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BROWNIAN MOTION AND TRANSIENT GROUPS

by Nicolas Th. VAROPOULOS (*)

0. Introduction.

Let M be a complete connected Riemannian manifold and let $\{\omega(t);t>0\}_{\omega\in\Omega}$ be the Brownian motion on M starting at some fixed point $m \in M$ (cf. [1] for the construction of that motion; in [2] the readers can find an overall review of some of the general facts on potential theory and diffusion on a manifold). For some fixed continuous Brownian path $\omega \in \Omega$ we can join $\omega(t)$ (for some fixed $t > 0$) with m by a minimal geodesic $\gamma(t)$ and obtain a closed loop $\Gamma(t)$. Such a minimal geodesic will in fact be unique unless $\omega(t) \in C(m)$ = the cut locus of m which is a closed set of measure zero on M. If $\gamma(t)$ is unique then $\Gamma(t)$ determines a unique element in $\pi_1(M;m)$ the fundamental group based at m. That element I shall also denote by $\Gamma(t) = \Gamma_\omega(t)$. The Brownian motion determines thus a process with values on $\pi_1(M)$ which is some kind of « generalized random walk ». The starting point of this paper was my effort to determine when the above « Brownian walk » on $\pi_1(M)$ is transient and when it is recurrent. In other words decide whether Brownian motion wind's itself more and more as time goes on or whether it comes back infinitely often close to its starting point and unwound.

I shall start with a precise definition, and to avoid the complications that arise from the cut locus, I shall pass to the universal covering (simply connected) manifold $\alpha: \tilde{M} \to M$ with its induced Riemannian structure and I shall also assume that M is *compact*. Let us fix $\tilde{m} \in M$ such that $\alpha(\tilde{m}) = m$ and let $\{\tilde{\omega}(t);t>0\}$ denote the Brownian motion on $\tilde{M}$ starting at $\tilde{m}$.

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DEFINITION. — *I shall say that the Brownian motion on* M *does not wind on* M *if the motion* $\{\tilde{\omega}(t);t>0\}$ *is recurrent on* $\tilde{\mathrm{M}}$. *This implies that for all* $\omega\in\Omega$ *there exists* $t_j\uparrow+\infty$ *a sequence such that* $\omega(t_j)\xrightarrow[j\to\infty]{} m$ *and* $\Gamma_\omega(t_j)=1\in\pi_1(\mathrm{M};m)$ *is the homotopically trivial loop.*

I shall say that Brownian motion winds on M *if the motion* $\{\tilde{\omega}(t);t>0\}$ *is transient on* $\tilde{\mathrm{M}}$. *This implies that for all* $\omega\in\Omega$ *and all* $\mathrm{F}\subset\pi_1$ *finite subset of the fundamental group we can find* $\mathrm{T}>0$ s.t. *the loop* $\Gamma_\omega(t)\notin\mathrm{F}$ *for almost all* $t>\mathrm{T}$.

[The almost all in t arises from the cut locus. Indeed observe that if we denote by $\tilde{\mathrm{G}}(\tilde{x},\tilde{y})$ the Green's function on $\tilde{\mathrm{M}}$, which exists by our hypotheses, then

$$\mathbf{E}_{\tilde{m}}\{\text{Leb mes}\,[t;\tilde{\omega}(t)\in\alpha^{-1}[\mathrm{C}(m)]]\}=\int_{\alpha^{-1}\mathrm{C}(m)}\mathrm{G}(\tilde{m},\tilde{x})\,d\tilde{\mathrm{V}}(\tilde{x})$$

$\tilde{\mathrm{V}}$ denotes Riemannian volume element].

Our problem is therefore to decide whether Brownian motion is recurrent or transient on $\tilde{\mathrm{M}}$. It will turn out convenient to consider a more general problem.

Let G be a discrete group generated by the finite set of generators $\{g_1,g_2,\dots,g_p\}$; any element $g\in\mathrm{G}$ can then be written as

$$(0.1)\qquad g=g_{i_1}^{\varepsilon_1}g_{i_2}^{\varepsilon_2}\dots g_{i_n}^{\varepsilon_n}\qquad \varepsilon_s=\pm1;$$

we shall denote by $|g|=\inf n$ the inf being taken under (0.1). I shall also denote by:

$$\gamma(n)=\text{Card}\,\{g\in\mathrm{G};|g|\leqslant n\}$$

the growth function of the group. A different set of generators $\{h_1,\dots,h_q\}$ will of course give a different $|\ |$ and γ. But the new and the old satisfy the simple relation:

$$\mathrm{A}^{-1}|g|^{\text{new}}\leqslant|g|^{\text{old}}\leqslant\mathrm{A}|g|^{\text{new}};\qquad\forall g\in\mathrm{G}$$

for some $\mathrm{A}>0$.

Let now G be a finitely generated group; I shall say that G is a transient group if there exists μ a symmetric

$$(\text{i.e. } \mu(\{g^{-1}\}) = \mu(\{g\}; \forall g \in \mathrm{G})$$

probability measure on G s.t.

$$\mu(\{g\}) \leqslant \mathrm{C}e^{-c|g|^2}; \qquad g \in \mathrm{G}$$

$$\sum_{n=0}^{\infty} \mu^n(\{1\}) < +\infty \quad (1 = \text{identity of G})$$

for some C, $c > 0$ where μ^n denotes the $n\overset{\text{th}}{=}$ convolution power of μ. It will turn out that if G is a transient finitely generated group then every probability measure μ on G s.t. $\mu(\{g\}) \geqslant \mathrm{C}e^{-c|g|^2}$ $(g \in \mathrm{G})$ for some C, $c > 0$ satisfies

$$\sum_{n=0}^{\infty} \mu^n(\{1\}) < +\infty .$$

The above definition is but a formulation of the transience of the random walk on G determined by the transition probability $\mathrm{P}_g(h) = \mu(\{gh^{-1}\})$. My definition of a transience on a group differs from the classical one only on the square exponential estimate. The reader at this stage should consult some of the classical literature in the subject (eg. [3], [4]) cf. Appendix.

I shall now state some of the theorems of this paper.

