} approximation to $h : U \longrightarrow \mathbf{R}^q$ in the fine C^{r-1} topology.

Proof. – Without loss of generality, one may assume that there is a neighbourhood N(T) of T in U such that: for each $x \in N(T)$, $g(x) = \dfrac{\partial^r h}{\partial t^r}(x)$.

Let $(A_p)_{p \geqslant 0}$, $(K_p)_{p \geqslant 0}$ be countable locally finite covers of U such that:

1) K_p is a translate in $\mathbf{R}^n$ of an n-cube $[0, a_p]^n$, $a_p > 0$, $p = 0, 1, 2, \ldots$.

2) A_p is a compact subset of int K_p, $p = 1, 2, 3, \ldots$

To prove the theorem, one constructs, inductively, an open neighbourhood M_m of $T \cup \left(\bigcup_{i=-1}^{m} A_i \right)$ in U, and a formal solution (f_m, g_m) to Y, $m = -1, 0, 1, 2, \ldots$, such that:

$1(m)$: $j^1(f_m)(\mathrm{U}) \subset \mathrm{W}(h)$.

$2(m)$: For each $x \in \mathrm{M}_m$, $g_m(x) = \dfrac{\partial^r f_m}{\partial t^r}(x)$.

In particular, for each $x \in \mathrm{M}_m$, $j^r(f_m)(x) \in \mathrm{Y}$. Thus the C^r map $f_m : \mathrm{U} \longrightarrow \mathbf{R}^q$ solves the condition Y over M_m.

$3(m)$: For each $x \in \mathrm{U}$, $g(x) \in \mathrm{Y}(j^1 f_m(x))$, and,

$$g_m(x) \in \mathrm{Arc}\,(\mathrm{Y}(j^1 f_m(x))\,;\, g(x)).$$

$4(m)$: For each $x \in \mathrm{U}$, $\dfrac{\partial^r f_m}{\partial t^r}(x) \in \mathrm{Env}\,(\mathrm{Y}(j^1 f_m(x))\,;\, g(x))$.

$5(m+1)$: (Coherence properties):

For each $x \in \mathrm{U} - \mathrm{K}_{m+1}$, $f_m(x) = f_{m+1}(x)$; $g_m(x) = g_{m+1}(x)$; $\mathrm{M}_m - \mathrm{K}_{m+1} = \mathrm{M}_m - \mathrm{K}_{m+1}$.

Remarks. – The inductively defined data begins with $m = -1$, where, $f_{-1} = h$; $g_{-1} = g$; $\mathrm{A}_{-1} = \phi$, $\mathrm{M}_{-1} = \mathrm{N}(\mathrm{T})$. Thus when $m = -1$, the above conditions obtain by the hypotheses of the theorem. Assuming the existence of neighbourhoods and formal solutions with the above properties, $m = -1, 0, 1, 2, 3, \ldots$, the theorem is proved as follows:

Let $f : \mathrm{U} \longrightarrow \mathbf{R}^q$ be the map $f(x) = \lim_{m \to \infty} f_m(x)$ (pointwise).

Since $(\mathrm{K}_p)_{p \geqslant 0}$ is a locally finite cover of U, it follows from the coherence properties, $5(m+1)$, that f is well-defined, is a C^r map, and that the conclusions of the theorem are all satisfied. The sequence of maps $(g_m : \mathrm{U} \longrightarrow \mathbf{R}^q)_{m \geqslant -1}$ is used for the refinements of Theorem 1 (cf. Complement 2 below.)

The construction of the inductively defined data proceeds as follows:

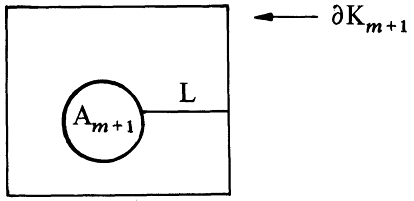
Assume that (f_m, g_m) and M_m have been constructed.

Working now within the n-cube $\mathrm{K}_{m+1} \equiv \mathrm{K}$, let

$$\mathrm{L} = \mathrm{K}_{m+1} \cap \Big(\mathrm{T} \cup \Big(\bigcup_{i=-1}^{m} \mathrm{A}_i\Big)\Big)\,; \quad \mathrm{M} = \mathrm{L} \cup \mathrm{A}_{m+1}.$$

Since $\mathrm{A}_{m+1} \subset \mathrm{int}\,\mathrm{K}_{m+1}$, it follows that $\overline{(\mathrm{M} - \mathrm{L})} \cap \partial\mathrm{K} = \phi$.

Let $\mathrm{N}(\mathrm{L}) = \mathrm{M}_m \cap \mathrm{K}_{m+1}$; thus $\mathrm{N}(\mathrm{L})$ is an open neighbourhood of L in K_{m+1}. Define $\mathrm{K}_0 = \mathrm{L} \cup \partial\mathrm{K}$ ($\mathrm{K}_0 \subset \mathrm{K}$ is a closed subset).



$M = L \cup A_{m+1}$.
$K_0 = L \cup \partial K$.

The n-cube $K_{m+1} \equiv K$

The formal solution (f_m, g_m) satisfies the following properties:

1) For each $x \in K_{m+1}$, $g_m(x) \in \mathrm{Arc}\,(Y(j^\perp f_m(x))\,;\, g(x))$

2) For each $x \in K_{m+1}$, $\dfrac{\partial^r f_m}{\partial t^r}(x) \in \mathrm{Env}\,(Y(j^\perp f_m(x))\,;\, g(x))$

3) For each $x \in N(L)$, $g_m(x) = \dfrac{\partial^r f_m}{\partial t^r}(x)$.

In what follows, one extends (f_m, g_m), within the cube K_{m+1}, to a formal solution $(\bar{f}_m, \bar{g}_m)$ to the open condition Y, with respect to the closed sets $L \subset M$. This is the basic induction step; the Local Extension Theorem (Appendix 6) is employed to complete this step. It is convenient to work over the submanifold $j^\perp f_m(K_{m+1}) \subset J^\perp(U, \mathbf{R}^q)$ in the Euclidean vector bundle $\pi : J^r(U, \mathbf{R}^q) \longrightarrow J^\perp(U, \mathbf{R}^q)$:

$$\begin{tikzcd}
K_{m+1} \times \mathbf{R}^q \arrow[r] \arrow[d, "\pi"] & J^r(U, \mathbf{R}^q) \arrow[d, "\pi"] \\
K_{m+1} \arrow[r, "j^\perp f_m"] & J^\perp(U, \mathbf{R}^q)
\end{tikzcd}$$

To this end, we pull-back via the embedding $j^\perp f_m$ to obtain the product Euclidean vector bundle, fiber $\mathbf{R}^q$ over K_{m+1}.

It is convenient to again denote by Y the (open) subset of $K_{m+1} \times \mathbf{R}^q$ obtained by pulling back $Y \subset J^r(U, \mathbf{R}^q)$ via the embedding $j^\perp f_m : K_{m+1} \longrightarrow J^\perp(U, \mathbf{R}^q)$.

Thus, for each $x \in K_{m+1}$, $Y(x) \equiv Y(j^\perp f_m(x)) \subset \mathbf{R}^q$.

To summarize, with respect to the product bundle $\pi : K_{m+1} \times \mathbf{R}^q \longrightarrow K_{m+1}$, the formal solution (f_m, g_m) satisfies the properties:

1) For each $x \in K_{m+1}$, $g_m(x) \in \mathrm{Arc}\,(Y(x), g(x))$.

2) For each $x \in K_{m+1}$, $\dfrac{\partial^r f_m}{\partial t^r}(x) \in \mathrm{Env}\,(Y(x), g_m(x))$.

3) For each $x \in N(L)$, $g_m(x) = \dfrac{\partial^r f_m}{\partial t^r}(x)$.

Applying now the Local Extension Theorem (Appendix 6), one obtains the following conclusion:

Let $\epsilon > 0$.

There is a C^r map $\overline{f}_m : K_{m+1} \longrightarrow \mathbf{R}^q$, a continuous map $\overline{g}_m : K_{m+1} \longrightarrow U$, a neighbourhood W of M in K_{m+1}, a neighbourhood V of K_0 in K_{m+1}, such that the following properties obtain:

a) $\sup_{x \in K_{m+1}} \|D^\alpha(\overline{f}_m - f_m)(x)\| \leqslant \epsilon$, where D^α runs over all differential operators on functions of n-variables, $|\alpha| \leqslant r$, and $D^\alpha \neq \dfrac{\partial^r}{\partial t^r}$.

b) For each $x \in K_{m+1}$,

$$\overline{g}_m(x) \in \operatorname{Arc}(Y(x); g_m(x)) \ (= \operatorname{Arc}(Y(x); g(x)).)$$

c) For each $x \in K_{m+1}$, $\dfrac{\partial^r \overline{f}_m}{\partial t^r}(x) \in \operatorname{Env}(Y(x); \overline{g}_m(x))$.

d) For each $x \in W$, $\overline{g}_m(x) = \dfrac{\partial^r \overline{f}_m}{\partial t^r}(x)$

e) For each $x \in V$, $\overline{f}_m(x) = f_m(x)$; $\overline{g}_m(x) = g_m(x)$.

f) $W \cap \partial K_{m+1} \subset N(L) \cap \partial K_{m+1}$.

Remarks. – 1) Employing e) above, $f_m = \overline{f}_m$; $g_m = \overline{g}_m$ in a neighbourhood of ∂K_{m+1} in K_{m+1}.

2) There is a neighbourhood U of L in K_{m+1} (in fact, $U = N(L) \cap V$) such that:

For each $x \in U$, $\dfrac{\partial^r \overline{f}_m}{\partial t^r}(x) = \dfrac{\partial^r f_m}{\partial t^r}(x) = \overline{g}_m(x) = g_m(x)$.

One concludes therefore, from the Local Extension theorem applied to the data (f_m, g_m) on K_{m+1}, that the formal solution $(\overline{f}_m, \overline{g}_m)$ extends (f_m, g_m) with respect to the closed sets $L \subset M$ in K_{m+1}.

Construction of the neighbourhood M_{m+1}, *and the maps* f_{m+1}, g_{m+1}.

Let $M_{m+1} = W \cup (M_m - K_{m+1})$.

Applying conclusion (f) above, it follows that M_{m+1} is a neighbourhood of $T \cup \left(\bigcup_{i=-1}^{m+1} A_i \right)$ in U.

Let $f_{m+1} : U \longrightarrow \mathbf{R}^q$ be the map,

$$f_{m+1}(x) = \begin{cases} f_m(x), & x \in U - K_{m+1} \\ \bar{f}_m(x), & x \in K_{m+1}. \end{cases}$$

Employing Remark 1, above, f_{m+1} is a C^r map.

Let $g_{m+1} : U \longrightarrow \mathbf{R}^q$ be the map,

$$g_{m+1}(x) = \begin{cases} g_m(x), & x \in U - K_{m+1} \\ \bar{g}_m(x), & x \in K_{m+1} \end{cases}$$

Employing Remark 1, above, g_{m+1} is a continuous map.

Furthermore for $\epsilon > 0$ sufficiently small, it is clear that the following properties obtain:

1) $j^\perp f_{m+1}(U) \subset W(h)$.

2) Employing the remark on compactness in Complement 2 of Appendix 5, it follows that for $\epsilon > 0$ sufficiently small (so that the maps $j^\perp f_m$, $j^\perp f_{m+1} : U \longrightarrow J^\perp(U, \mathbf{R}^q)$ are sufficiently close):

a) For each $x \in U$, $g(x) \in Y(j^\perp f_{m+1}(x))$;

$$g_{m+1}(x) \in \operatorname{Arc}(Y(j^\perp f_{m+1}(x)) ; g(x)).$$

b) For each $x \in U$, $\dfrac{\partial^r f_{m+1}}{\partial t^r}(x) \in \operatorname{Env}(Y(j^\perp f_{m+1}(x)) ; g(x))$.

3) For each $x \in M_{m+1}$, $\dfrac{\partial^r f_{m+1}}{\partial t^r}(x) = g_{m+1}(x)$.

In particular, (f_{m+1}, g_{m+1}) is a formal solution to Y which extends formal solution (f_m, g_m) to Y with respect to the closed sets, $T \cup \left(\bigcup_{i=-1}^{m} A_i \right) \subset T \cup \left(\bigcup_{i=-1}^{m+1} A_i \right)$.

Evidently properties $1(m+1)$ to $4(m+1)$ are satisfied by the formal solution (f_{m+1}, g_{m+1}) as well as the coherence properties $5(m+1)$.

This completes the inductive step and hence the proof of Theorem 1 is complete.

Complement 1.

Let $Y \subset J^r(U, \mathbf{R}^q)$ be an open set.

Let $\sigma : U \longrightarrow (0, \infty)$ be a continuous map.

Suppose $h : U \longrightarrow \mathbf{R}^q$ is a C^r map, $g : U \longrightarrow \mathbf{R}^q$ a continuous map such that (h, g) is a formal solution to Y within the tolerance σ (cf. the definition of formal solution in this section).

In this case, the conclusions of Theorem 1 may be strengthened to include the following (cf. Complement 3, Appendix 6):

For each $x \in U$, $\dfrac{\partial^r f}{\partial t^r}(x) \in \mathrm{Arc}_{\sigma(x)}(Y(j^\perp f(x)) ; g(x))$.

(This strengthened conclusion is used in § 3 to prove convergence results.)

Complement 2.

Since $\{K_p\}_{p \geq 0}$ is a locally finite cover, one employs Complement 1 to the Local Extension Theorem (Appendix 6) to prove that there is a homotopy of formal solutions, rel T, to the open condition $Y \subset J^r(U, \mathbf{R}^q)$ which connects (h, g) to $\left(f, \dfrac{\partial^r f}{\partial t^r}\right)$.

Complement 3.

Suppose in addition, $h : U \longrightarrow \mathbf{R}^q$ is a C^∞ map. Employing Complement 4 to the Local Extension Theorem (Appendix 6) one easily proves the following additional conclusions to Theorem 1:

Fix an integer $s \geq r$ and a neighbourhood W of the constant map equal to zero, $0 : U \longrightarrow \mathbf{R}^q$, in the space of continuous maps from U to $\mathbf{R}^q$ in the fine C^0-topology.

Then, $D^\alpha(f - h) \in W$, for all differential operators D^α on functions of n-variables $|\alpha| \leq s$, and such that D^α involves differentiation at most $(r - 1)$-times of the t-variable.

3. Solution to Closed Conditions in Jet Spaces.

In this section the main result (theorem 2) of this paper is proved. Suppose $\Omega \subset J^r(U, \mathbf{R}^q)$ is a closed condition in the space of r-jets which satisfies the geometrical properties G(1) to G(4) stated below. If there is a formal solution to Ω, then by theorem 2, there is an exact solution to Ω which satisfies given approximations on all lower order derivatives.

Briefly, one constructs a suitable sequence of metaneighbourhoods $(\mathrm{Met}_m\,\Omega)_{m\geqslant -1}$ of Ω in $\mathrm{J}^r(\mathrm{U},\mathbf{R}^q)$.

For each m, $\mathrm{Met}_m\,\Omega$ is open in $\mathrm{J}^r(\mathrm{U},\mathbf{R}^q)$ and $\bigcap_m \overline{\mathrm{Met}_m\,\Omega} = \Omega$. One constructs a sequence (S) of formal solutions $(h_m, g_m)_{m\geqslant -1}$ to the closed condition Ω such that:

1) For each m, $j^r h_m(\mathrm{U}) \subset \mathrm{Met}_m\,\Omega$; h_m, h satisfy given approximations on lower order derivatives.

2) For each m, the metaneighbourhood $\mathrm{Met}_{m+1}\,\Omega$ is so chosen so that derivative maps $\dfrac{\partial^r h_m}{\partial t^r}, \dfrac{\partial^r h_{m+1}}{\partial t^r} : \mathrm{U} \longrightarrow \mathbf{R}^q$ satisfy given estimates.

These estimates ensure the uniform convergence of the sequence of derivative maps $\left(\dfrac{\partial^r h_m}{\partial t^r} : \mathrm{U} \longrightarrow \mathbf{R}^q\right)_{m\geqslant -1}$.

Let $\mathrm{U} \subset \mathbf{R}^n$ be open $n \geqslant 1$.

Coordinates in $\mathbf{R}^n$ are denoted by $(s_1, s_2, \ldots, s_{n-1}, t)$.

Recall the product bundle $s\colon \mathrm{J}^r(\mathrm{U},\mathbf{R}^q) \longrightarrow \mathrm{U}$ of r-jets of C^r maps from U to $\mathbf{R}^q$, and also the product Euclidean vector bundle, fiber $\mathbf{R}^q$, $\pi\colon \mathrm{J}^r(\mathrm{U},\mathbf{R}^q) \longrightarrow \mathrm{J}^\perp(\mathrm{U},\mathbf{R}^q)$ (cf. § 1 for complete definitions of these bundles). In particular, if $f\colon \mathrm{U} \longrightarrow \mathbf{R}^q$ is a C^r map, then $j^r f\colon \mathrm{U} \longrightarrow \mathrm{J}^r(\mathrm{U},\mathbf{R}^q)$ is the continuous section,

$$j^r f(x) = \left(j^\perp f(x), \frac{\partial^r f}{\partial t^r}(x)\right) \in \mathrm{J}^\perp(\mathrm{U},\mathbf{R}^q) \times \mathbf{R}^q = \mathrm{J}^r(\mathrm{U},\mathbf{R}^q).$$

Statement of the Problem.

Let $\Omega \subset \mathrm{J}^r(\mathrm{U},\mathbf{R}^q)$ be a closed subset. We seek reasonable conditions which ensure the existence of a C^r map $f\colon \mathrm{U} \longrightarrow \mathbf{R}^q$ which solves the condition Ω; ie., $j^r f(\mathrm{U}) \subset \Omega$.

As in the case of open conditions in $\mathrm{J}^r(\mathrm{U},\mathbf{R}^q)$ treated in § 1, these conditions are stated in terms of the Euclidean vector bundle $\pi\colon \mathrm{J}^r(\mathrm{U},\mathbf{R}^q) \longrightarrow \mathrm{J}^\perp(\mathrm{U},\mathbf{R}^q)$.

DEFINITION. — $Q \subset \mathbf{R}^q$ *is nowhere flat if the intersection of* Q *with each hyperplane in* $\mathbf{R}^q$ *is a nowhere dense subspace of* Q. *Thus, for example, the unitsphere* S^n *in Euclidean space* $\mathbf{R}^{n+1}$ *is nowhere flat.*

In this section, we suppose $\Omega \subset J^r(U, \mathbf{R}^q)$ satisfies the following geometrical properties:

G(1): The restriction map, $\pi : \Omega \longrightarrow J^\perp(U, \mathbf{R}^q)$ is a topological fiber bundle with locally arc-wise connected fibers.

G(2): For each $z \in J^\perp(U, \mathbf{R}^q)$ the fiber $\Omega(z) = \Omega \cap \pi^{-1}(z)$ is nowhere flat in $z \times \mathbf{R}^q \equiv \mathbf{R}^q$.

G(3): There is a neighbourhood $N(\Omega)$ of Ω in $J^r(U, \mathbf{R}^q)$ and a retraction $R : N(\Omega) \longrightarrow \Omega$ which respects the fibers:

$$\pi \circ R = \pi : N(\Omega) \longrightarrow J^\perp(U, \mathbf{R}^q).$$

Remarks. — Let $\mathfrak{S}$ be an r-th order system of m partial differential equations in q unknown functions defined on an open set $U \subset \mathbf{R}^n$.

Let $h : U \longrightarrow \mathbf{R}^q$ be a C^r map. In the introduction, a surface $\Omega_h(x) \subset \mathbf{R}^q$ was defined by the system $\mathfrak{S}$, for each $x \in U$.

The geometry of these surfaces is now explained within the context of the product bundle $\pi : J^r(U, \mathbf{R}^q) \longrightarrow J^\perp(U, \mathbf{R}^q)$.

For each $x \in U$, let $z = j^\perp h(x) \in J^\perp(U, \mathbf{R}^q)$. Evidently $\Omega_h(x)$ is a surface in the fiber $z \times \mathbf{R}^q \equiv \mathbf{R}^q$ of this bundle.

Let $\Omega = \bigcup_{h,x} \Omega_h(x) \subset J^r(U, \mathbf{R}^q)$.

The properties G(1), G(2), G(3), above, make precise the informally presented properties P(1) to P(4) in the introduction.

DEFINITION. — 1) *A subset* $Q \subset \mathbf{R}^q$ *in locally ample if for each* $x \in Q$, *and for each* $\epsilon > 0$, *the convex hull of the arc-component of* x *in* $Q \cap D(x ; \epsilon)$ *contains* x *in its interior, where,* $D(y ; \delta)$ *is the open q-ball in* $\mathbf{R}^q$ *center* y *and radius* $\delta > 0$.

2) *Let* $\Omega \subset J^r(U, \mathbf{R}^q)$.

Ω *is locally ample if for each* $z \in J^\perp(U, \mathbf{R}^q)$, *the fiber* $\Omega(z) = \Omega \cap (z \times \mathbf{R}^q) \subset z \times \mathbf{R}^q \equiv \mathbf{R}^q$ *is locally ample.*

The main result of the paper is as follows:

THEOREM 2. – *Let* $h : \mathrm{U} \longrightarrow \mathbf{R}^q$ *be a* C^r *map, and* g : $\mathrm{U} \longrightarrow \mathbf{R}^q$ *a continuous map such that the following conditions obtain:*

(i) *For each* $x \in \mathrm{U}$, $(j^{\perp} h(x), g(x)) \in \Omega$.

(ii) *For each* $x \in \mathrm{U}$, $\dfrac{\partial^r h}{\partial t^r}(x) \in \mathrm{Env}\,(\Omega(j^{\perp} h(x)) ; g(x))$. *Thus* (h, g) *is a formal solution to the closed condition* $\Omega \subset \mathrm{J}^r(\mathrm{U}, \mathbf{R}^q)$.

(iii) *In case* $\Omega \subset \mathrm{J}^r(\mathrm{U}, \mathbf{R}^q)$ *is locally ample, let* $\mathrm{T} \subset \mathrm{U}$ *be the closed set* $\left\{ x \in \mathrm{U} \mid g(x) = \dfrac{\partial^r h}{\partial t^r}(x) \right\}$. *(In particular,* h *solves the condition along the closed set* T *in* U).

Let $\mathrm{W}(h)$ *be a neighbourhood of* $j^{\perp} h(\mathrm{U}) \subset \mathrm{J}^{\perp}(\mathrm{U}, \mathbf{R})^q$.

There is a C^r *map* $f : \mathrm{U} \longrightarrow \mathbf{R}^q$ *such that the following properties obtain:*

1) *For each* $x \in \mathrm{U}$, $\dfrac{\partial^r f}{\partial t^r}(x) \in \mathrm{Arc}\,(\Omega(j^{\perp} f(x)) ; g(x))$. *(In particular* f *solves the condition* Ω ; *ie.,* $j^r f(\mathrm{U}) \subset \Omega$).

2) $j^{\perp} f(\mathrm{U}) \subset \mathrm{W}(h)$.

3) *In case* $\Omega \subset \mathrm{J}^r(\mathrm{U}, \mathbf{R}^q)$ *is locally ample, then for each* $x \in \mathrm{T}$, $j^r f(x) = j^r h(x)$.

Remarks. – 1) Theorem 2 states that a formal solution to $\Omega \subset \mathrm{J}^r(\mathrm{U}, \mathbf{R}^q)$ can be transformed into a C^r solution to Ω which satisfies given approximations on lower order derivatives. As explained in Complement 1 below, this transformation is performed by a homotopy of formal solutions to Ω.

2) It follows from conclusion (2) above that, in particular, f is a C^{r-1} approximation to h in the fine C^{r-1} topology.

Proof of Theorem 2. – One fixes a convergent series of positive numbers $\sum_{m=0}^{\infty} a_m$.

Recall the Euclidean vector bundle $\pi : \mathrm{J}^r(\mathrm{U}, \mathbf{R}^q) \longrightarrow \mathrm{J}^{\perp}(\mathrm{U}, \mathbf{R}^q)$ with fiber $\mathbf{R}^q$. Let d be a metric on $\mathrm{J}^{\perp}(\mathrm{U}, \mathbf{R}^q)$.

Applying Proposition 1.1 (Appendix 1) to this bundle, there is a continuous map $\mu \equiv \mu(m) : \Omega \longrightarrow (0, \infty)$, $m = -1, 0, 1, 2, \ldots$

such that: Let $\mathrm{Met}_m \Omega = \mathrm{Met}_\mu \Omega$, the metaneighbourhood of Ω in $J^r(U, \mathbf{R}^q)$ associated to the map μ.

M(1): $\mu \leqslant \inf\{a_m, a_{m+1}\}$; $\mathrm{Met}_m \Omega \subset N(\Omega)$, if $m \geqslant 0$.

$\mu \leqslant a_0$; $\mathrm{Met}_{-1} \Omega \subset N(\Omega)$, if $m = -1$.

M(2): Let $z \in J^\perp(U, \mathbf{R}^q)$; $w \in \Omega(z)$; $v \in \mathrm{Env}_{\mu(w)}(\Omega(z); w)$.

Then $v \in \mathrm{Env}_{a_{m+1}}(\Omega(z); R(v))$;

(the definitions and properties of metaneighbourhoods are provided in Appendix 1).

Evidently, $\bigcap_m \overline{\mathrm{Met}_m \Omega} = \Omega$, and $\mathrm{Met}_m \Omega$ is an open subset of $J^r(U, \mathbf{R}^q)$, $m = -1, 0, 1, 2, 3, \dots$.

Let $W_1(h)$ be a neighbourhood if $j^\perp h(U)$ in $J^\perp(U, \mathbf{R}^q)$ such that $\overline{W_1(h)} \subset W(h)$.

In order to prove theorem 2, one constructs the following sequences (S) of maps:

1) A sequence of C^r maps $(h_m : U \longrightarrow \mathbf{R}^q)_{m \geqslant 0}$.

2) A sequence of continuous maps $(g_m : U \longrightarrow \mathbf{R}^q)_{m \geqslant 0}$.

These sequences (S) are required to satisfy the following properties (recall the retraction map $R: N(\Omega) \longrightarrow \Omega$):

$1(m)$: $j^r h_m(U) \subset \mathrm{Met}_m \Omega (\subset N(\Omega))$;

$j^\perp h_m(U) \subset W_1(h)$.

$2(m)$: For each $x \in U$, $g_m(x) = R(j^\perp h_m(x)) \in \Omega(j^\perp h_m(x))$.

$3(m)$: For each $x \in U$, $\dfrac{\partial^r h_m}{\partial t^r}(x) \in \mathrm{Env}_{a_m}(\Omega(j^\perp h_m(x)); g_m(x))$.

$4(m)$: In case Ω is locally ample :

For each $x \in T$, $j^r h_m(x) = j^r h(x)$.

$5(m)$: For each $x \in U$, $d(j^\perp h_m(x), j^\perp h_{m+1}(x)) \leqslant a_m$ where d is the metric on $J^\perp(U, \mathbf{R}^q)$.

$6(m)$: For each $x \in U$, $g_{m+1}(x) \in D(g_m(x); 2a_m + 2a_{m+1})$, where $D(y; \delta)$ is the open q ball in $\mathbf{R}^q$, centre y, radius $\delta > 0$.

Assuming that the above sequences (S) have been constructed, theorem 2 is proved as follows:

a) From properties $6(m)$, $3(m)$ it is evident that the sequences of continuous maps $\left(\frac{\partial^r h_m}{\partial t^r} : \mathrm{U} \longrightarrow \mathbf{R}^q\right)_{m \geqslant 0}$, $(g_m : \mathrm{U} \longrightarrow \mathbf{R}^q)_{m \geqslant 0}$ converge uniformly (in the topology of uniform convergence on compact sets) to a continuous map from U to $\mathbf{R}^q$, and that

$$\lim_{m \to \infty} g_m = \lim_{m \to \infty} \frac{\partial^r h_m}{\partial t^r} : \mathrm{U} \longrightarrow \mathbf{R}^q .$$

b) From property $5(m)$ and a) above, one easily concludes that $f = \lim_{m \to \infty} h_m$ is a C^r map from U to $\mathbf{R}^q$.

In particular, $\frac{\partial^r f}{\partial t^r} = \lim_{m \to \infty} \frac{\partial^r h_m}{\partial t^r} = \lim_{m \to \infty} g_m : \mathrm{U} \longrightarrow \mathbf{R}^q$.

c) Since $\Omega \subset \mathrm{J}^r(\mathrm{U}, \mathbf{R}^q)$ is closed it follows from properties $2(m)$ and b) above that f solves the condition Ω; ie., $j^r(\mathrm{U}) \subset \Omega$.

d) In case Ω is locally ample, if follows from $4(m)$ above that for all $x \in \mathrm{T}$, $j^r f(x) = j^r h(x)$.

e) Evidently, from property $1(m)$ above, $j^{\perp} f(\mathrm{U}) \subset \mathrm{W}(h)$.

Thus the map $f = \lim_{m \to \infty} h_m : \mathrm{U} \longrightarrow \mathbf{R}^q$ satisfies all the conclusions of Theorem 2. It therefore remains to construct the sequence (S).

Remark. – The sequences (S) are defined inductively. However one cannot set $h_0 = h$ since, in general $j^r(h)(\mathrm{U}) \not\subset \mathrm{N}(\Omega)$.