Let $\tilde{\mathrm{M}} \to \mathrm{M}$ a normal Riemannian covering mapping and let G be the group of deck transformations. [By *normal* we mean that there exists $\mathrm{N} \lhd \pi_1(\mathrm{M})$ a normal subgroup of $\pi_1(\mathrm{M})$ so that $\mathrm{G} \cong \pi_1/\mathrm{N}$ and that G is transitive on the fiber.]

THEOREM 1. — *Let* $\tilde{\mathrm{M}}$, M *and* G *be as above and let us suppose that* M *is compact. Then Brownian motion on* $\tilde{\mathrm{M}}$ *is transient if and only if* G *is a transient group.*

The story at this point takes a rather interesting twist:

Start from any finitely generated discrete group G. It is possible then to find $\tilde{\mathrm{M}} \to \mathrm{M}$ a normal Riemannian covering for which G is exactly the group of deck transformations. Indeed let $\mathrm{M} = \mathrm{M}_0 \# \mathrm{M}_0 \# \ldots \# \mathrm{M}_0$ be the gluing together of $\mathrm{M}_0 \cong \mathrm{S}^1 \times \mathrm{S}^2$ n times with itself; we have $\pi_1(\mathrm{M}) \cong \mathbf{Z}^* \ldots {}^*\mathbf{Z} = \mathbf{Z}^{*n}$ the $n\overset{\text{th}}{=}$ free power. Since G is the quotient of $\mathbf{Z}^{*n}$ for some n the assertion follows.

Using then the necessary condition proved in [2] for the transience of Brownian motion on a manifold we obtain :

THEOREM 2. — *A necessary condition for the transience of the finitely generated group* G *is that the growth function* $\gamma(n)$ *of* G *should satisfy*

$$\sum_{n=1}^{\infty} \frac{n}{\gamma(n)} < +\infty .$$

If we assume that M is positively curved then we can obtain a better result :

THEOREM 3. — *Let* M, $\tilde{\mathrm{M}}$ *and* G *be as above and let us assume that* M *is compact and that* $\mathrm{Ric}(\mathrm{M}) \geqslant 0$ *then* G *is transient if and only if*

$$\sum_{n=1}^{\infty} \frac{n}{\gamma(n)} < +\infty$$

where γ *denotes the growth function of* G.

This theorem was proved in [5].

In the same (technical) spirit of theorem 1 we can examine the amenability of G.

Towards that end let us denote by $\tilde{\Delta}$ the Laplace-Beltrami operator of $\tilde{\mathrm{M}}$, let

$$\text{top sp.}(\tilde{\Delta}) = \sup \{\lambda ; \lambda \in \mathrm{Sp}\, \tilde{\Delta}\},$$

notice that top $\mathrm{sp}(\tilde{\Delta}) \leqslant 0$. We have

THEOREM 4. — *Let us suppose that* $\tilde{\mathrm{M}}$, M *and* G *are as above and that* M *is compact. Then* G *is amenable if only if :*

$$\text{top sp}(\tilde{\Delta}) = 0 .$$

After I wrote this paper I found out that the above theorem has been proved recently in [14].

As a corollary we obtain the well known facts that :

(i) M compact with negative sectional curvature implies that $\pi_1(\mathrm{M})$ not amenable.

(ii) $\mathrm{Ric}(\mathrm{M}) \geqslant 0$ implies that $\pi_1(\mathrm{M})$ amenable.

Before I embark with the proofs I would like to point out that a reasonable conjecture is that a finitely generated group is transient if and only if

$$\sum_{n=1}^{\infty} \frac{n}{\gamma(n)} < +\infty .$$

The only if is the content of Theorem 2.

The above conjecture holds when G can be embedded as a discrete subgroup of a connected Lie group (cf. [14]). The conjecture also holds when G is soluble (cf. [15]) and thus also it holds if we assume that the growth of G is polynomial (cf. [6]) i.e. if we assume that there exist some C and q s.t. $\gamma(n) \leqslant Cn^q$ $\forall n \in \mathbf{Z}$. Fairly trivially also the conjecture holds when G is *not* amenable (cf. [4]).

At the end of this paper (§ 7 and 8) I finally show how the methods developed adapt to cope with some non compact manifolds M.

I shall restrict myself to one specific example that was examined recently by Lyons-McKean [12]. In that example M is the complex plane minus two points and the metric is any conformal metric (e.g. the Poincare metric or the flat metric which is not complete). $\tilde{\mathrm{M}} \to \mathrm{M}$ is the covering obtained by the group of deck transformations $\mathrm{G} = \pi_1/[\pi_1,\pi_1] = \mathrm{H}_1(\mathrm{M})$. (The homology group.) The transience of Brownian motion on $\tilde{\mathrm{M}}$ (which is now a conformal invariant) is what was proved in [12]. In § 8 I offer an alternative proof of that fact.

1. S-operators.

Let us fix $(\mathrm{X};dx)$ a measure space and G a discrete group. We shall consider then $(\Omega;d\omega) = (\mathrm{X};dx) \otimes (\mathrm{G};dg)$ the cartesian product where dg is the discrete normalized Haar measure on G. For every $1 \leqslant p \leqslant +\infty$ we can then identify $\mathrm{L}^p(\Omega)$ with $\mathrm{L}^p(\mathrm{X};\mathrm{B})$ the space of B-valued L^p-functions where B is the Banach space $\mathrm{L}^p(\mathrm{G})$. This allows us to identify the canonical scalar product between $\mathrm{L}^{p_i}(\Omega)$ $\left(i = 1,2;\right.$ with $\left.\dfrac{1}{p_1} + \dfrac{1}{p_2} = 1\right)$ with

$$(1.1) \qquad \langle \mathrm{F}_1,\mathrm{F}_2 \rangle = \int_{\mathrm{X}} (\mathrm{F}_1(x),\mathrm{F}_2(x))\, dx$$

with $F_i \in L^{p_i}(X;L^{p_i}(G))$, $F_i(x) \in L^{p_i}(G)$, $x \in X$ and $(\cdot,\cdot)$ the scalar product between $L^{p_1}(G)$ and $L^{p_2}(G)$.