That is, condition $1(0)$ is violated, in general, if $h_0 = h$. In what follows the first term, h_0, of the sequence $(h_m : \mathrm{U} \longrightarrow \mathbf{R}^q)_{m \geqslant 0}$ is constructed inductively along with the remaining terms.

Set $h_{-1} = h$, $g_{-1} = g$.

Suppose that C^r maps $h_{-1}, h_0, \ldots, h_m : \mathrm{U} \longrightarrow \mathbf{R}^q$ and continuous maps $g_{-1}, g_0, g_1, \ldots, g_m : \mathrm{U} \longrightarrow \mathbf{R}^q$ have been constructed and satisfy properties $1(k)$ to $4(k)$, $k = 0, 1, 2, \ldots, m$ (if $m \geqslant 0$) and properties $5(k)$, $6(k)$ $k = 0, 1, 2, 3, \ldots m - 1$ (if $m \geqslant 1$).

Set $\mathrm{Y} = \mathrm{Met}_m(\Omega) \subset \mathrm{J}^r(\mathrm{U}, \mathbf{R}^q)$.

Remark. – The pair of maps (h_m, g_m) is a formal solution to the closed condition Ω but, in general, (h_m, g_m) is not a formal solution to the open condition $\mathrm{Y} \subset \mathrm{J}^r(\mathrm{U}, \mathbf{R}^q)$ $[\Omega \subset \overline{\mathrm{Met}_m(\Omega)}]$.

One now perturbs the map g_m, in a controlled way, to achieve a formal solution to Y.

To this end, applying Corollaries 1.7.1, 1.7.2 (Appendix 1), there is a continuous map $g' : U \longrightarrow \mathbf{R}^q$ such that:

1) For each $x \in U$, $g'(x) \in D(g_m(x) ; a_m) \subset \mathbf{R}^q$, if $m \geqslant 0$ ... E(1)

2) For each $x \in U$, $(j^\perp h_m(x), g'(x)) \in Y$. ie.,
$$g'(x) \in Y(j^\perp h_m(x)) \subset \mathbf{R}^q .$$

3) For each $x \in U$,

$$\frac{\partial^r h_m}{\partial t^r}(x) \in \mathrm{Env}_{2a_m}(Y(j^\perp h_m(x)) ; g'(x)), \text{ if } m \geqslant 0 \qquad \ldots E(2)$$

$$\frac{\partial^r h_m}{\partial t^r}(x) \in \mathrm{Env}(Y(j^\perp h_m(x)) ; g'(x)), \text{ if } m = -1 .$$

4) In case Ω is locally ample, $g' = \dfrac{\partial^r h_m}{\partial t^r}$ in a neighbourhood of T in U.

From (3), above, the pair of maps (h_m, g') is a formal solution to the open condition $Y \subset J^r(U, \mathbf{R}^q)$ (to within the tolerance $2a_m$, if $m \geqslant 0$).

Applying theorem 1, § 2 (Complement 1, if $m \geqslant 0$) one concludes that there is a C^r map $h_{m+1} : U \longrightarrow \mathbf{R}^q$ such that:

(i) $j^\perp h_{m+1}(U) \subset W_1(h)$. For each $x \in U$,

$d(j^\perp h_m(x), j^\perp h_{m+1}(x)) \leqslant a_m$, if $m \geqslant 0$.

(ii) For each $x \in U$,

$$\frac{\partial^r h_{m+1}}{\partial t^r}(x) \in \mathrm{Arc}_{2a_m}(Y(j^\perp h_{m+1}(x)) ; g'(x)), \text{ if } m \geqslant 0 \qquad \ldots E(3)$$

$$\frac{\partial^r h_{m+1}}{\partial t^r}(x) \in \mathrm{Arc}(Y(j^\perp h_{m+1}(x)) ; g'(x)), \text{ if } m = -1 .$$

In particular the C^r map $h_{m+1} : U \longrightarrow \mathbf{R}^q$ solves the open condition Y.

(iii) In case Ω is locally ample, $j^r h_{m+1} = j^r h$ in a neighbourhood of T in U.

Let $g_{m+1} : \mathrm{U} \longrightarrow \mathbf{R}^q$ be the continuous map defined by retracting onto Ω : For each $x \in \mathrm{U}$,

$$g_{m+1}(x) = \mathrm{R}(j^r h_{m+1}(x)) \in \Omega(j^\perp h_{m+1}(x)).$$

Employing property M(2) for the metaneighbourhood $\mathrm{Met}_m \Omega$, it follows that property $3(m+1)$ is satisfied and that,

(iv) For each $x \in \mathrm{U}$, $\dfrac{\partial^r h_{m+1}}{\partial t^r}(x) \in \mathrm{D}(g_{m+1} ; a_{m+1}) \subset \mathbf{R}^q$ ... E(4)

The pair of maps (h_{m+1}, g_{m+1}) evidently satisfy all the properties $1(m+1)$ to $4(m+1)$ and property $5(m)$ (in case $m \geqslant 0$) for the sequences (S).

In case $m \geqslant 0$, property $6(m)$ follows from the estimates E(1), E(3), E(4), established above. More precisely :

For each $x \in \mathrm{U}$, $g'(x) \in \mathrm{D}(g_m(x) ; a_{m+1})$, if $m \geqslant 0$.

$$\frac{\partial^r h_{m+1}}{\partial t^r}(x) \in \mathrm{D}(g'(x) \quad 2a_m), \text{ if } m \geqslant 0.$$

$$\frac{\partial^r h_{m+1}}{\partial t^r}(x) \in \mathrm{D}(g_{m+1}(x) ;\ a_{m+1}).$$

Consequently, if $m \geqslant 0$:

For each $x \in \mathrm{U}$, $g_{m+1}(x) \in \mathrm{D}(g_m(x) ;\ 2a_m + 2a_{m+1})$.

This completes the inductive step in the construction of the sequences (S) (including the construction of the first terms h_0, g_0). Hence the proof of Theorem 2 is complete.

Complément 1.

Employing Complement 2 of Theorem 1, § 2, one may construct a homotopy of formal solutions, rel T, to the closed condition $\Omega \subset \mathrm{J}^r(\mathrm{U}, \mathbf{R}^q)$, which connects (h, g) to $\left(f, \dfrac{\partial^r f}{\partial t^r}\right)$, where $f = \lim_{m \to \infty} h_m : \mathrm{U} \longrightarrow \mathrm{R}^q$ is the solution to Ω constructed in theorem 2 above.

Suppose Ω is locally ample. It follows from the above complement, and standard arguments, that there is a weak homotopy equivalence between the space of formal solutions to Ω and the space of exact (holonomic) solutions to Ω.

Complément 2.

Suppose in addition that $h: \mathrm{U} \longrightarrow \mathbf{R}^q$ is a C^∞ map. Employing Complement 3 to theorem 1, § 2, the following additional conclusion is easily proved: The map $f: \mathrm{U} \longrightarrow \mathbf{R}^q$ in the conclusion of theorem 2 is of class $\mathrm{C}^{\infty, r}$. That is, f is a C^r map and $\mathrm{D}^\alpha f$: $\mathrm{U} \longrightarrow \mathbf{R}^q$. exists and is continuous map for all differential operators D^α on functions of n-variables which involves differentiation in the t-variable at most $(r-1)$-times.

Note, however, by Remark 2 in the Introduction, § 1, it is not known whether one can arrange for $f: \mathrm{U} \longrightarrow \mathbf{R}^q$ to be of class C^{r+1} if Ω is also a C^∞ submanifold of $\mathrm{J}^r(\mathrm{U}, \mathbf{R}^q)$.

4. Results on Mixed Derivatives.

In the previous sections open and closed conditions in jet spaces have been solved when they admit formal solutions defined with respect to the "pure" derivative $\dfrac{\partial^r}{\partial t^r}\cdot$

In this section, simple algebraic transformations are introduced to partially extend our results to open and closed conditions in jet spaces which admit formal solutions defined with respect to a fixed differential operator on functions on n-variables $\mathrm{D}^\alpha \equiv \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$. The results are less comprehensive in that one cannot control *all* the other derivatives D^β, $|\beta| \leqslant |\alpha|$ and $\beta \neq \alpha$. Indeed, one should not expect to be able to do this; thus, for example, employing elementary inequalities in analysis, one cannot make large perturbations in the derivative $\dfrac{\partial^2 f}{\partial x\, \partial y}$ while keeping the perturbations of $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$ arbitrarily small.

Nevertheless, as will be seen below, all derivatives defined by operators D^β, where $|\beta| < |\alpha|$, may be suitably controlled.

One fixes a differential operator $\mathrm{D}^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_p^{\alpha_p}}$ on functions of n-variables $(n \geqslant p)$ such that $\alpha_i > 0$, $1 \leqslant i \leqslant p$. Let

J_α be the set of all n-tuples $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ such that either,

a) $|\beta| < |\alpha|$ $\left(\text{if } \gamma = (\gamma_1, \gamma_2, \dots, \gamma_n), \text{ then } |\gamma| = \sum_{i=1}^{n} \gamma_i\right)$,

or

b) $|\beta| = |\alpha|$, $\sum_{i=1}^{p} \beta_i < |\alpha|$, and $\beta_i \leqslant \alpha_i$ $i = 1, 2, 3, \dots, p$.

Example. – $D^\alpha = \dfrac{\partial^3}{\partial x^2\, \partial y}\cdot$

Let D^β be a differential operator on functions of three variables such that $|\beta| \leqslant 3$, then $\beta \notin J_\alpha$ if and only if,

$$D^\beta \in \left\{ \frac{\partial^3}{\partial x^3}, \frac{\partial^3}{\partial y^3}, \frac{\partial^3}{\partial y^2\, \partial z}, \frac{\partial^3}{\partial x\, \partial y^2} \right\}.$$

In particular, if $|\beta| \leqslant 2$, then $\beta \in J_\alpha$.

Remark. – Let $D^\alpha = \dfrac{\partial^r}{\partial t^r}$ (the case of a "pure" r-th order derivative). Then $\beta \in J_\alpha$ if and only if $|\beta| \leqslant r$ and D^β involves differentiation in the t-variable with multiplicity q, where $q < r$. This is exactly the case considered in Theorems 1, 2 of this paper.

The following observation explains the above definition of J_α.

Observation. – Let A be the invertible linear map on $\mathbf{R}^n$,

$$t = x_1 + x_2 \dots + x_p \qquad u_j = x_j, \quad 2 \leqslant j \leqslant n.$$

Let $\beta \in J_\alpha$, and $f\colon \mathbf{R}^n \longrightarrow \mathbf{R}^q$ a C^r map ($r = |\alpha|$). Then the following conditions obtain:

1) $D^\beta f$ is a linear combination of terms of the form:

$$\frac{\partial^{|\beta|}(f \circ A^{-1})}{\partial t^{a_1}\, \partial u_2^{a_2} \dots \partial u_n^{a_n}}, \quad \text{where } a_1 < r.$$

2) $D^\alpha f = \dfrac{\partial^r (f \circ A^{-1})}{\partial t^r} +$ a linear combination of terms as in (1).

It follows from the above observation that the techniques of this paper carry over to mixed derivatives provided one is required to control only those derivatives D^β, where $\beta \in J_\alpha$.

Rather than state precise theorems for open and closed conditions corresponding to mixed derivatives, we illustrate with the following example:

Let $D^\alpha = \dfrac{\partial^3}{\partial x^2\, \partial y}$.

Consider the following system $\mathcal{S}$ consisting of one partial differential equation for a function of three variables defined on an open set $U \subset \mathbf{R}^3$, $f: U \longrightarrow \mathbf{R}^3$ $(f = (f_1, f_2, f_3))$.

$$\mathcal{S} \equiv \frac{\partial^3 f_1}{\partial x^2\, \partial y} - \left[\frac{\partial^3 f_2}{\partial x^2\, \partial y}\right]^2 + \left[\frac{\partial^3 f_3}{\partial x^2\, \partial y}\right]^2$$

$$= F\left(j^2 f, \frac{\partial^3 f}{\partial x\, \partial y\, \partial z}, \frac{\partial^3 f}{\partial x^2\, \partial z}, \frac{\partial^3 f}{\partial y\, \partial z^2}, \frac{\partial^3 f}{\partial z^2\, \partial x}, \frac{\partial^3 f}{\partial z^3}\right),$$

where F is a continuous map.

Note that the derivatives D^β included on the right-hand side are defined by the condition: $\beta \in J_\alpha$.

Let $h: U \longrightarrow \mathbf{R}^3$ be any C^3 map.

Then there is a C^3 map $f: U \longrightarrow \mathbf{R}^3$ such that f solves $\mathcal{S}$ on U and such that $D^\beta f$, $D^\beta h$ satisfy given approximations for all derivatives D^β, $\beta \in J_\alpha$. In particular f is a C^2 approximation of h in the fine C^2-topology.

Proof. – Set $t = x + y$; $u = y$, $v = z$.

Let $g(t, u, v) = f(t - u, u, v)$; $g = (g_1, g_2, g_3)$.

In (t, u, v) coordinates, $\mathcal{S}$ is written,

$$\mathcal{S}' \equiv \frac{\partial^3 g_1}{\partial t^3} - \left[\frac{\partial^3 g_2}{\partial t^3}\right]^2 + \left[\frac{\partial^3 g_3}{\partial t^3}\right]^2 = G(j^\perp g)$$

where G is a continuous map. Applying Theorem 2, § 3, there is a C^3 map g which solves $\mathcal{S}'$ and such that $j^\perp g, j^\perp h_0$ satisfy given approximations where $h_0(t, u, v) = h(t - u, u, v)$.

Transforming back to (x, y, z) coordinates, $\mathcal{S}$ is solved by a C^3 map which satisfies the required approximations.

Appendix 1.

In this appendix, the definition and properties of metaneighbourhoods are established. The principal result, Proposition 1.1, proves that metaneighbourhoods of Ω can be constructed so that their constituent convex hulls have diameters of controlled size and also that each point, v, in the metaneighbourhood is "enveloped" by a convex hull, centred at $R(v) \in \Omega$, with preassigned diameter. In this way, properties $M(1)$, $M(2)$ are established for the sequence of metaneighbourhoods $(\mathrm{Met}_m \Omega)_{m \geqslant -1}$ in the proof of Theorem 2, § 3. This appendix makes precise the sketch of metaneighbourhoods and extensive metaneighbourhoods in Gromov [1].

DEFINITION 1.1. – 1) *A subspace* $Q \subset \mathbf{R}^q$ *is nowhere flat if the intersection of* Q *with each hyperplane in* $\mathbf{R}^q$ *is nowhere dense in* Q. $D(x;r)$ *denotes the open ball, centre* x, *radius* $r > 0$, *in* $\mathbf{R}^q$.