Let now $K(x,y) \in M(G)$, $(x,y \in X)$ be a measurable function on $X \times X$ with values in $M(G) = \ell^1(G)$ that satisfies

$$\sup_x \int \|K(x,y)\| \, dy \leqslant 1$$

$$\sup_y \int \|K(x,y)\| \, dx \leqslant 1$$

where $\|\ \|$ indicates the $M(G)$ total mass norm. It is evident that K induces a norm decreasing operator on $L^\infty(\Omega)$ by the formula

$$KF(x) = \int_X F(y)*K(x,y) \, dy$$

where $*$ indicates the convolution operator on G (observe that $F(y)$; $KF(x) \in L^\infty(G)$, $\forall x,y \in X$). It is also clear that the transposed operator of the above K with respect to the scalar product (1.1) is of the same form and is given by a new kernel $K^*(x,y) \in M(G)$ where $K^*(x,y) = \check{K}(y,x)$ $(x,y \in X)$ and where $\check{}$ is the operation on $\ell^1(G)$ that is defined by $\check{\gamma}(\{g\}) = \gamma(\{g^{-1}\})$. It follows therefore that $K(x,y)$ also induces a norm decreasing operator on $L^1(\Omega)$ and therefore by the Riesz-Thorin theorem on all the $L^p(\Omega)$ spaces.

DEFINITION. — *I sall say that* K *a norm decreasing operator on all the* $L^p(\Omega)$ *spaces is an* S*-operator if it is given as above by some kernel* $K(x,y) \in M(G)$ *that satisfies*

(i) $$K(x,y) \geqslant 0 \quad \forall x,y \in X$$

i.e. positive in the order relation of $M(G)$.

(ii) $\displaystyle\int_X \|K(x,y)\| \, dy = 1 \quad \forall x \in X$

(iii) $\displaystyle\int_X \|K(x,y)\| \, dx = 1 \quad \forall y \in X.$

If in addition $K(x,y)$ *satisfies*

(iv) $K(x,y) = \check{K}(y,x)$.

I shall say that K *is a symmetric* S*-operator. It is clear that* K *is* *symmetric if and only if it is self adjoint on* $L^2(\Omega)$. *The following order relation will be introduced on the space of* S-*operators : We shall say that* $K_1 \gg K_2$ *for two* S-*operators* K_1 *and* K_2 *if there exists some* $0 < \alpha \leqslant 1$ *such that :*

$$K_1(x,y) \geqslant \alpha K_2(x,y) \qquad \forall x,y \in X$$

in the order relation of $\ell^1(G)$.

The following obvious proposition is critical.

PROPOSITION. — *Let* K_1, K_2 *be two* S-*operators and let the* L^p-*operator norm of* K_1 *be* 1 ($\|K_1\|=1$) *for some fixed* $1 \leqslant p \leqslant +\infty$. *Then if* $K_1 \gg K_2$ *we also have* $\|K_2\| = 1$.

Proof. — By our hypothesis there exists some $0 < \alpha < 1$ such that :

$$K_1 = \alpha K_2 + (1-\alpha)\frac{K_1 - \alpha K_2}{1-\alpha} = \alpha K_2 + (1-\alpha)K_3$$

and it is clear that K_3 is also an S-operator and therefore satisfies $\|K_3\| \leqslant 1$. It follows by convexity that unless $\|K_2\| = 1$ we could not have $\|K_1\| = 1$.

2. Operators on covering Riemannian manifolds.

Let $\tilde{M}$, M be two Riemannian manifolds and $\tilde{M} \to M$ a normal Riemannian covering (i.e. a local isometry). Let G be the discrete group of deck transformations. We can then identify the measure space $(\tilde{M};d\tilde{V})$ with the product space $(M;dV) \times (G;dg)$ [where dV and $d\tilde{V}$ are the canonical volume elements of M and $\tilde{M}$ respectively]. We shall abusively say then that T an operator on $L^p(\tilde{M};d\tilde{V})$ is an S (or symmetric S) operator if it becomes such an operator after the above identification. Such an operator is easily seen to commute with the action of the group G on $L^p(\tilde{M};d\tilde{V})$.

Notice that the above identification is not unique but depends on the choice of the fundamental domain $D \subset \tilde{M}$. We shall fix such a domain D. When M is compact we shall choose D to be compact and a nice subset of $\tilde{M}$ with a nice boundary.

Examples.

(i) Any bimarkovian G-invariant kernel on $\tilde{M}$ i.e. any

$$\tilde{K}(\tilde{x},\tilde{y}) \geqslant 0 \quad (\tilde{x},\tilde{y} \in \tilde{M})$$

that satisfies:

$$\int \tilde{K}(\tilde{x},\tilde{y})\, d\tilde{V}(\tilde{y}) = 1, \qquad \forall \tilde{x} \in \tilde{M}$$

$$\int \tilde{K}(\tilde{x},\tilde{y})\, d\tilde{V}(\tilde{x}) = 1, \qquad \forall \tilde{y} \in \tilde{M}$$

$$\tilde{K}(g\tilde{x},g\tilde{y}) = \tilde{K}(\tilde{x},\tilde{y}); \qquad \forall g \in G; \qquad \tilde{x}, \tilde{y} \in \tilde{M}$$

is an S-operator, in fact it is easy to see that every S-operator is of that form. If in addition $\tilde{K}(\tilde{x},\tilde{y}) = \tilde{K}(\tilde{y},\tilde{x})$ then it is a symmetric S-operator.

(ii) More specifically when M is compact the Heat diffusion Kernel $\tilde{p}_t(\tilde{x},\tilde{y})(t>0;\tilde{x},\tilde{y} \in \tilde{M})$ on $\tilde{M}$ is a symmetric S-operator. $\tilde{p}_t(\tilde{x},\tilde{y})$ is by definition the minimal positive solution of $\dfrac{\partial}{\partial t} - \Delta_{\tilde{y}} = 0$ that satisfies $\tilde{p}_t(\tilde{x},.) \xrightarrow[t\to 0]{} \delta_{\tilde{x}}(.)$ (cf. [1], [2]). The non trivial fact that is needed here is that under the hypotheses that M is compact [or more generally that $\mathrm{Ric}(M) \geqslant -A$ (some $A \geqslant 0$); or even more general conditions cf. [2]] we have:

$$\int \tilde{p}_t(\tilde{x},\tilde{y})\, d\tilde{V}(\tilde{y}) = 1, \qquad \forall t > 0, \qquad \tilde{x} \in \tilde{M}.$$

i.e. that heat diffusion is conservative.

(iii) Let $k(x,y)$ be a scalar valued S-kernel on M i.e. $k(x,y) \geqslant 0$

$$\int k(x_0,y)\, dV(y) = \int k(x,y_0)\, dV(x) = 1, \qquad \forall x_0,\ y_0 \in M$$

and let $\mu \in M(G)$ be a fixed probability measure on G. We can then define $K = k \otimes \mu$, an S-kernel on $\tilde{M}$, by $K(x,y) = k(x,y).\mu$. If $k(x,y) = k(y,x)$ and $\mu = \check{\mu}$ then $k \otimes \mu$ is a symmetric S-kernel.

Let us now assume that M is of finite volume, by renormalization we can then assume that

$$\mathrm{Vol}(M) = 1.$$

I shall then (fix some fundamental domain D) define two mappings

$$\mathrm{I}: \ell^p(\mathrm{G}) \to \mathrm{L}^p(\tilde{\mathrm{M}};d\tilde{\mathrm{V}}); \qquad \mathrm{P}: \mathrm{L}^p(\tilde{\mathrm{M}};d\tilde{\mathrm{V}}) \to \ell^p(\mathrm{G}) \quad (1 \leqslant p \leqslant +\infty)$$

by:

$$\mathrm{I}f(\tilde{x}) = f(g), \text{ for } \tilde{x} \in g\mathrm{D},\ \tilde{x} \in \tilde{\mathrm{M}},\ g \in \mathrm{G};\ \forall f \in \ell^p(\mathrm{G})$$

$$\mathrm{PF}(g) = \int_{g\mathrm{D}} \mathrm{F}(\tilde{x})\, d\tilde{\mathrm{V}}(\tilde{x}), \qquad g \in \mathrm{G}, \qquad \forall \mathrm{F} \in \mathrm{L}^p(\tilde{\mathrm{M}};d\tilde{\mathrm{V}}).$$

The above two mappings are clearly norm decreasing.

PROPOSITION. – *Let* $\tilde{\mathrm{M}}$, M *and* G *be as above and let* K *be an* S-*operator on* $\mathrm{L}^p(\tilde{\mathrm{M}};d\tilde{\mathrm{V}})$. *Then the operator* $k = \mathrm{P} \circ \mathrm{K} \circ \mathrm{I}$ *is given on* $\ell^p(\mathrm{G})$ *by the convolution of a probability measure* $\kappa \in \mathrm{M}(\mathrm{G})$ [*i.e.* $k(f) = f * \kappa$] *further more if the operator* •K *is symmetric then the measure* κ *is symmetric* [*i.e.* $\kappa = \check{\kappa}$].

Proof. – Indeed k is obviously positive norm decreasing on all the $\ell^p(\mathrm{G})$ $(1 \leqslant p \leqslant +\infty)$ translation invariant and also satisfies $k(1) = 1$ (Alt. $\|k(f)\| = \|f\|$, $\forall 0 \leqslant f \in \ell^1(\mathrm{G})$).

Further more P and I are transposes of each other so k is self adjoint on ℓ^2 as soon as K is symmetric. The proposition follows.

3. The spectrum of the Laplacian and amenability.

I shall give here a first illustration of the notions introduced in § 1 and 2 by giving the proof of Theorem 4.

Let $\tilde{\mathrm{M}} \to \mathrm{M}$ be a normal Riemannian covering and let G be the group of deck transformations. Let $\tilde{\Delta}$ be the Laplacian of $\tilde{\mathrm{M}}$ and let us assume that topsp. $(\tilde{\Delta}) = 0$. By basic spectral theory this implies that the operator norm on $\mathrm{L}^2(\tilde{\mathrm{M}};d\tilde{\mathrm{V}})$ of $\tilde{p}_t(\tilde{x},\tilde{y})$, the heat diffusion kernel, is equal to 1. (i.e. $\|\tilde{p}_t\| = 1$ for $t > 0$). Now, if we assume that $\tilde{p}_t(\tilde{x},\tilde{y})$ is conservative (cf. § 2, Example (ii)) which is certainly the case if M is compact [in fact much weaker conditions will ensure this] then we are in the situation of § 2, Example (ii) with a symmetric S-operator of norm 1.

Let now $\mathbf{1} = \mathbf{1}(x,y) \equiv 1$ $(\forall x,y \in \mathrm{M})$ be the function that is identically equal to 1 and let $\mathrm{K} = \mathbf{1} \otimes \mu$ be the S-kernel defined from the probability measure $\mu \in M(G)$ of compact support as in § 2 Example (iii). [We assume, as we may, that $\mathrm{Vol}(M) = 1$.]

It is then clear that $\tilde{p}_t \gg K$ $(t>0)$ and this with the proposition in § 1 implies that $\|K\| = 1$. The upshot is that μ as a convolution operator on $\ell^2(G)$ has norm 1. This implies the amenability of G (cf. [7]).

Conversely let us assume that $\mathrm{topsp.}(\tilde{\Delta}) < 0$ and that therefore $\|\tilde{p}_t\| < 1$ $(t>0)$ as an L^2-operator. Let also $k = P \circ \tilde{p}_1 \circ I$ be the operator defined in § 2 it follows that $\|k\| < 1$ and from this it follows that κ the probability measure it defines on G by the proposition in § 2 has convolution operator norm less than 1. G is therefore not amenable (cf. [7]).

Remark. – The above argument holds as soon as $\tilde{p}_t(\tilde{x},\tilde{y})$ is conservative *and* M has finite volume.

4. The Hilbert space argument.

Most of this section is taken out from [8].

Let H be a *real* Hilbert space and let A, B be two invertible operators on H that satisfy the following conditions:

$$0 \leqslant \langle Ax,x\rangle \leqslant \langle Bx,x\rangle; \qquad \langle Ax,y\rangle = \langle x,Ay\rangle, \qquad \forall x,y \in H$$

then we have:

LEMMA [8]. – *With* A *and* B *as above we have*

$$\langle B^{-1}x,x\rangle \leqslant A^{-1}x,x\rangle, \qquad \forall x \in H.$$

Proof. – Fix $x \in H$ and set

$$y_1 = A^{-1}x, \qquad y_2 = B^{-1}x$$

we then have

$$\langle B^{-1}x,x\rangle^2 = \langle y_2,Ay_1\rangle^2 \leqslant \langle Ay_1,y_1\rangle\langle Ay_2,y_2\rangle \\ \leqslant \langle Ay_1,y_1\rangle\langle By_2,y_2\rangle = \langle A^{-1}x,x\rangle\langle B^{-1}x,x\rangle.$$

The lemma follows.