2) *Let* $Q \subset \mathbf{R}^q$ *be a subspace.*

Fix $x \in Q$.

(i) $\mathrm{Arc}(Q;x)$ *is the arc component in* Q *to which* x *belongs.*

(ii) $\mathrm{Env}(Q;x)$ *is the interior, in* $\mathbf{R}^q$, *of the convex hull ("enveloppe convexe") of* $\mathrm{Arc}(Q;x)$.

(iii) $\mathrm{Arc}_\epsilon(Q;x) = \mathrm{Arc}(Q \cap D(x;\epsilon);x)$.

(iv) $\mathrm{Env}_\epsilon(Q;x) = \mathrm{Env}(Q \cap D(x;\epsilon);x)$.

The following geometrical observations is useful for understanding the proofs in this section.

Observation. – Let $y \in \mathrm{Env}(Q;x)$. There is a finite set $z_1, z_2, \ldots, z_N$, $z_i \in \mathrm{Arc}(Q;x)$, $i = 1, 2, 3, \ldots, N$ such that the convex hull of the set $\{z_1, z_2, \ldots, z_N\}$ is a neighbourhood of y in $\mathbf{R}^q$.

Indeed let $y \in \mathrm{int}(\Delta^q)$ where Δ^q is a q-simplex in $\mathbf{R}^q$ contained in $\mathrm{Env}(Q;x)$. Since the vertices of Δ^q are in the convex hull of $\mathrm{Arc}(Q;x)$ it follows that there are a finite number of points in $\mathrm{Arc}(Q;x)$ whose convex hull is a neighbourhood of y in $\mathbf{R}^q$.

The following properties are easily verified:

1) If $Q \subset \mathbf{R}^q$ is open, then $\mathrm{Arc}(Q;x) \subset \mathrm{Env}(Q;x)$, for all $x \in Q$.

2) Let $Q \subset \mathbf{R}^q$ be locally arc-wise connected, and nowhere flat. Fix $\epsilon > 0$.

Then for each $x \in Q$, $x \in \overline{\mathrm{Env}_\epsilon(Q;x)}$.

Metaneighbourhood.

Let $Q \subset \mathbf{R}^q$ be locally arc-wise connected, and nowhere flat. Fix a continuous map $\epsilon : Q \longrightarrow (0, \infty)$.

The ϵ-metaneighbourhood of Q in $\mathbf{R}^q$ is the subset, $\mathrm{Met}_\epsilon Q = \bigcup_{x \in Q} \mathrm{Env}_{\epsilon(x)}(Q ; x)$.

$\mathrm{Met}_\epsilon Q$ is open in $\mathbf{R}^q$; $Q \subset \overline{\mathrm{Met}_\epsilon Q}$, but in general $\mathrm{Met}_\epsilon Q$ is not a neighbourhood of Q in $\mathbf{R}^q$.

Thus for example if $\epsilon \in (0,1)$, the ϵ-metaneighbourhood of the unit circle S^1 in $\mathbf{R}^2$ is an open annulus in the interior of the unit disc, whose closure contains S^1.

The following lemmas are easily established.

LEMMA 1.1. – *Let $Q \subset \mathbf{R}^q$ be closed, locally arc-wise connected and nowhere flat. Then $Q = \bigcap_\epsilon \overline{\mathrm{Met}_\epsilon Q}$.*

LEMMA 1.2. – *Let $Q \subset \mathbf{R}^q$ be locally arc-wise connected, and nowhere flat. Let $h : [0,1] \longrightarrow Q$ be a continuous map.*

Let $\epsilon : Q \longrightarrow (0, \infty)$ be a continuous map.

Then $\bigcup_{t \in [0,1]} \mathrm{Env}_{\epsilon(h(t))}(Q ; h(t))$ is arc-wise connected.

COROLLARY 1.2. – *Let $\mathrm{Met}_\epsilon Q$ be an ϵ-metaneighbourhood of Q.*

Fix $x \in Q$; $y \in \mathrm{Env}_{\epsilon(x)}(Q ; x)$.

For each $a > 0$, $\mathrm{Env}_a(Q ; x) \subset \mathrm{Env}_{\epsilon(x)+a}(\mathrm{Met}_\epsilon Q ; y)$.

The above notions in $\mathbf{R}^q$ are now generalized to the context of Euclidean vector bundles with fiber $\mathbf{R}^q$.

Let X be a locally compact metric space.

Let $\pi : E \longrightarrow X$ be a Euclidean vector bundle over X, fiber $\mathbf{R}^q$. $\Gamma(E)$ denotes the space of continuous sections of this bundle. Let Q be a subspace of E; $Q(x) = Q \cap \pi^{-1}(x)$, for all $x \in X$.

DEFINITION 1.2. – *Let $\Omega \subset E$ be a subspace and $h \in \Gamma(E)$ such that $h(X) \subset \Omega$.*

a) $\mathrm{Arc}(\Omega ; h) = \bigcup_{x \in X} \mathrm{Arc}(\Omega(x) ; h(x))$.

b) $\operatorname{Env}(\Omega\,;h) = \bigcup_{x\in X} \operatorname{Env}(\Omega(x)\,;h(x))$.

The notation $\operatorname{Env}(\Omega(x)\,;h(x))$ *indicates that the interior is taken in the fiber* $E(x) \simeq \mathbf{R}^q$.

The following lemma is easily established.

LEMMA 1.3. – *If* $\Omega \subset E$ *is open, then* $\operatorname{Arc}(\Omega\,;h)$ *is open in* E.

COROLLARY 1.3. – *If* $\Omega \subset E$ *is open, then* $\operatorname{Env}(\Omega\,;h)$ *is open in* E *and* $\operatorname{Arc}(\Omega\,;h) \subset \operatorname{Env}(\Omega\,;h)$.

Suppose now $\Omega \subset E$ is a closed subspace which satisfies the following geometrical properties (cf. the solution of closed conditions § 3).

G(1) : $\pi : \Omega \longrightarrow X$ is a topological fiber bundle, with locally arc-wise connected fibers.

G(2) : For each $x \in X$, the subset $\Omega(x) \subset E(x)$ $(\simeq \mathbf{R}^q)$ is nowhere flat. (Since $\pi : E \longrightarrow X$ is a vector bundle, this condition makes sense).

G(3) : There is a neighbourhood $N(\Omega)$ of Ω in E and a continuous retraction map $R : N(\Omega) \longrightarrow \Omega$ which respects the fibers: $\pi \circ R = \pi : N(\Omega) \longrightarrow X$.

Fiberwise Metaneighbourhood.

Fix a continuous map $\epsilon : \Omega \longrightarrow (0, \infty)$.

For each $x \in X$, let $\epsilon(x) : \Omega(x) \longrightarrow (0, \infty)$ be the restriction of the map ϵ to the fiber $\Omega(x)$.

The ϵ-fiberwise metaneighbourhood of Ω in E, denoted by $\operatorname{Met}_\epsilon \Omega$, is the union of all the $\epsilon(x)$-metaneighbourhoods in each fiber:

$$\operatorname{Met}_\epsilon \Omega = \bigcup_{x\in X} \operatorname{Met}_{\epsilon(x)} \Omega(x) \left(= \bigcup_{x\in X} \bigcup_{w\in\Omega(x)} \operatorname{Env}_{\epsilon(w)}(\Omega(x)\,;w)\right).$$

The following lemma is easily established.

LEMMA 1.4. – a) $\operatorname{Met}_\epsilon \Omega \subset E$ *is open.*

b) $\Omega = \bigcap_\epsilon \overline{\operatorname{Met}_\epsilon \Omega}$.

The main result of this Appendix is the following

PROPOSITION 1.1. – *Let* $\Omega \subset E$ *be a closed set which satisfies the geometrical properties* G(1), G(2), G(3) *above.*

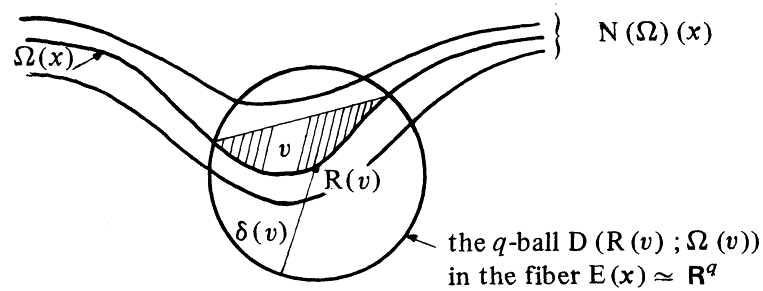
Let $\epsilon : \Omega \longrightarrow (0, \infty)$, $\delta : N(\Omega) \longrightarrow (0, \infty)$ *be continuous maps.*

There is a continuous map $\mu : \Omega \longrightarrow (0, \infty)$ *such that:*

(i) $\mu \leqslant \epsilon$; $\mathrm{Met}_\mu \Omega \subset N(\Omega)$.

(ii) *Let* $x \in X$; $w \in \Omega(x)$; $v \in \mathrm{Env}_{\mu(w)}(\Omega(x) ; w)$.

Then $v \in \mathrm{Env}_{\delta(v)}(\Omega(x) ; R(v))$ *(cf. Figure 1.1).*



The shaded area is $\mathrm{Env}_{\delta(v)}(\Omega(x) ; R(v))$.

Remarks. – 1) Conclusion (ii) ensures that each point, v, in a sufficiently "thin" metaneighbourhood, $\mathrm{Met}_\mu \Omega$, is enveloped by the points in $\Omega(x)$ which are contained in a q-ball with pre-assigned radius centred at $R(v) \in \Omega$. When applied to the context of the solution of closed conditions, § 3, it is this property which ensures the converges of the derivative maps $\left(\dfrac{\partial^r h_m}{\partial t^r} : U \longrightarrow \mathbf{R}^q\right)_{m \geqslant 0}$.

2) Let $Y \subset X$ and $h \in \Gamma(E(Y))$ (i.e., h is a continuous section over the subspace Y of X), such that $h(Y) \subset \mathrm{Met}_\mu \Omega$.

Define $V(h) = \bigcup_{y \in Y} \mathrm{Env}_{\delta(h(y))}(\Omega(y) ; R(h(y)))$.

It follows from conclusion (ii) above that $h(Y) \subset V(h)$. In Gromov [1], $V(h)$ is called a fiberwise extensive neighbourhood of $h(Y) \subset E$.

Proof of Proposition 1.1. – The proof of the proposition results from the following elementary lemmas in point set topology.

LEMMA 1.5. – *Let* $\delta : \mathrm{N}(\Omega) \longrightarrow (0,\infty)$ *be a continuous map. There is a continuous map* $\nu : \Omega \longrightarrow (0,\infty)$ *such that:*

(i) $\mathrm{Met}_\nu \Omega \subset \mathrm{N}(\Omega)$.

(ii) Let $x \in \mathrm{X}$, $z \in \Omega(x)$, $v \in \mathrm{Env}_{\nu(z)}(\Omega(x);z)$.

Then $\nu(z) \leqslant \delta(v)$.

LEMMA 1.6. – *Let* $\epsilon : \Omega \longrightarrow (0,\infty)$ *be a continuous map. There is a continuous map* $\mu : \Omega \longrightarrow (0,\infty)$ *such that:*

(i) $\mu \leqslant \epsilon$.

(ii) $\mathrm{Met}_\mu \Omega \subset \mathrm{N}(\Omega)$.

(iii) *Let* $x \in \mathrm{X}$; $z \in \Omega(x)$; $v \in \mathrm{Env}_{\mu(z)}(\Omega(x);z)$.

Then $\mathrm{R}(v) \in \mathrm{Arc}_{\epsilon(z)}(\Omega(x);z)$.

(*In particular* $z \in \mathrm{D}(\mathrm{R}(v);\epsilon(z))$).

Let us first prove the Proposition from the above Lemmas.

a) Applying Lemma 1.5, there is a continuous map $\nu : \Omega \longrightarrow (0,\infty)$ such that $\mathrm{Met}_\nu(\Omega) \subset \mathrm{N}(\Omega)$ and such that:

Let $x \in \mathrm{X}$; $z \in \Omega(x)$; $v \in \mathrm{Env}_{\nu(z)}(\Omega(x);z)$.

Then $\nu(z) \leqslant \dfrac{\delta(v)}{2}$. ... (1)

b) From Lemma 1.6 there is a continuous map $\mu : \Omega \longrightarrow (0,\infty)$ such that $\mu \leqslant \inf(\nu,\epsilon)$, $\mathrm{Met}_\mu \Omega \subset \mathrm{N}(\Omega)$, and such that:

Let $x \in \mathrm{X}$; $z \in \Omega(x)$; $v \in \mathrm{Env}_{\mu(z)}(\Omega(x);z)$.

Then $\mathrm{R}(v) \in \mathrm{Arc}_{\nu(z)}(\Omega(x);z)$. ... (2)

Fix points $z_1, z_2, \ldots, z_{\mathrm{N}}$ in $\mathrm{Arc}_{\mu(z)}(\Omega(x);z)$ such that the convex hull of $\{z_1, z_2, \ldots, z_{\mathrm{N}}\}$ in $\mathrm{E}(x)$ ($\simeq \mathbf{R}^q$) is a neighbourhood of v. ... (3)

From (1), (2) $\mathrm{D}(z;\nu(z)) \subset \mathrm{D}(\mathrm{R}(v);\delta(v))$.

Since $\mu \leqslant \nu$, it follows from (2), (3), above, that

$$z_i \in \mathrm{Arc}_{\delta(v)}(\Omega(x);\mathrm{R}(v)), \quad i = 1,2,3,\ldots,\mathrm{N}.$$

Consequently $v \in \mathrm{Env}_{\delta(v)}(\Omega(x);\mathrm{R}(v))$.

This completes the proof of the Proposition.

Turning now to the proofs of the lemmas, recall that $\Omega \subset \mathrm{E}$ is closed, and, X is a locally compact metric space. Evidently Ω is a locally compact metric space. Thus there is a countable relatively compact open cover $\{\mathrm{Q}_j\}_{j \geqslant 1}$ of Ω such that $\{\mathrm{Q}_j\}_{j \geqslant 1}$ is locally finite, and such that the bundle $\pi : \mathrm{E} \longrightarrow \mathrm{X}$ is trivial over $\pi(\mathrm{Q}_j) \subset \mathrm{X}$, $j = 1, 2, 3, \ldots$. ... (4)

In what follows, when working in Q_j, j fixed, one may assume from (4) that $\mathrm{E} = \mathrm{X} \times \mathbf{R}^q$ the product bundle.

Proof of Lemma 1.5. – For fixed j, since Q_j is compact, it is clear that $\bar{\mathrm{Q}}_j$ is contained in a finite union of compact product neighbourhoods, each of which is contained in $\mathrm{N}(\Omega)$. Evidently there is a constant $\epsilon_j > 0$ such that:

(i) $\epsilon_j \leqslant \delta(v)$, for each v in any of the above compact product neighbourhoods.