Let now T_1, T_2 and T_3 be three contractions on H (i.e. $\|T_i\| \leqslant 1$,

$i = 1,2,3)$ and let us assume that T_2 is symmetric

$$(\text{i.e. } \langle T_2 x,y\rangle = \langle x,T_2 y\rangle : \ x,y \in H)$$

that satisfy

$$T_1 = \alpha T_2 + (1-\alpha)T_3$$

for some $0 < \alpha < 1$.

It is clear then that for all $0 \leqslant \lambda < 1$ we have

$$0 \leqslant \langle (I-\lambda T_2)\varphi,\varphi\rangle$$

$$0 \leqslant (1-\alpha)[\|\varphi\|^2 - \lambda\langle T_3\varphi,\varphi\rangle] = \langle (1-\lambda T_1)\varphi,\varphi\rangle - \alpha\langle (1-\lambda T_2)\varphi,\varphi\rangle$$

for all $\varphi \in H$. It follows therefore that in the previous lemma we can set

$$A = \alpha(1-\lambda T_2), \qquad B = (1-\lambda T_1)$$

and conclude that:

$$\sum_{n=0}^{\infty} \lambda^n \langle T_1^n \varphi,\varphi\rangle \leqslant \frac{1}{\alpha} \sum_{n=0}^{\infty} \lambda^n \langle T_2^n \varphi,\varphi\rangle \tag{4.1}$$

for all $\varphi \in H$.

If now we assume that $f \in H$ is a fixed element of the Hilbert space for which:

$$\langle T_1^n f,f\rangle \geqslant 0; \qquad \langle T_2^n f,f\rangle \geqslant 0, \qquad n = 0,1,\ldots \tag{4.2}$$

we can let $\lambda \to 1$ in (4.1) and conclude that:

$$\sum_{n=0}^{\infty} \langle T_1^n f,f\rangle \leqslant \frac{1}{\alpha} \sum_{n=0}^{\infty} \langle T_2^n f,f\rangle \tag{4.3}$$

we shall apply the above when T_i, $i = 1,2,3$ are S-operators and T_2 a symmetric one. Any $f \in L^2(\Omega)$ that is non negative (i.e. $f(\omega) \geqslant 0$, $\forall \omega \in \Omega$) satisfies the conditions (4.2) and we deduce therefore at once:

Proposition. — *Let K_1, K_2 be two S-operators such that K_2 is symmetric and $K_1 \gg K_2$. Then there exists some $\alpha > 0$ s.t. for any non negative $f \in L^2(\Omega)$ we have:*

$$\sum_{n=0}^{\infty} \langle K_1^n f,f\rangle \leqslant \frac{1}{\alpha} \sum_{n=0}^{\infty} \langle K_2^n f,f\rangle .$$

Proof. – There exists some $0 < \alpha < 1$ for which $K_1 \geqslant \alpha K_2$ and for which therefore $K_1 = \alpha K_2 + (1-\alpha)K_3$ with $K_3 = \dfrac{K_1 - \alpha K_2}{1-\alpha}\cdot$ The inequality (4.3) therefore applies.

5. The Green's function.

Let M be a Riemannian manifold, let dV denote its canonical volume element and $p_t(x,y)$ its heat diffusion kernel. Let

$$G(x,y) = \int_0^\infty p_t(x,y)\,dt$$

be the Green's function on M. For every $\theta > 0$ I shall also define:

$$H_\theta(x,y) = \int_0^\theta p_t(x,y)\,dt; \qquad G_\theta(x,y) = \sum_{n=1}^\infty p_{n\theta}(x,y).$$

Both G and G_θ could be identically $= +\infty$. But if $G(x,y) < +\infty$ for some pair $x,y \in G$ then for every fixed $x_0 \in M$ we have $G(x_0,y) \in L^1_{\mathrm{loc}}(M;dV(y))$ (for this classical fact, cf. [2] or the general literature).

We have

$$\int_M H_\theta(x,y)\,dV(y) = \theta, \qquad \forall x \in M$$

and (with the possible interpretation $+\infty = +\infty$) we also have:

$$(5.1)\quad G(x,y) - H_\theta(x,y) = \int_\theta^\infty p_t(x,y)\,dt$$
$$= \int_M G_\theta(x,z)H_\theta(z,y)\,dV(z), \qquad \forall x,y \in M$$

$$(5.2)\qquad \int_\alpha^\beta G_\theta(x,y)\,d\theta = \sum_{n=1}^\infty \frac{1}{n}\int_{n\alpha}^{n\beta} p_t(x,y)\,dt;$$
$$x,y \in M, \qquad \beta > \alpha > 0.$$

An immediate consequence of (5.1) is that if $G(x,y) < +\infty$ then $G_\theta(x_0,y) \in L^1_{\mathrm{loc}}(M;dV(y))$ for all $x_0 \in M$ [observe that $H_\theta(x,y) > 0$, $\forall x$,

$y \in M]$ a converse assertion can be obtained with the use of (5.2). Indeed we obtain

$$\int_{\alpha}^{\beta} G_{\theta}(x,y)\, d\theta = \int_{0}^{\infty} \chi_{\alpha,\beta}(t) p_t(x,y)\, dt$$

with

$$\chi_{\alpha,\beta}(t) = [\Sigma\, 1/n; t/\beta \leqslant n \leqslant t/\alpha]$$

and the obvious estimate $\chi_{\alpha,\beta}(t) \sim \log \beta/\alpha$ for large t shows that the *uniform* boundedness of G_{θ} for $\theta \in [\alpha,\beta]$ in some interval implies the existence of the Green's function on M. More explicitly let us assume that $\varphi(x)$, $\psi(y) \geqslant 0$ are continuous *non* zero functions that satisfy

$$\int\int G_{\theta}(x,y)\varphi(x)\psi(y)\, dV(x)\, dV(y) \leqslant K, \qquad \forall \theta \in [\alpha,\beta]$$

for some $K > 0$ and $0 < \alpha < \beta$. Then the Green's function exists on M. We shall need this fact in the next section.

6. The transience of Brownian motion.

In this paragraph I shall need the following deep estimate on the Heat kernel of a Riemannian manifold.

Estimate [9], [10]. – Let $\tilde{M}$ be a complete connected Riemannian manifold and let us assume that the curvature of $\tilde{M}$ is uniformly bounded and that the injectivity radius is bounded from below.

For all $0 < \alpha < \beta$ then there exists $C > 0$, s.t.

$$(6.1)\quad \frac{1}{C} e^{-\frac{C_0 \tilde{d}^2}{t}} \leqslant \tilde{p}_t(\tilde{x},\tilde{y}) \leqslant C e^{-\frac{\tilde{d}^2}{C_0 t}}; \qquad \tilde{x}, \tilde{y} \in \tilde{M}, \qquad t \in [\alpha,\beta]$$

$\tilde{d} = \tilde{d}(\tilde{x},\tilde{y})$ is the distance on $\tilde{M}$ between the points $\tilde{x}$, $\tilde{y}$, and $C_0 > 0$ only depends on $\tilde{M}$ and not on α and β.

The above estimate holds in particular when $\tilde{M} \to M$ covers a compact manifold. Let us assume that we are in this situation and that G is the deck transformation group of $\tilde{M} \to M$. The following two estimates are then critical but very easy to obtain (cf. [11]):

$$\tilde{d}(\tilde{x}_0,\tilde{x}_g) \leqslant C_1|g| + C_4$$
$$|g| \leqslant C_2\tilde{d}(\tilde{x}_0,\tilde{x}_g) + C_3$$

where $g \in G$, $\tilde{x}_0 \in D$ (= some fixed compact fundamental domain) and $\tilde{x}_g \in gD$, and where C_i $(1 \leqslant i \leqslant 4)$ are positive constants independent of g, $\tilde{x}_0$, and $\tilde{x}_g$. Let now $\tilde{M} \to M$ be a normal Riemannian covering mapping with G as a group of deck transformations and M compact (G is then automatically finitely generated). Let us assume that μ is a symmetric probability measure on G that satisfies

$$\mu(\{g\}) \leqslant Ce^{-c|g|^2}; \qquad \sum_{n=0}^{\infty} \mu^n(\{1\}) < +\infty$$

for some C, $c > 0$. It follows then from the general theory that

$$\nu(\{g\}) = \sum_{n=0}^{\infty} \mu^n(\{g\}) < +\infty, \qquad \forall g \in G. \tag{6.2}$$

i.e. that ν is a measure on G (cf. [4]).

Let $\mathbf{1}(x,y) = 1$, $\forall x, y \in M$ and let $K = \mathbf{1} \otimes \mu$ the symmetric S-kernel that we can construct as in § 2 Example (iii).

It then follows from (6.1) and our hypothesis on μ that:

$$\tilde{p}_\theta(\tilde{x},\tilde{y}) \gg K(\tilde{x},\tilde{y}) \qquad \forall\theta \in [\alpha,\beta]$$

for some $0 < \alpha < \beta$. This is because for $\tilde{x} \in gD$, $\tilde{y} \in hD$ we have $\tilde{d}(\tilde{x},\tilde{y}) \approx |g^{-1}h|$.

The proposition in § 4 applies and we deduce therefore from (6.2) that

$$\sum_{n=1}^{\infty} \langle \tilde{p}_{n\theta}\tilde{f},\tilde{f}\rangle < K; \qquad \theta \in [\alpha,\beta]$$

for all $\tilde{f}$ that are positive, bounded and of compact support on $\tilde{M}$, and some $K > 0$ that depends on $\tilde{f}$. This and the final remark in § 5 implies that Brownian motion is transient on $\tilde{M}$, and proves one half of Theorem 1.

Remark. — Observe that if we make on μ the additional hypothesis that it is of compact support then we do not need in the above proof the estimate (6.1).

Conversely now let us assume that Brownian motion is transient and that μ is some probability measure on G s.t. $\mu(\{g\}) \geqslant Ce^{-c|g|^2}$ for some C, $c > 0$. Let $K(x,y) = \mathbf{1} \otimes \mu$ be the S-kernel constructed as before.

By the estimate (6.1) it follows then that $K \gg \tilde{p}_\theta$ for some $\theta > 0$. By the use of § 4 we conclude that $\sum_{n=0}^{\infty} \langle \mu^n f, f\rangle < +\infty$ for all f positive and of finite support on G. The second half of Theorem 1 follows.

We have at the same time proved the assertion that was implicit in the definition of recurrent groups.

7. The construction of a discrete Random walk.

Let G be a discrete group, let $a_1, a_2 \in G$ be a couple of generators of G, which we shall keep fixed throughout, and let $a_3 = a_2^{-1}a_1^{-1} \in G$. Let also D be the disjoint union of the three copies of the non negative integers $D_i = \{x_j^i\}$ $(i=1,2,3, j=0,1,\dots)$ where we identify $x_0^1 = x_0^2 = x_0^3 = x$ (but keep all the other points distinct).

We first define on D the random walk that sends any point x_j^i with $j \neq 0$, to one of the two points x_{j-1}^i, x_{j+1}^i with probability $1/2$ each and sends the point x to one of the three points x_1^1, x_1^2, x_1^3 with probability $1/3$ each.

Let then $\Theta = D \times G$ and let us define a new random walk on Θ by giving the transition matrix $P(\theta,\theta')$ $(\theta,\theta' \in \Theta)$

$$P((d,g), (d',g')) = p(d,d')\mu_{d,d'}(\{g^{-1}g'\})$$

where $p(d,d')$ is the transition matrix of the random walk we just defined on D and $\mu_{d,d'} \in M(G)$ is a measure on G

$$\mu_{d,d'} = \frac{1}{2}(\delta_{a_i} + \delta_{a_i^{-1}}) \qquad \text{for} \qquad d, d' \in D_i, \qquad i = 1,2,3.$$

The above matrix P is not symmetric (nor is p on D) but we can symmetrize it.