(ii) For each $z \in \bar{\mathrm{Q}}_j$, the q-ball $\mathrm{D}(z ; \epsilon_j)$ is contained in at least one of the above compact neighbourhoods.

Let $\bar{\epsilon} : \Omega \longrightarrow (0,\infty)$ be the lower semi-continuous map $\bar{\epsilon}(x) = \inf \{\epsilon_j \mid x \in \mathrm{Q}_j\}$. One easily constructs a continuous map $\nu : \Omega \longrightarrow (0,\infty)$ such that $\nu \leqslant \bar{\epsilon}$. Evidently, the metaneighbourhood, $\mathrm{Met}_\nu \Omega$, satisfies the conclusions of Lemma 1.5.

Proof of Lemma 1.6. – Recall that $\pi : \Omega \longrightarrow \mathrm{X}$ is a topological fiberbundle whose fiber is locally arc-wise connected.

For fixed j, one easily verifies that for each

$$z = (x , t) \in \bar{\mathrm{Q}}_j \subset \mathrm{X} \times \mathbf{R}^q$$

there is a neighbourhood $\mathrm{U}(z)$ of z in Ω and a number $\mu(z) > 0$ such that: If $y \in \mathrm{U}(z)$ then $\mathrm{D}(y ; \mu(z)) \cap \Omega(x) \subset \mathrm{Arc}_{\epsilon(y)}(\Omega(x) ; y)$.

Since $\bar{\mathrm{Q}}_j$ is compact, one concludes that there is a number $\epsilon_j > 0$ such that:

For each $z = (x , t) \in \bar{\mathrm{Q}}_j \subset \mathrm{X} \times \mathbf{R}^q$,

$$\Omega(x) \cap \mathrm{D}(z ; \epsilon_j) \subset \mathrm{Arc}_{\epsilon(z)}(\Omega(x) ; z) \qquad \ldots (5)$$

Let $\bar{\epsilon} : \Omega \longrightarrow (0,\infty)$ be the lower semi-continuous map $\bar{\epsilon}(z) = \inf \{\epsilon_j \mid z \in \bar{\mathrm{Q}}_j\}$. Let $\alpha : \Omega \longrightarrow (0,\infty)$ be a continuous map such that $\alpha \leqslant \bar{\epsilon}$ and such that $\mathrm{Met}_\alpha \Omega \subset \mathrm{N}(\Omega)$ (cf. Lemma 1.5).

Finally, since $R : N(\Omega) \longrightarrow \Omega$ is a continuous retraction map, one again may employ the cover $\{Q_j\}_{j \geqslant 1}$ to construct a continuous map $\mu : \Omega \longrightarrow (0, \infty)$ such that:

$$\mu \leqslant \inf(\alpha, \epsilon). \qquad \ldots (6)$$

For each $x \in X$, $z \in \Omega(x)$,

$$R(D(z; \mu(z))) \subset D(z; \alpha(z)) \subset E(x) \simeq \mathbf{R}^q. \qquad \ldots (7)$$

To prove Lemma 1.6, one proceeds as follows:

a) Since $\alpha \leqslant \overline{\epsilon}$, it follows from (5) above that:

For each $x \in X$, $z \in \Omega(x)$,

$$\Omega(x) \cap D(z; \alpha(z)) \subset \mathrm{Arc}_{\epsilon(z)}(\Omega(x); z) \qquad \ldots (8)$$

b) Since $\mu \leqslant \alpha$, then $\mathrm{Met}_\mu \Omega \subset N(\Omega)$.

c) Fix $x \in X$; $z \in \Omega(x)$; $v \in \mathrm{Env}_{\mu(z)}(\Omega(x); z)$.

From (7) above, one observes that $R(v)$ is contained in the q-ball $D(z; \alpha(z)) \subset E(x) \simeq \mathbf{R}^q$.

Consequently from (8) above, $R(v) \in \mathrm{Arc}_{\epsilon(z)}(\Omega(x); z)$.

This completes the proof of Lemma 1.6.

To conclude this appendix, some additional topological properties of metaneighbourhoods are established.

LEMMA 1.7. – *Let* $Y \subset X$ *be a closed subspace. Recall that* $\Gamma(E(Y))$ *is the space of continuous sections of the bundle* $\pi : E \longrightarrow X$ *over the subspace* Y.

Let $\sigma : Y \longrightarrow (0, \infty)$ *be a continuous map.*

Suppose $\Phi \in \Gamma(E(Y))$ *is a continuous section over the subspace* Y, *such that* $\Phi(Y) \subset \Omega$.

Then there is a continuous section $g \in \Gamma(E(Y))$ *such that: For each* $y \in Y$, $g(y) \in \mathrm{Env}_{\sigma(y)}(\Omega(y); \Phi(y))$.

Proof. – One easily verifies that there is a locally finite open cover $\{Y_j\}_{j \geqslant 1}$ of Y; a family of continuous sections $\{g_j \in \Gamma(E(Y))\}_{j \geqslant 1}$ such that:

For each $y \in Y_j$, $g_j(y) \in \mathrm{Env}_{\sigma(y)}(\Omega(y); \Phi(y))$, $j = 1, 2, 3, \ldots$.

Let $g \in \Gamma(E(Y))$ be the continuous section, $g = \sum_{j=1}^{\infty} p_j \cdot g_j$

where $(p_j : Y \longrightarrow [0,1])_{j \geqslant 1}$ is a partition of unity subordinate to the cover $\{Y_j\}_{j \geqslant 1}$. Since $\mathrm{Env}_{\sigma(y)}(\Omega(y), \Phi(y))$ is a convex set, for all $y \in Y$, it follows that the section g satisfies the conclusions of the lemma.

COROLLARY 1.7.1. – *Let* $\epsilon : \Omega \longrightarrow (0,\infty)$ *be a continuous map and* $\mathrm{Met}_\epsilon \Omega$ *the corresponding metaneighbourhood of* Ω *in* E.

Let $Y \subset X$ *be a closed subspace and* $\Phi \in \Gamma(E(Y))$ *a continuous section over* Y *such that* $\Phi(Y) \subset \Omega$.

Fix a continuous map $\delta : Y \longrightarrow (0,\infty)$.

Then there is a continuous section $g \in \Gamma(E(Y))$ *such that* $g(Y) \subset \mathrm{Met}_\epsilon \Omega$ *and for each* $y \in Y$, $d(g(y), \Phi(y)) \leqslant \delta(y)$ [*the metric on the Euclidean vector-bundle* E *is* d].

Indeed, set $\sigma = \inf\{\delta, \epsilon \circ \Phi : Y \longrightarrow (0,\infty)\}$, and apply Lemma 1.7.

COROLLARY 1.7.2. – *Suppose, in addition to the hypotheses of Corollary 1.7.1, there is a continuous section* $h \in \Gamma(E(Y))$ *such that:*

For each $y \in Y$, $h(y) \in \mathrm{Env}_{\epsilon(\Phi(y))}(\Omega(y); \Phi(y)) \subset \mathrm{Met}_\epsilon \Omega$. *Set* $\delta = \epsilon \circ \Phi : Y \longrightarrow (0,\infty)$. *Then, in addition, for each* $y \in Y$, $h(y) \in \mathrm{Env}_{2\epsilon(\Phi(y))}(\mathrm{Met}_\epsilon \Omega(y); g(y))$.

Indeed, the q-ball $D(g(y); 2\epsilon(\Phi(y)))$ contains the q-ball $D(\Phi(y); \epsilon(\Phi(y)))$ for all $y \in Y$. Employing Lemma 1.2, this corollary is easily proved.

To conclude this appendix, we state the following result the proof of which is left to the reader.

PROPOSITION 1.2. – *Let* $Y \subset X$ *be a closed subspace and* $\Phi \in \Gamma(E(Y))$ *a continuous section over* Y *such that* $\Phi(Y) \subset \Omega$.

Let $\sigma : Y \longrightarrow (0,\infty)$ *be a continuous map.*

Let $Z = \bigcup_{y \in Y} \mathrm{Env}_{\sigma(y)}(\Omega(y); \Phi(y))$.

Then $\pi : Z \longrightarrow Y$ *is a fiberbundle, with fiber the open* q-*ball* $D(0;1) \subset \mathbf{R}^q$.

Appendix 2.

The auxiliary functions.

1) Let K be a compact set.

Let $\epsilon \in (0,1)$ and let $P = (p_i : K \longrightarrow [0,1])_{1 \leqslant i \leqslant n}$ be a sequence of continuous functions such that:

For each $x \in K$, $\sum_{i=1}^{n} p_i(x) = 1$.

Following Gromov [1], one constructs a continuous map $\theta_\epsilon \equiv \theta_\epsilon^p$ from $K \times [0,1] \longrightarrow [0,1]$ such that:

(i) For each $x \in K$, $\theta_\epsilon(x, t)$ is monotone increasing and constant $\left(= \frac{i}{n+1}\right)$ on a subinterval of $[0,1]$ of length $(1-\epsilon)\, p_i(x)$.

(ii) For each $x \in K$, $\theta_\epsilon(x, 0) = 0$, $\theta_\epsilon(x, 1) = 1$.

Remark. – In case K is a C^∞ manifold and $p_i : K \longrightarrow [0,1]$ is a C^∞ map, $i = 1, 2, 3, \ldots, n$, one may construct θ_ϵ^p to also be a C^∞ map.

2) Let $b_i \in [0,1]$, $1 \leqslant i \leqslant N$ $(N \geqslant 2)$, such that $\sum_{i=1}^{N} b_i = 1$.

Let $\mathbf{N}$ be the set of non-negative integers.

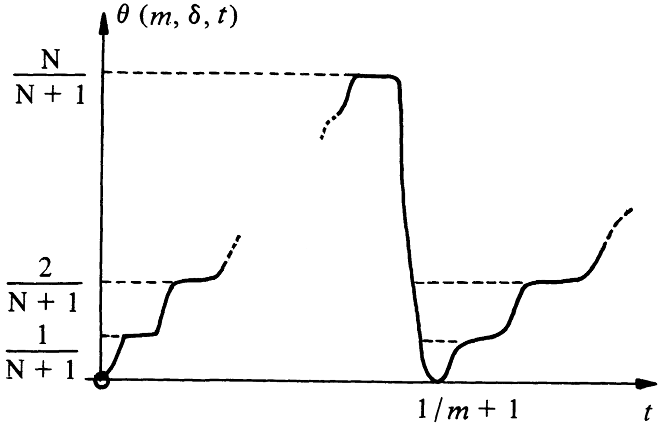
One easily constructs a C^∞ map,

$\theta : \mathbf{N} \times (0, 1/2) \times [0,1] \longrightarrow [0,1]$ such that the following properties obtain: Fix $m \in \mathbf{N}$, $\delta \in (0, 1/2)$.

a) The map $\theta(m, \delta, t) \equiv \theta(t)$ is periodic with period $\frac{1}{m+1}$; $\theta(m, \delta, 0) = 0 = \theta\left(m, \delta, \frac{1}{m+1}\right)$.

b) $\theta(m, \delta, t)$ is constant $\left(= \frac{i}{N+1}\right)$ on a subinterval, S_i, of $\left[0, \frac{1}{m+1}\right]$ of length $\frac{1-2\delta}{m+1}\, b_i$, $i = 1, 2, 3, \ldots, N$.

c) The interval separating S_i, S_{i+1} has length $\frac{2\delta}{(m+1)N}$, $i = 1, 2, 3, \ldots, N-1$.



Appendix 3.
The One-dimensional Lemma.

In this appendix we prove the principal approximation result (the one-dimensional lemma in Gromov [1]) in a general setting. The central feature of the proof is an explicit uniform computation of the approximation. This is useful for further applications of the convex integration technique.

Let $\mathbf{B}$ be a Banach space, norm $\| \ \|$. We consider a compact set of the form $K \times [0,1]$, K compact.

Let $Q \subset K \times [0,1] \times \mathbf{B}$. (For the purposes of the approximation result, below, Q need not be open.)

Let $\gamma : K \times [0,1] \times [0,1] \longrightarrow \mathbf{B}$ be a continuous map (a homotopy on $K \times [0,1]$) such that:

For each (x, t, s) in $K \times [0,1] \times [0,1]$, $(x, t, \gamma(x, t, s)) \in Q$.

Let $f_i : K \times [0,1] \longrightarrow \mathbf{B}$ be the continuous map,

$$f_i(x, t) = \gamma\left(x, t, \frac{i}{N+1}\right), \quad i = 1, 2, 3, \ldots, N.$$

Let $g : K \times [0,1] \longrightarrow \mathbf{B}$ be the continuous map (a convex combination), $g = \sum_{i=1}^{N} b_i f_i$, where $b_i \in [0,1]$, $1 \leqslant i \leqslant N$, and $\sum_{i=1}^{N} b_i = 1$. Recall the auxiliary map $\theta : \mathbf{N} \times (0,1/2) \times [0,1] \longrightarrow [0,1]$ (Appendix 2).

Fix $(m, \delta) \in \mathbf{N} \times (0,1/2)$.

Define $\Psi : K \times [0,1] \longrightarrow \mathbf{B}$ as follows:

$$\Psi(y, t) = \gamma(y, t, \theta(t)) \equiv \gamma(y, t, \theta(m, \delta, t)).$$

Note that the graph of Ψ is contained in Q, and, $\Psi(y, t) = f_i(y, t)$ on a subinterval of $\left[\frac{j}{m+1}, \frac{j+1}{m+1}\right]$ of length $\frac{(1-2\delta)b_i}{m+1}$, $1 \leqslant i \leqslant N$; $0 \leqslant j \leqslant m$.

APPROXIMATION THEOREM (One-dimensional lemma). – *Fix* $\epsilon > 0$. *Suppose the derivative* $\frac{\partial \gamma}{\partial t}$ is a continuous map.

There is an integer $J \in \mathbf{N}$ *and a number* $\delta_0 \in (0,1/2)$ *such that:*

Let $m \in \mathbf{N}$, $m \geqslant J$; $\delta \in (0, \delta_0)$.

The map $\Psi(y, t) = \gamma(y, t, \theta(m, \delta, t))$ *satisfies the property:*

For each $(y, t) \in K \times [0,1]$, $\left\| \int_0^t (\Psi - g)(y, s)\, ds \right\| \leqslant \epsilon$.

Remarks. – Another way of stating the above conclusion is as follows:

$$\text{Let } F(y, t) = \int_0^t \Psi(y, s)\, ds\,; \quad G(y, t) = \int_0^t g(y, s)\, ds\,.$$

Then the continuous map $F : K \times [0,1] \longrightarrow \mathbf{B}$ is a C^0 approximation to the continuous map $G : K \times [0,1] \longrightarrow \mathbf{B}$ such that the graph of the derivative map $\frac{\partial F}{\partial t} : K \times [0,1] \longrightarrow \mathbf{B}$ is contained in Q.

Proof. – Let

$$C = \sup\{\|\gamma(y, t, s)\| \mid (y, t, s) \in K \times [0,1] \times [0,1]\}.$$

The proof of the following lemma is straightforward and is omitted.

LEMMA 3.1. – *Fix* $\epsilon > 0$.

There is an integer R *such that:*

Let $q \in \{R, R+1, R+2, \ldots\}$. *Let* $p \in \{0, 1, 2, \ldots, q\}$. *Then, for all* $y \in K$,

(i) $\| \int_{\frac{p}{q+1}}^{t} g(y, s)\, ds \| \leqslant \epsilon \,; \quad t \in \left[\frac{p}{q+1}, \frac{p+1}{q+1}\right]$.

(ii) $\| \int_{\frac{p}{q+1}}^{t} \gamma(y, s, \theta(m, \delta, s))\, ds \| \leqslant \epsilon \,; \; t \in \left[\frac{p}{q+1}, \frac{p+1}{q+1}\right]$.

It follows from the above lemma that one need only prove the following approximation result:

Fix $\epsilon > 0$.

There is an integer $J \in \mathbf{N}$ and a number $\delta_0 \in (0, 1/2)$ such that:

Let $m \in \mathbf{N}$, $m \geqslant J$, $\delta \in (0, \delta_0)$.

The map $\Psi(y, t) = \gamma(y, t, \theta(m, \delta, t))$ satisfies the property:

Let $p \in \{0, 1, 2, 3, \ldots, m\}$.

Then $\| \int_0^{\frac{p}{m+1}} (\Psi - g)(y, s)\, ds \| \leqslant \epsilon$. ... (A.1)

The proof of (A.1) proceeds by a series of estimates.

Notation. – Fix $p \in \{0, 1\ 2, 3, \ldots, m\}$. Let $I(p)$ denote the interval $\left[\frac{p}{m+1}, \frac{p+1}{m+1}\right]$.

$$1) \int_{I(p)} g(y, s)\, ds = \int_{I(p)} \sum_{i=1}^{N} b_i f_i(y, s)\, ds$$
$$= \sum_{i=1}^{N} b_i \int_{I(p)} f_i(y, s)\, ds\,.$$

Let $f_i(y, s) = f_i(y, \xi_i) + (f_i(y, s) - f_i(y, \xi_i))$ where $\xi_i \in I(p)$. A suitable choice of ξ_i will be made below, $i = 1, 2, 3, \ldots .\, N$.

Thus $\int_{I(p)} g(y, s)\, ds = \sum_{i=1}^{N} \frac{b_i}{m+1} f_i(y, \xi_i) + E_1$ ... (A.2)

where $\quad E_1 = \sum_{i=1}^{N} b_i \int_{I(p)} (f_i(y, s) - f_i(y, \xi_i))\, ds\,.$

Evidently, $\| E_1 \| = \dfrac{1}{m+1} O_1\left(\dfrac{1}{m+1}\right)$ where

$$\lim_{m \to \infty} O_1\left(\frac{1}{m+1}\right) = 0 .$$

2) $\displaystyle\int_{I(p)} \Psi(y, s)\, ds = \int_{I(p)} \gamma(y, s, \theta(m, \delta, s))\, ds$

$$= \sum_{i=1}^{N} \int_{S_i} \gamma(y, s, \theta(s))\, ds + \int_T \gamma(y, s, \theta(s))\, ds ,$$

where a) $S_i \subset I(p)$ is the subinterval, length $\dfrac{(1-2\delta)\, b_i}{m+1}$, on which $\theta(m, \delta, s) = \dfrac{i}{N+1}$, $i = 1, 2, 3, \dots, N$ (cf. the construction of the auxiliary function θ, Appendix 2).

Recall that $f_i(y, s) = \gamma\left(y, s, \dfrac{i}{N+1}\right)$, $i = 1, 2, \dots, N$.

b) T is a disjoint union of subintervals of $I(p)$, whose total length is $\dfrac{2\delta}{m+1}$; $T \cap \left(\bigcup_{i=1}^{N} \operatorname{int} S_i\right) = \phi$.

Thus, $\displaystyle\int_{I(p)} \Psi(y, s)\, ds = \sum_{i=1}^{N} \int_{S_i} f_i(y, s)\, ds$

$$+ \int_T \gamma(y, s, \theta(s))\, ds.$$

Let $f_i(y, s) = f_i(y, \eta_i) + (f_i(y, s) - f_i(y, \eta_i))$, where

$\eta_i \in S_i$, $i = 1, 2, \dots, N$.

Then $$\int_{I(p)} \Psi(y, s)\, ds = \sum_{i=1}^{N} f_i(y, \eta_i) \frac{(1-2\delta)\, b_i}{m+1} + E_2 + E_3 \qquad \dots (A.3)$$

where,
$$E_2 = \int_T \gamma(y, s, \theta(s))\, ds .$$
$$E_3 = \sum_{i=1}^{N} \int_{S_i} (f_i(y, s) - f_i(y, \eta_i))\, ds .$$

Evidently, $\| E_2 \| \leqslant \dfrac{2\delta}{m+1} C$, and, $\| E_3 \| \leqslant \sum_{i=1}^{N} \frac{(1-2\delta) b_i}{m+1} O_2\left(\frac{1}{m+1}\right) = \frac{1-2\delta}{m+1} \; O_2\left(\frac{1}{m+1}\right)$,

where, $\lim_{m \to \infty} O_2\left(\frac{1}{m+1}\right) = 0$.

Since $S_i \subset I(p)$, all i, one may choose $\xi_i = \eta_i$, $i = 1, 2, 3, \ldots, N$. Hence from (A.2), (A.3), above

$$\int_{I(p)} (g(y,s) - \Psi(y,s))\, ds = \sum_{i=1}^{N} \frac{2\delta}{m+1} b_i f_i(y, \xi_i) + E_1 - E_2 - E_3 .$$

Consequently,

$$\begin{aligned} \| \int_{I(p)} (g(y,s) - \Psi(y,s))\, ds \| &\leqslant \frac{2\delta C}{m+1} + \frac{1}{m+1} O_1\left(\frac{1}{m+1}\right) \\ &\quad + \frac{2\delta C}{m+1} + \frac{1-2\delta}{m+1} O_2\left(\frac{1}{m+1}\right) \\ &= \frac{4\delta C}{m+1} + \frac{1}{m+1} O_3\left(\frac{1}{m+1}\right) , \end{aligned}$$

where, $\lim_{m \to \infty} O_3\left(\frac{1}{m+1}\right) = 0$. $\qquad \ldots$ (A.4)

To prove the approximation result (A.1), one notes the following: Let $p \in \{1, 2, 3, \ldots, m\}$.

$$\begin{aligned} \| \int_0^{\frac{p}{m+1}} (g - \Psi)(y,s)\, ds \| &\leqslant \sum_{j=0}^{p-1} \| \int_{I(j)} (g - \Psi)(y,s)\, ds \| \\ &\leqslant m \left[\frac{4\delta C}{m+1} + \frac{1}{m+1} O_3\left(\frac{1}{m+1}\right) \right] \\ &\leqslant 4\delta C + O_3\left(\frac{1}{m+1}\right). \end{aligned}$$

Clearly then, the approximation (A.1) obtains for $m \in \mathbf{N}$ sufficiently large and $\delta \in (0, 1/2)$ sufficiently small.

This completes the proof of the Approximation Theorem.

Complement 1.

Let $\Phi : K \times [0,1] \longrightarrow \mathbf{B}$ be the continuous map,

$$\Phi(y, t) = \gamma(y, t, 0) .$$

Let $H: K \times [0,1] \times [0,1] \longrightarrow \mathbf{B}$ be the homotopy on $K \times [0,1]$, $H(y, t, s) = \gamma(y, t, s \,.\, \theta(m, \delta, t)))$.

Clearly, the homotopy H connects the map Φ, Ψ, and is such that the graph of the corresponding map at each stage of the homotopy is contained in Q.

Complement 2.

Suppose $U \subset K \times [0,1]$ is a subset such that:

For each $(x, t) \in U \times [0,1]$, $\gamma(x, t) = 0 \in \mathbf{B}$.

Thus in particular, for each $x \in U$, $f_i(x) = g(x) = 0$, $1 \leqslant i \leqslant N$.

Recall that by construction $\Psi: K \times [0,1] \longrightarrow \mathbf{B}$ is the map, $\Psi(y, t) = \gamma(y, t, \theta(m, \delta, t))$, for a suitable auxiliary map θ. Consequently, for each $x \in U$, $\Psi(x) = 0 \in \mathbf{B}$.

Complement 3.

Suppose now $K \times [0,1] = [0,1]^n$, the n-cube in $\mathbf{R}^n$ $(K = [0,1]^{n-1})$, and $\gamma: [0,1]^n \times [0,1] \longrightarrow \mathbf{B}$ is a C^{r+1} map. Coordinates in $\mathbf{R}^n$ are denoted by $(s_1, s_2, \ldots, s_{n-1}, t)$. The Approximation Theorem easily extends to the following

C^r-APPROXIMATION THEOREM (C^r-one dimensional Lemma). – *Fix an integer $r > 0$ and a real number $\epsilon > 0$. There is an integer $J \in \mathbf{N}$ and a number $\delta_0 \in (0, 1/2)$ such that:*

Let $m \in \mathbf{N}$, $m \geqslant J$; $\delta \in (0, \delta_0)$.

The C^{r+1} map $\Psi(y, t) = \gamma(y, t, \theta(m, \delta, t))$ satisfies the following property:

For each $(y, t) \in [0,1]^{n-1} \times [0,1] = [0,1]^n$,

$$\left\| \int_0^t D^\alpha (g - \Psi)(y, s)\, ds \right\| \leqslant \epsilon,$$

where D^α runs over all differential operators on functions of n-variables such that $|\alpha| \leqslant r$ and D^α involves no derivatives in the t-variable.

The proof of the C^r-one-dimensional lemma is analogous to the proof of the one-dimensional lemma subject to the following observation.

Observation. – Let D^α be a differential operator as above. $D^\alpha \Psi(y,t) = D^\alpha(\gamma(y,t,\theta(m,\delta,t)))$ does not introduce the derivative $\dfrac{d}{dt}\theta(m,\delta,t)$. ...(1).

Let $C_r = \sup\{\|D^\alpha \gamma(x,s)\| \mid (x,s)\in[0,1]^n \times [0,1]$ and D^α as above$\}$. Then, from (1), above,

$\sup\{\|D^\alpha \Psi(y,t)\| \mid (y,t)\in[0,1]^{n-1}\times[0,1]=[0,1]^n$, and D^α as above$\} \leqslant C_r$, (*independent* of the choice of $(m,\delta)\in \mathbf{N}\times(0,1/2)$ in the auxiliary function θ).

Consequently, there is a corresponding Lemma 3.1 applied to the maps $D^\alpha g$, $D^\alpha\Psi : [0,1]^n \longrightarrow \mathbf{B}$, D^α as above.

(The corresponding Lemma 3.1 provides an integer R independent of $(m,\delta)\in\mathbf{N}\times(0,1/2)$ in the auxiliary function θ).

With Lemma 3.1 in hand, the proof of the C^r one-dimensional lemma is an exact parallel to the above proof of the one-dimensional lemma.

Complement 4.

The C^r one-dimensional lemma is easily reformulated and proved in case the n-cube $[0,1]^n$ is replaced by a translate in $\mathbf{R}^n$ of the n-cube $[0,a]^n$, $a>0$.

Appendix 4.

Introduction. – Let X be compact and $\pi : E \longrightarrow X$ a Euclidean vector bundle over X, norm $\| \ \|$.

Let $Q \subset E$ be open. $\Gamma_Q(E)$ is the space of continous section of E which map X into Q. Recall that $Q(x) = Q \cap \pi^{-1}(x)$, $x \in X$. Fix $\Phi \in \Gamma_Q(E)$, and open sets U, U_1 in X, $\overline{U} \subset U_1$.

Suppose g is a continuous section of E such that:

$$x \in X \Longrightarrow g(x) \in \mathrm{Env}(Q(x);\Phi(x)) \text{ (cf. Appendix 1)}$$

$$x \in U_1 \Longrightarrow g(x) = \Phi(x).$$

In the Convex Hull lemmas below, it is proved firstly, that the above convexity conditions on the section g imply that g is a convex linear combination (variable coefficients, in general) of sections in $\Gamma_Q(E)$. Secondly, it is proved that this convex linear combination may be replaced by a convex linear combination, *constant* coefficients, of sections in $\Gamma_Q(E)$.

In this way the Convex Hull Lemma translates the above convexity conditions on the map g into the requisite convex linear combination conditions of the hypotheses in the one-dimensional lemma (Appendix 3).

CONVEX HULL LEMMA I. – *There are sections* $h_i \in \Gamma_Q(E)$ *and continuous maps,* $p_i : X \longrightarrow [0,1]$, $i = 1, 2, 3, \ldots, n$, $\sum_{i=1}^{n} p_i = 1$, *such that* :

a) $g = \sum_{i=1}^{n} p_i . h_i$.

b) *For each* $x \in U$, $h_i(x) = \Phi(x)$, $i = 1, 2, 3, \ldots, n$.

c) *For each* $x \in X$, $h_i(x) \in \mathrm{Arc}(Q(x) ; \Phi(x))$.

Proof. – Since Q is open, it follows from Lemma 1.3 (Appendix 1) that $\mathrm{Arc}(Q ; \Phi) = \bigcup_{x \in X} \mathrm{Arc}(Q(x) ; \Phi(x))$ is open in E.

One easily constructs open covers of X, $\{V(x)\}_{x \in X}$, $\{W(x)\}_{x \in X}$, $\overline{V(x)} \subset W(x)$, all x, such that :

(i) $W(x) \subset U_1$ if $x \in \overline{U}$; $W(x) \cap \overline{U} = \phi$ if $x \notin \overline{U}$.

(ii) For each $x \in X$, there are sections $h_i^x \in \Gamma_Q(E)$, $1 \leqslant i \leqslant n(x)$ such that:

a) For each $y \in X$, $h_i^x(y) \in \mathrm{Arc}(Q(y) ; \Phi(y))$.

b) if $y \in X - W(x)$, then $h_i^x(y) = \Phi(y)$.

c) if $y \in V(x)$, then $g(y) = \sum_{i=1}^{n} p_i^x(y)\, h_i^x(y)$, where $p_i^x : X \longrightarrow [0,1]$ is a continuous function, $i = 1, 2, 3, \ldots, n$ and $\sum_{i=1}^{n} p_i^x(y) = 1$, all $y \in V(x)$.

d) if $x \in \overline{U}$, then $h_1^x = \Phi$; $p_1^x = 1$ (that is, in case $x \in \overline{U}$, $n(x) = 1$). Since X is compact, the cover $\{V(x)\}_{x \in X}$ contains a finite sub-cover. An easy partition of unity argument completes the proof.