More explicitely let $\lambda \in M(D)$ be given by $\lambda(\{x\}) = 3/2$, $\lambda(\{d\}) = 1$, $\forall d \neq x \in D$ and let $\tilde{\lambda} = \lambda \otimes h \in M(\Theta)$ where h is the discrete Haar measure on G. Let also $\tilde{P}: C_0(\Theta) \to C_0(\Theta)$ be the linear transformation

$$\tilde{P}f(\theta) = \sum_{\theta' \in \Theta} P(\theta,\theta')f(\theta') \quad f \in C_0(\Theta)$$

$\tilde{\lambda}$ is then a symmetrizing measure in the sense that

$$\int_{\Theta} \tilde{P}f(\theta)g(\theta)\, d\tilde{\lambda}(\theta) = \int_{\Theta} f(\theta)\tilde{P}g(\theta)\, d\tilde{\lambda}(\theta), \qquad f, g \in C_0(\Theta).$$

Or, which amounts to the same thing, $\tilde{P}$ induces a self adjoint linear transformation on $L^2(\Theta; d\tilde{\lambda})$. Indeed the kernel of $\tilde{P}$ on $L^2(d\tilde{\lambda})$ is $Q(\theta,\theta') = P(\theta,\theta')\tilde{\lambda}(\theta')^{-1}$ and it is easily seen to be symmetric $Q(\theta,\theta') = Q(\theta',\theta)$. Q in fact induces a symmetric S-operator on $(\Theta; d\tilde{\lambda})$.

The question arises whether the above random walk is transient or not. It all depends, of course, on G and the two generators a_1, a_2, but I know of no good general answer to that.

PROPOSITION. — *Let* $G = \mathbf{Z}^2$ *and* $a_1 = (1,0)$, $a_2 = (0,1)$ *be the two canonical generators. Then the above random walk is transient.*

The proof is elementary I shall only give the outline : We clearly have $p^{(n)}(x,y) \leqslant \frac{C}{\sqrt{n}}$ $n = 1,2, \ldots$.

Let us now denote by n_i $(i=1,2,3)$ the number of edges (steps if you prefer) of the original random walk on D that lie on D_i (we start that walk at x). Then *conditional* that at time $2n$ we are back at x the probabilities P $[n_1 = 2k_1, n_2 = 2k_2 \| \text{cond.}]$ are independent of k_1, k_2 for all $k_1 + k_2 \leqslant n$. This follows from a repeated (but not entirely trivial) application of [13] III.*g* and it is a good Feller type of exercise.

These two facts put together give us at once the estimate :

$$Q^{(n)}[(x,1),(x,1)] \leqslant \frac{C}{n^{1+\varepsilon}}$$

(for some small $\varepsilon > 0$) where by 1 I denote the identity of G. The proposition follows.

The above proposition shows that G can be a non-transient group and yet give rise to a transient random walk on Θ.

If we know however that G is transient in the sense that there exists a symmetric probability measure μ on G with supp μ *finite* for which $\sum_{n=1}^{\infty} \mu^n(\{1\}) < +\infty$ then it is easy to see that our random walk on Θ is transient.

The best way to see that is to go back to the original random walk on D which at $t = 0$ starts at x and consider $0 = T_0 < T_1 < \cdots < T_n < \cdots$ the successive return times to the point x. Denote by $g_n \in G$ the random group element defined by $\theta(T_n) = (x, g_n)$ [I denote here by $\theta(m)$ the position of the random walk on $\Theta = D \times G$ at time $t = m$]. g_n performs a translation invariant symmetric random walk on G. Let Γ denote the probability distribution of g_1. It is easy to see that $Gp(\operatorname{supp}\Gamma) = G$. From this and our hypotheses on G if we use the machinery of § 4 we conclude that $\sum_{n=1}^{\infty} \Gamma^n(\{1\}) < +\infty$. Our assertion follows. [It is worth observing that the original argument of [8] was devised to cope with a situation just as above.] Γ can in fact be computed very easily, it is the sum of three Cauchy distributions on the each of the three subgroups $Gp(a_i)$ $i = 1,2,3$ of G. The Cauchy distribution fails to have a first moment and this explains the apparent discrepancy between our proposition and what has just been said.

8. The Riemann surface.

Let M be the Riemann surface that we obtain by removing three points z_i $(i = 1,2,3)$ from the unit sphere of $\mathbf{R}^3$. Let $B^i \subset M$ be three small disjoint ponctured discs centered at z_i $(i = 1,2,3)$. I shall give on each B^i the flat metric $ds^2 = \sin^{-2}[d(z,z_i)]\, ds_0^2$ [to help you see that the above ds^2 is flat observe that the length of all small circles centered at z_i is the same !] where ds_0^2 and $d(z,z_i)$ denote the metric and the angular distance induced by $\mathbf{R}^3$, I shall then extend the definition of ds^2 to the rest of M so as to obtain a smooth complete conformal Riemannian manifold. M looks like a sphere with three straight pipes welded to it. The main point of the above metric is that its restriction to each B^i is flat. I shall now subdivide each B^i into regions

$$(8.1) \qquad B_n^i = \{z \in B^i;\ \alpha_{n+1}^{(i)} < d(z,z_i) \leqslant \alpha_n^{(i)}\}; \quad n = 1,2,\ldots,$$

and for convenience, I shall denote the complement

$$C = M \backslash \cup \{B_n^i; i=1,2,3, n=1,2,\ldots\}$$

by B_0^i for *any* $i = 1,2,3$. A proper choice of the above subdivision and a possible renormalization of the metric will ensure me that:

$$\mathrm{Vol}(C) = 3/2; \qquad \mathrm{Vol}(B_n^i) = 1; \quad i = 1,2,3, \quad n = 1,2,\dots.$$

Let now $\tilde{M} \to M$ be a covering of the above surface and let G be the deck transformation group. We must think of G as a quotient of $\pi_1(M)$ this allows us to identify two generators of G, say a_1 and a_2, which correspond to the loops of $\pi_1(M)$ that go once around z_i $(i = 1,2)$ in a clockwise direction. I shall fix then a fundamental domain in $\tilde{M}$ by cutting M from z_i $(i = 1,2,3)$ to some fixed $u \notin B^1 \cup B^2 \cup B^3$ along great circles. Using that fundamental domain I shall then identify $\tilde{M}$ with $M \times G$. From the subdivision (8.1) of M I can obtain then a subdivision of $\tilde{M}$

$$\begin{cases} gB_n^i;\ g \in G,\ i = 1,2,3,\ n = 1,2,\dots \\ gC;\ g \in G. \end{cases}$$