Complement 1.

There are homotopies $F_i : [0,1] \longrightarrow \Gamma_Q(E)$ connecting Φ, $h_i, i = 1, 2, 3, \ldots, n$, such that :

1) For each $(x, t) \in U \times [0,1]$, $F_i(x, t) = \Phi(x)$, $i = 1, 2, 3, \ldots, n$.

2) For each $(x, t) \in X \times [0,1]$, $F_i(x, t) \in \mathrm{Arc}(Q(x); \Phi(x))$, $i = 1, 2, 3, \ldots, n$.

Complement 2.

In case X is a compact C^∞ manifold (in particular $X = [0,1]^n$, the n-cube in $\mathbf{R}^n$), and Φ, g are C^∞ sections, one may also arrange that $h_i \in \Gamma_Q(E)$, $F_i : [0,1] \longrightarrow \Gamma_Q(E)\, p_i : X \longrightarrow [0,1]$ are C^∞ maps, $i = 1, 2, 3, \ldots, n$.

CONVEX HULL LEMMA II. – *Suppose now* $g = \sum_{i=1}^{n} p_i h_i$, $h_i \in \Gamma_Q(E)$ *and* $F_i : [0,1] \longrightarrow \Gamma_Q(E)$, $i = 1, 2, 3, \ldots, n$, *as in the conclusions of Convex Hull Lemma I.*

There is a sequence $(b_i \in [0,1])_{1 \leq i \leq N}$, $\sum_{i=1}^{N} b_i = 1$, *and maps* $f_i \in \Gamma_Q(E)$ *such that* :

1) *For each* $x \in X$, $g(x) = \sum_{i=1}^{N} b_i f_i(x)$.

2) *For each* $x \in U$, $f_i(x) = \Phi(x)$, $i = 1, 2, 3, \ldots, N$.

3) *For each* $x \in X$, $f_i(x) \in \mathrm{Arc}(Q(x); \Phi(x))$, $i = 1, 2, 3, \ldots, N$.

Proof. – Employing the homotopies $F_i : [0,1] \longrightarrow \Gamma_Q(E)$, one easily constructs a homotopy $\gamma : [0,1] \longrightarrow \Gamma_Q(E)$ such that : $\gamma(0) = \Phi$; $\gamma\left(\frac{i}{n+1}\right) = h_i$, $i = 1, 2, 3, \ldots, n$; for each $(x, s) \in U \times [0,1]$, $\gamma(x, s) = \Phi(x)$ [the homotopy γ runs "back and forth" along each homotopy F_i, $i = 1, 2, 3, \ldots, n$], and, for each $(x, s) \in X \times [0,1]$, $(x, \gamma(x, s)) \in \mathrm{Arc}(Q(x); \Phi(x))$. Fix $\epsilon > 0$.

Recall the auxiliary map $\theta_\epsilon : X \times [0,1] \longrightarrow [0,1]$ (Appendix 2) associated to the continuous maps $p_i : X \longrightarrow [0,1]$, $i = 1, 2, 3, \ldots, n$, $\sum_{i=1}^{n} p_i = 1$.

Let $\Psi_\epsilon : [0,1] \longrightarrow \Gamma_Q(E)$ be the homotopy of sections, $\Psi_\epsilon(x, t) = \gamma(x, \theta_\epsilon(x, t))$. In particular, $\Psi_\epsilon(x, t) = h_i(x)$, on a subinterval of $[0,1]$ of length $(1 - \epsilon) p_i(x)$, $i = 1, 2, 3, \ldots, n$.

Thus for each $x \in X$,

$$\int_0^1 \Psi_\epsilon(x, t)\, dt = \sum_{i=1}^{n} (1 - \epsilon) p_i(x) h_i(x) + E_\epsilon(x),$$

where $\lim_{\epsilon \to 0} E_\epsilon(x) = 0$. The map $\gamma : [0,1] \times X \longrightarrow E$ has compact image in E. Thus $\lim_{\epsilon \to 0} \int_0^1 \Psi_\epsilon(x, t)\, dt = g(x)$, uniformly in X ...(1).

Since X is compact, and $Q \subset E$ is open, the following properties obtain : Fix $\epsilon_1 > 0$.

1) There is a number $\delta > 0$ such that :

Let $P \equiv 0 = s_1 \leqslant s_2 \leqslant s_3 \ldots \leqslant s_{N+1} = 1$ be a partition of $[0,1]$, mesh $P \leqslant \delta$. For each $x \in X$,

$$\left\| \int_0^1 \Psi_\epsilon(x, t)\, dt - \sum_{i=1}^{N} (s_{i+1} - s_i) \Psi_\epsilon(x, s_i) \right\| \leqslant \epsilon_1 \qquad \ldots (2).$$

2) There is a number $r > 0$ such that :

Let $f \in \Gamma(E)$, $\|f\| \leqslant r$. Then for each $(x, s) \in X \times [0,1]$, $\gamma(x, s) + f(x) \in \mathrm{Arc}(Q(x); \Phi(x))$. Fix a partition P, as above; mesh $P \leqslant \delta$.

Let $f \in \Gamma(E)$ be the section,

$$f(x) = g(x) - \sum_{i=1}^{N} (s_{i+1} - s_i) \Psi_\epsilon(x, s_i).$$

From (1), (2), above, one may suppose that $\|f\| \leqslant r$. Note also that, for each $x \in U$, $f(x) = 0$. Define $f_i \in \Gamma_Q(E)$ to be the section,

$$f_i(x) = f(x) + \Psi_\epsilon(x, s_i),\ i = 1, 2, 3, \ldots, N.$$

Set $b_i = s_{i+1} - s_i \in [0,1]$, $i = 1, 2, 3, \ldots, N$, then $\sum_{i=1}^{N} b_i = 1$.

Clearly, for each $x \in X$, $g(x) = \sum_{i=1}^{N} b_i f_i(x)$.

Evidently, all the conclusions of the lemma are satisfied. Hence the proof of the Convex Hull Lemma is complete.

Complement 1.

Let $H_\gamma : [0,1] \times [0,1] \longrightarrow \Gamma_Q(E)$ be the continuous map. For each $x \in X$, $H_\gamma(x, t, s) = \gamma(x, t.s)$.

Evidently, for each $(x, t) \in X \times [0,1]$, $H_\gamma(x, t, 0) = \Phi(x)$; $H_\gamma(x, t, 1) = \gamma(x, t)$.

For each $(x, t, s) \in U \times [0,1] \times [0,1]$, $H_\gamma(x, t, s) = \Phi(x)$.

Employing the map H_γ, one easily constructs homotopies $H_i : [0,1] \longrightarrow \Gamma_Q(E)$ connecting f_i to Φ, such that for each $(x, t) \in U \times [0,1]$, $H_i(x, t) = \Phi(x)$, $i = 1, 2, 3, \ldots N$.

Complement 2.

In case X is a C^∞ manifold (in particular if X is the n-cube $[0,1]^n$ in $\mathbf{R}^n$) and g, p_i, h_i are C^∞ maps, $i = 1, 2, \ldots, n$, one may arrange that the maps γ, f_i, H_i are C^∞ maps, $i = 1, 2, 3, \ldots, N$.

Remark. – The construction of the maps γ, H_γ associated to the data h_i, F_i, $i = 1, 2, \ldots, n$, may be repeated for the data f_i, H_i, $i = 1, 2, \ldots, N$. In particular there are corresponding continuous maps $\gamma : [0,1] \longrightarrow \Gamma_Q(E)$, $H_\gamma : [0,1] \times [0,1] \longrightarrow \Gamma_Q(E)$ such that :

(i) For each $x \in X$, $\gamma(x, 0) = \Phi(x)$; $\gamma\left(x, \dfrac{i}{N+1}\right) = f_i(x)$, $i = 1, 2, 3, \ldots, N$.

(ii) For each $(x, t) \in U \times [0,1]$, $\gamma(x, t) = \Phi(x)$.

(iii) For each $(x, t, s) \in X \times [0,1] \times [0,1]$,

$$H_\gamma(x, t, s) = \gamma(x, t.s).$$

Consequently $H_\gamma(x, t, 0) = \Phi(x)$; $H_\gamma(x, t, 1) = \gamma(x, t)$.

Appendix 5.

The Cube Lemma.

In this appendix, the C^r-cube Lemma (stated in the Introduction, § 1) is solved.

By hypothesis, for each $x \in [0,1]^n$,

$$\frac{\partial^r G}{\partial t^r}(x) \in \operatorname{Env}(Q(x)\,;\Phi(x))$$

(cf. Appendix 1 for this notation) and $\dfrac{\partial^r G}{\partial t^r} = \Phi$ on a neighbourhood U of the closed set K_0 in K.

Reduction of the Problem.

For each $x \in [0,1]^n$, let $Q'(x) = Q(x) - \dfrac{\partial^r G}{\partial t^r}(x)$ [the translate in $\mathbf{R}^q$ of the set $Q(x)$ by the vector $\dfrac{\partial^r G}{\partial t^r}(x) \in \mathbf{R}^q$]; let $\Phi' : U \longrightarrow \mathbf{R}^q$ be the continuous map

$$\Phi'(x) = \Phi(x) - \frac{\partial^r G}{\partial t^r}(x).$$

Note that $\Phi' = 0$ in U.

Thus $Q' = \bigcup_{x \in [0,1]^n} Q'(x)$ is open in $[0,1]^n \times \mathbf{R}^n$, and $0 \in \operatorname{Env}(Q'(x), \Phi'(x))$.

If necessary, one may replace the map Φ' by a C^∞ map Φ_0 : $[0,1]^n \longrightarrow \mathbf{R}^q$ such that : For each $x \in [0,1]^n$,

$\Phi_0(x) \in \operatorname{Arc}(Q'(x)\,;\Phi'(x))\,;\Phi_0(x) = 0$ in a neighbourhood U_0 of K_0 in K, $\bar{U}_0 \subset U$. (Note that, (Appendix 1), $\operatorname{Arc}(Q'(x), \Phi'(x)) \subset \mathbf{R}^q$ is open for all $x \in [0,1]^n$.) Fix a neighbourhood U_1 of K_0, $\bar{U}_1 \subset U_0$.

In what follows, the Cube Lemma is solved for the special case when $G : [0,1]^n \longrightarrow \mathbf{R}^q$ is the constant map equal to zero. By translating back to the original data, the Cube Lemma is solved in general.

Applying the Convex Hull Lemma, Complement 2 (Appendix 4) one concludes that there are constants $b_i \in [0,1]$ $1 \leqslant i \leqslant N$, C^∞ maps $f_i : [0,1]^n \longrightarrow \mathbf{R}^q$, $1 \leqslant i \leqslant N$ and a C^∞ map, γ : $[0,1]^n \times [0,1] \longrightarrow \mathbf{R}^q$ such that :

(i) For each $x \in [0,1]^n$, $0 = \sum_{i=1}^{N} b_i f_i(x)$.

(ii) For each $(x, t) \in U_1 \times [0,1]$, $\gamma(x, t) = 0$;
For each $(x, t) \in [0,1]^n \times [0,1]$,

$$\gamma(x, t) \in \operatorname{Arc}(Q'(x)\,;\Phi_0(x)).$$

(iii) For each $x \in [0,1]^n$, $\gamma\left(x, \frac{i}{N+1}\right) = f_i(x)$, $1 \leqslant i \leqslant N$.

Fix $\epsilon > 0$. Coordinates in $\mathbf{R}^n$ are $(s_1, s_2, \ldots, s_{n-1}, t)$. Applying the C^r-one dimensional lemma (Appendix 3), one concludes there is a C^∞ map $\Psi: [0,1]^n \longrightarrow \mathbf{R}^n$ (in fact, the map Ψ is of the form $\Psi(y, t) = \gamma(y, t, \theta(t))$ for a suitable auxiliary map $\theta(t))$, such that :

(iv) For each $x \in U_1$, $\Psi(x) = 0$.

(v) For each $(y, t) \in [0,1]^{n-1} \times [0,1] = [0,1]^n$,

$\| \int_0^t D^\alpha \Psi(y, s)\, ds \| \leqslant \epsilon$, where D^α runs over all differential operators on functions of n-variables such that $|\alpha| \leqslant r$ and D^α involves no differentiation in the t-coordinate.

$$\text{Let } f_0 = \underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{r} \Psi(y, s)\,(ds)^r .$$

Evidently $f_0 : [0,1]^n \longrightarrow \mathbf{R}^q$ is a C^∞ map ;

$$\frac{\partial^r f_0}{\partial t^r}(x) \in \operatorname{Arc}(Q'(x); \Phi_0(x))$$

for all $x \in [0,1]^n$, and :

For each $(y, t) \in [0,1]^{n-1} \times [0,1] = [0,1]^n$, $\|D^\beta f_0(y, t)\| \leqslant \epsilon$, where D^β runs over all differential operators on functions of n-variables such that $|\beta| \leqslant r$ and D^β involves at most $(r-1)$ differentiations in the t-coordinate $\left(\text{that is, } |\beta| \leqslant r \text{ and } D^\beta \neq \frac{\partial^r}{\partial t^r}\right)$.

Thus $f_0 : [0,1]^n \longrightarrow \mathbf{R}^q$ solves the Cube lemma for the (reduced data) $G = 0$; $[0,1]^n \longrightarrow \mathbf{R}^q$ except possibly for the condition that f_0 must equal zero in a neighbourhood of K_0 in $[0,1]^n$.

To achieve this last property, let $h: [0,1]^n \longrightarrow [0,1]$ be a C^∞ map and U_2 a neighbourhood of K_0 in $[0,1]^n$, $\overline{U}_2 \subset U_1$, such that : $K_0 \subset \operatorname{int}(h^{-1}(1)) \subset \operatorname{supp} h \subset U_2$.

Define $V = \operatorname{int} h^{-1}(1)$, and let $f: [0,1]^n \longrightarrow \mathbf{R}^q$ be the C^∞ map $f = (1 - h).f_0$. Evidently, $f = 0$ in V and $f = f_0$ on the complement of U_2 in $[0,1]^n$.

For $\epsilon > 0$ sufficiently small in the estimate (v) above, the corresponding map $f = (1 - h) f_0$ satisfies the additional property.

For each $x \in [0,1]^n$, $\dfrac{\partial^r f}{\partial t^r}(x) \in \mathrm{Arc}(Q'(x), \Phi_0(x))$ (recall here that $Q' \subset [0,1]^n \times \mathbf{R}^q$ is open and $\dfrac{\partial^r f_0}{\partial t^r} = \Psi = 0$ in U_1).