The above subdivision of $\tilde{M}$ allows me to define two mappings

$$T:\ L^2(\tilde{M};d\tilde{V}) \to L^2(\Theta;d\tilde{\lambda}); \qquad T^*:\ L^2(\Theta;d\tilde{\lambda}) \to L^2(\tilde{M};d\tilde{V})$$

where Θ and $d\tilde{\lambda}$ are as in the previous paragraph.

$$Tf[(x_n^i,g)] = \frac{1}{\mathrm{Vol}(B_n^i)} \int_{gB_n^i} f(\tilde{m})\, d\tilde{V}(\tilde{m})$$

$$T^*f(\tilde{m}) = f[(x_n^i, g)] \qquad \text{if} \qquad \tilde{m} \in gB_n^i.$$

The above two mapping are clearly norm decreasing they are adjoint to each other and $TT^* = I\,\partial\Theta$.

Using the two generators $a_i (i = 1,2)$ of G defined above I can then define $Q(\theta,\theta')$ as in the previous section and also $K = T^*QT$. K is then a symmetric S-operator on $L^2(\tilde{M})$ (in the sense of § 2). The best way to see that it is an S-operator is to extend the definition of T and T^* to L^p $(1 \leqslant p \leqslant +\infty)$ and observe that K (together with its adjoint) is positive, norm decreasing and has $K1 = 1$.

Let now $\tilde{p}_1(\tilde{x},\tilde{y})(\tilde{x},\tilde{y} \in \tilde{M})$ be the heat diffusion kernel on $\tilde{M}$ at time $t = 1$. We have then:

PROPOSITION. — *If we denote by* K *again the kernel of the operator* T^*QT *on* $\tilde{M}$ *we then have* $\tilde{p}_1 \gg K$.

Proof. — By looking at the Geometry of M and of the subdivision (8.1) we see from the definition of Q that there exists some $A > 0$ such that :

$$\begin{cases} K(\tilde{x},\tilde{y}) = 0; & \tilde{x},\tilde{y} \in \tilde{M}, \quad \tilde{d}(\tilde{x},\tilde{y}) > A \\ |K(\tilde{x},\tilde{y})| < A \end{cases}$$

where $\tilde{d}$ denotes the distance on $\tilde{M}$ induced by our metric ds^2 on M.

The proposition then follows from the estimate :

$$\inf \{\tilde{p}_1(\tilde{x},\tilde{y});\ \tilde{d}(\tilde{x},\tilde{y}) < A\} > 0 .$$

The above estimate is a consequence of the estimates of § 6, but of course in this case a direct proof can be given by the standard argument of following Brownian motion through a chain of discs and using the Markov property.

THE LYONS-MCKEAN THEOREM. — *The transience of Brownian motion on* $\tilde{M}$ *when* $G = \pi_1/[\pi_1,\pi_1] \cong \mathbf{Z}^2$ *follows at once from the above proposition together with the proposition in* § 7 *and the machinery developed in* § 4 *and* § 5.

9. Concluding remarks.

Without proofs I shall state here some further partial results that can be obtained in the context of § 8 (using the same methods).

PROPOSITION. — *Let* $\tilde{M} \to M$ *be the Riemann surfaces as in* § 8, *let* G *be the deck transformation group and let us assume that Brownian motion is transient on* $\tilde{M}$. *Then :*

(i) $\sum_{n=0}^{\infty} \frac{1}{\gamma(n)} < +\infty$ *where* $\gamma(n)$ *is the growth function of* G,

(ii) *For every* μ *probability measure on* G *that satisfies*

$$\mu(\{g\}) \geqslant Ce^{-c|g|^2} \ (g \in G)$$

for some $C,\ c > 0$ *we have*

$$\sum_{n=1}^{\infty} \frac{\mu^n(\{1\})}{\sqrt{n}} < +\infty .$$

It is tempting to conjecture that both (i) and (ii) are also sufficient conditions for the transience of Brownian motion on $\tilde{M}$. This however I have not been able to prove.

Appendix.

Let G be a group that is finitely generated and that is transient in the sence of § 0 i.e. there exists $\mu_0 \in \mathbf{P}(G)$ a symmetric probability measure on G such that $\mu_0(\{g\}) \leqslant Ce^{-c|g|^2}$ $(\forall g \in G)$ for some C, $c > 0$ and such that

$$\sum_{n=0}^{\infty} \mu_0^{*n}(\{1\}) < +\infty;$$

then G is also transient for the more standard definition (or for any other reasonable definition). Indeed we have:

PROPOSITION. – *Let* G *be as above, let* $g_1, \ldots, g_k \in G$ *be a finite set of generators of* G *and let* $\nu \in \mathbf{P}(G)$ *a probability measure on* G *that satisfies* $\nu(g_j^{\pm 1}) > 0$ $(j=1,\ldots,k)$ *then we have*

$$\sum_{n=0}^{\infty} \nu^{*n}(\{1\}) < +\infty.$$

Proof (I shall be brief). – Let ν be as above, then the measure $\lambda = e^{-1}e^{\nu} \in \mathbf{P}(G)$ satisfies

$$\lambda(\{g\}) \geqslant \frac{1}{A}|g|^{-|g|}e^{-A|g|} \geqslant \frac{1}{\alpha}e^{-\alpha|g|^2}; \qquad \forall g \in G, \quad (g \neq e)$$

for some A, $\alpha > 0$. From this and what we have already proved it follows that

$$\sum_{n=0}^{\infty} \lambda^{*n}(\{1\}) < +\infty.$$

It remains to observe that:

$$\sum_{n=0}^{\infty} \nu^{*n} \leqslant C \sum_{n=0}^{\infty} \lambda^{*n}$$

for some numerical $C > 0$. $\left[\text{Expand } \lambda^{*n} = e^{-n}\sum_{n=0}^{\infty}\frac{(n\nu)^p}{p!}\right.$ and substitute].

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