Thus the cube lemma is proved with respect to the reduced data $G = 0 : [0,1]^n \longrightarrow \mathbf{R}^q$.

It is evident that the map $F = G + f : [0,1]^n \longrightarrow \mathbf{R}^q$ is a C^r map which yields all the conclusions of the Cube lemma for the original data.

Complement 1.

Employing the homotopy H_γ associated to the map $\gamma : [0,1]^n \times [0,1] \longrightarrow \mathbf{R}^q$ (Convex Hull Lemma, Appendix 4), one easily verifies that there is a homotopy $H : [0,1]^n \times [0,1] \longrightarrow \mathbf{R}^q$ such that $H_0 = \Phi$; $H_1 = \dfrac{\partial^r (G + f)}{\partial t^r}$; $H(x, t) = \Phi(x)$ for all $(x, t) \in V \times [0,1]$, and also such that :

For each $(x, t) \in [0,1]^n \times [0,1]$,

$$(x, H(x, t)) \in \mathrm{Arc}(Q(x); \Phi(x)).$$

Complement 2 (Compactness).

Let $\mathrm{Arc}(Q; \Phi) = \bigcup_{x \in [0,1]^n} \mathrm{Arc}(Q(x); \Phi(x))$ (cf. Appendix 1).

Let $A = \{(x, \gamma(x, t)) \in [0,1]^n \times \mathbf{R}^q \mid (x, t) \in [0,1]^n \times [0,1]\}$.

Evidently A is a compact subset of $\mathrm{Arc}(Q, \Phi)$. Consequently the graph of the homotopy H_t, $0 \leqslant t \leqslant 1$ which connects Φ, $\dfrac{\partial^r (G + f)}{\partial t^r}$, (cf. Complement 1) lies in a preassigned compact subset of $\mathrm{Arc}(Q; \Phi)$; that is, a compact subset which is *independent* of the auxiliary function θ used to define the map $\Psi : [0,1]^n \longrightarrow \mathbf{R}^q$.

This complement is employed in §2 in the inductive construction of a solution to an open condition $Y \subset J^r(U, \mathbf{R}^q)$, where $U \subset \mathbf{R}^n$ is open.

Complement 3.

The C^r-Cube Lemma is easily reformulated and proved in case $[0,1]^n$ is replaced by a translate of the n-cube $[0,a]^n$ in $\mathbf{R}^n$, $a>0$.

Complement 4.

Suppose in addition the map $G:[0,1]^n \longrightarrow \mathbf{R}^q$ of the hypotheses of the Cube Lemma is of class C^∞.

Then, in addition, the following conclusion obtains : Fix an integer $s \geqslant r$.

One may choose the map $F: [0,1]^n \longrightarrow \mathbf{R}^q$ in the conclusion of the Cube Lemma to be of class C^∞, and also so that the following estimate obtains : For each $(y,t) \in [0,1]^{n-1} \times [0,1] = [0,1]^n$,

$$\| D^\alpha (F-G)(y,t) \| \leqslant \epsilon,$$

where D^α runs overall differential operators on functions of n-variables such that $|\alpha| \leqslant s$ and D^α involves at most $(r-1)$ differentiations in the t-variable.

Appendix 6.
The Local Extension Theorem.

In this final appendix, the Local Extension Theorem is completely proved. This is the principal local result on which depends the inductive procedure for solving open conditions in jet spaces (§ 2). After performing some preliminary technical modification, the proof of the Local Extension Theorem is obtained by applying the Cube Lemma (Appendix 5). Let L, M be subspaces of $[0,1]^n$, $L \subset M$, such that $\overline{M-L}$ is contained in the interior of $[0,1]^n$.

Let $Y \subset [0,1]^n \times \mathbf{R}^q$ be an open set.

Recall the notation, $Y(x) = Y \cap (x \times \mathbf{R}^q) \subset x \times \mathbf{R}^q \equiv \mathbf{R}^q$, $x \in [0,1]^n$.

LOCAL EXTENSION THEOREM. – *Let* $h:[0,1]^n \longrightarrow \mathbf{R}^q$ *be a* C^r *map,* $\Phi:[0,1]^n \longrightarrow \mathbf{R}^q$ *be a continuous map and* $N(L)$ *a* *neighbourhood of* L *in* $[0,1]^n$ *such that the following properties obtain :*

1) *The graph of* $\Phi(x)$ *is contained in* Y.

2) *For each* $x \in [0,1]^n$, $\dfrac{\partial^r h}{\partial t^r}(x) \in \mathrm{Env}(\mathrm{Y}(x)\,;\Phi(x))$.

3) *For each* $x \in \mathrm{N(L)}$, $\dfrac{\partial^r h}{\partial t^r}(x) = \Phi(x)$.

Thus (h,Φ) *is a formal solution to* Y *which solves the open condition* Y *in* N(L).

Let $\mathrm{K}_0 = \mathrm{L} \cup \partial([0,1]^n)$.

Fix $\epsilon > 0$. *Coordinates in* $\mathbf{R}^n$ *are* $(s_1, s_2, \ldots, s_{n-1}, t)$.

Then there is a C^r *map* $f\colon [0,1]^n \longrightarrow \mathbf{R}^q$, *a continuous map* $\Phi_1 : [0,1]^n \longrightarrow \mathbf{R}^q$, *a neighbourhood* W *of* M *in* $[0,1]^n$, *a neighbourhood* V *of* K_0 *in* $[0,1]^n$ *such that the following properties obtain :*

(i) *For each* $x \in [0,1]^n$, $\|\mathrm{D}^\alpha(f-h)(x)\| \leqslant \epsilon$, *where* D^α *runs over all differential operators on functions of n-variables such that* $|\alpha| \leqslant r$, *and* $\mathrm{D}^\alpha \neq \dfrac{\partial^r}{\partial t^r}$.

(ii) *For each* $x \in [0,1]^n$, $\Phi_1(x) \in \mathrm{Arc}(\mathrm{Y}(x)\,;\Phi(x))$.

(iii) *For each* $x \in [0,1]^n$, $\dfrac{\partial^r f}{\partial t^r}(x) \in \mathrm{Env}(\mathrm{Y}(x)\,;\Phi(x))$.

(iv) *For each* $x \in \mathrm{W}$, $\dfrac{\partial^r f}{\partial t^r}(x) = \Phi_1(x)$.

(v) *For each* $x \in \mathrm{V}$, $f(x) = h(x)\,;\ \Phi_1(x) = \Phi(x)$.

(vi) $\mathrm{W} \cap \partial[0,1]^n \subset \mathrm{N(L)} \cap \partial[0,1]^n$.

Thus the formal solution (f,Φ_1) *to* Y *extends the formal solution* (h,Φ) *to* Y *with respect to the closed sets* $\mathrm{L} \subset \mathrm{M}$ *in the n-cube* $[0,1]^n$.

Proof. – Note that, in general, $\partial[0,1]^n$ is not a subset of L; consequently, in general, the maps $\dfrac{\partial^r h}{\partial t^r}$, $\Phi : [0,1]^n \longrightarrow \mathbf{R}^q$ are not equal in a neighbourhood of $\partial[0,1]^n$ in $[0,1]^n$. To remedy this, some preliminary modifications on Y are introduced.

Let $Q \subset [0,1]^n \times \mathbf{R}^q$ be the subset defined as follows : For each $x \in M$, $Q(x) = \mathrm{Arc}(Y(x)\,;\,\Phi(x))$.

For each x in the complement of M in $[0,1]^n$, $Q(x) = \mathrm{Env}(Y(x)\,;\,\Phi(x))$.

Let $Q = \bigcup_{x \in [0,1]^n} Q(x)$.

Since Y is open, applying Lemma 1.3 (Appendix 1) it follows that $\mathrm{Arc}(Y, \Phi) \subset [0,1]^n \times \mathbf{R}^q$ is open and $\mathrm{Arc}(Y, \Phi) \subset \mathrm{Env}(Y, \Phi)$. Evidently then, Q is open in $[0,1]^n \times \mathbf{R}^q$.

Note that the graph of Φ is contained in Q and, for each $x \in [0,1]^n$, $\dfrac{\partial^r h}{\partial t^r}(x) \in \mathrm{Env}(Q(x)\,;\,\Phi(x))$.

Furthermore, for each x in the complement of M in $[0,1]^n$, $\dfrac{\partial^r h}{\partial t^r}(x) \in \mathrm{Arc}(Q(x)\,;\,\Phi(x))$.

One easily constructs a continuous map $\Phi_0 : [0,1]^n \longrightarrow \mathbf{R}^q$ such that :

(i) $\Phi_0 = \dfrac{\partial^r h}{\partial t^r}$ in a neighbourhood of $\partial[0,1]^n$ in $[0,1]^n$.

(ii) $\Phi_0 = \Phi$ in a neighbourhood of M in $[0,1]^n$ (recall that $\overline{M - L} \subset \mathrm{int}([0,1]^n)$).

(iii) For each $x \in [0,1]^n$, $\Phi_0(x) \in \mathrm{Arc}(Q(x)\,;\,\Phi(x))$.

Thus the pair of maps (h, Φ_0) is a formal solution to the open condition Q, which solves Q in a neighbourhood of $K_0 = L \cup \partial[0,1]^n$. Applying the Cube Lemma, Appendix 5, to the data (h, Φ_0) and $Q \subset [0,1]^n \times \mathbf{R}^q$, one concludes that there is a C^r map $f\colon [0,1]^n \longrightarrow \mathbf{R}^q$, a neighbourhood V of K_0 in $[0,1]^n$ such that :

(i) For each $x \in [0,1]^n$ $\dfrac{\partial^r f}{\partial t^r}(x) \in \mathrm{Arc}(Q(x)\,;\,\Phi_0(x))$.

(ii) For each $x \in V$, $f(x) = h(x)\,;\, \dfrac{\partial^r f}{\partial t^r}(x) = \Phi_0(x)$.

(In particular $f = h$ in a neighbourhood of $\partial[0,1]^n$ in $[0,1]^n$).

(iii) For each $x \in [0,1]^n$, $\|D^\alpha(f - h)\| \leqslant \epsilon$, where D^α runs over all differential operators on functions of n-variables, such that

$|\alpha| \leqslant r$, and $D^\alpha \neq \dfrac{\partial^r}{\partial t^r}$. From the definition of Q and (i) above, it is clear that:

For each $x \in [0,1]^n$, $\dfrac{\partial^r f}{\partial t^r}(x) \in \mathrm{Env}(Y(x)\,;\,\Phi(x))$.

For each $x \in M$, $\dfrac{\partial^r f}{\partial t^r}(x) \in \mathrm{Arc}(Y(x)\,,\,\Phi(x))$.

(Thus, over M, the graph of $\dfrac{\partial^r f}{\partial t^r} : [0,1]^n \longrightarrow \mathbf{R}^q$ is contained in Y).

Furthermore, employing Complement 1 to the Cube Lemma (Appendix 4), there is a homotopy $H : [0,1]^n \times [0,1] \longrightarrow \mathbf{R}^q$ such that :

(iv) $H_0 = \Phi_0$; $H_1 = \dfrac{\partial^r f}{\partial t^r}$; $H_t = \Phi_0$ on the neighbourhood V of K_0 in $[0,1]^n$, $0 \leqslant t \leqslant 1$.

(v) For each $t \in [0,1]$, the graph of H_t is contained in $\mathrm{Arc}(Y\,;\,\Phi)$.

One easily "truncates" the homotopy H in a suitable neighbourhood of M in $[0,1]^n$ (recall that $\overline{M - L} \cap \partial[0,1]^n = \emptyset$) to obtain a continuous map $\Phi_1 : [0,1]^n \longrightarrow \mathbf{R}^q$, a neighbourhood W of M in $[0,1]^n$, such that the following properties obtain :

a) For each $x \in [0,1]^n$, $\Phi_1(x) \in \mathrm{Arc}(Y(x)\,;\,\Phi(x))$.

b) $\Phi_1 = \dfrac{\partial^r f}{\partial t^r}$ in W.

c) $\Phi_1 = \Phi$ in V.

d) $W \cap \partial[0,1]^n \subset N(L) \cap \partial[0,1]^n$.

Evidently the C^r map $F : [0,1]^n \longrightarrow \mathbf{R}^q$, the continuous map $\Phi_1 : [0,1]^n \longrightarrow \mathbf{R}^q$, the neighbourhoods W of M, V of K_0 in $[0,1]^n$ satisfy all the conclusions of the Local Extension Theorem.

This completes the proof of the Theorem.

Complement 1.

One easily constructs a homotopy of formal solutions to Y, rel a neighbourhood of K_0 in $[0,1]^n$, connecting $(h\,,\Phi)$ to $(f\,,\Phi_1)$.

Complement 2.

The local Extension Theorem is easily reformulated and proved in case $[0,1]^n$ is replaced by a translate of the n-cube $[0,a]^n$ in $\mathbf{R}^n$, $a > 0$.

Complement 3.

Let $\sigma : [0,1]^n \longrightarrow (0, \infty)$ be a continuous map.

Suppose (h, Φ) is a formal solution to Y within the tolerance σ: For each $x \in [0,1]^n$, $\dfrac{\partial^r h}{\partial t^r}(x) \in \mathrm{Env}_{\sigma(x)}(\mathrm{Y}(x)\,;\,\Phi(x))$. Then, in addition, one may construct the C^r map $f: [0,1]^n \longrightarrow \mathbf{R}^q$ and the continuous map $\Phi_1 : [0,1]^n \longrightarrow \mathbf{R}^q$ to satisfy the conclusion :

For each $x \in [0,1]^n$, $\dfrac{\partial^r f}{\partial t^r}(x) \in \mathrm{Env}_{\sigma(x)}(\mathrm{Y}(x)\,;\,\Phi(x))$.

For each $x \in [0,1]^n$, $\Phi_1(x) \in \mathrm{Arc}_{\sigma(x)}(\mathrm{Y}(x)\,;\,\Phi(x))$.

Complement 4.

Suppose, in addition to the hypotheses of the Local Extension Theorem, $h : [0,1]^n \longrightarrow \mathbf{R}^q$ is a C^∞ map.

Fix an integer $s \geqslant r$.

Then one may construct the map $f: [0,1]^n \longrightarrow \mathbf{R}^q$ in the conclusions of the Local Extension Theorem to be a C^∞ map and such that the following estimates obtain :

For each $(y, t) \in [0,1]^{n-1} \times [0,1] = [0,1]^n$,

$\|\mathrm{D}^\alpha(f-h)(y,t)\| \leqslant \epsilon$, where D^α runs over all differential operators on functions of n-variables such that $|\alpha| \leqslant s$ and D^α involves differentiation in the t-variable at most $(r-1)$ times (cf. Complement 4, Appendix 5).

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David SPRING,
York University
Faculty of Arts
Department of Mathematics
4700 Keele street
Downsview, Ontario M3J 1P3 (Canada).