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FAMILIES OF FUNCTIONS DOMINATED BY DISTRIBUTIONS OF $\mathscr{C}$-CLASSES OF MAPPINGS

by **Goo ISHIKAWA**

Introduction.

A subsheaf $\mathscr{D}$ of the sheaf $\mathscr{E}_\Omega$ of $\mathbf{R}$-valued C^∞ functions over an open subset Ω in $\mathbf{R}^n$ is called a *sheaf of sub* C^∞*-rings* of $\mathscr{E}_\Omega$ if, for each open subset U of Ω, for each $h_1, \ldots, h_r \in \mathscr{D}(U)$ and for each C^∞ function τ on $\mathbf{R}^r$, the composition $\tau \circ (h_1, \ldots, h_r) \in \mathscr{E}_\Omega(U)$ also belongs to $\mathscr{D}(U)$.

Our problem is to establish certain conditions for « finite presentability » of the sheaf of sub C^∞-rings $\mathscr{D}_f$ of $\mathscr{E}_\Omega$, which will be called the *family of functions dominated by the distribution of* $\mathscr{C}$*-classes of a* C^∞ *mapping* $f : \Omega \to \Omega'$ into an open subset Ω' of $\mathbf{R}^p$, and is defined as follows.

At first, for each point $x \in \Omega$, we take the proper ideal

$$\mathscr{C}_{f,x} = (f_{1,x} - f_1(x), \ldots, f_{p,x} - f_p(x)) \,.\, \mathscr{E}_x$$

of $\mathscr{E}_x = \mathscr{E}_{\Omega,x}$, where f_i is the i-th component of f. Then we define $\mathscr{D}_f$ by

$$\begin{aligned} \mathscr{D}_f(U) &= \{h \in \mathscr{E}_\Omega(U); \quad \mathscr{C}_{h,x} \subseteq \mathscr{C}_{f,x}, \quad \text{for all} \quad x \in U\} \\ &= \{h \in \mathscr{E}_\Omega(U); \quad h_x - h(x) \in \mathscr{C}_{f,x}, \quad \text{for all} \quad x \in U\}, \end{aligned}$$

for each open subset U of Ω. This restriction $h \in \mathscr{D}_f(U)$ on a function h is an analogue (in the case $n \leqslant p$) to the condition of R. Moussu-J.-Cl. Tougeron [7] that $df_1 \wedge \cdots \wedge df_p \wedge dh = 0$ (on the regular locus of f) in the case $n > p$. We note that $\mathscr{D}_f = \mathscr{E}_\Omega$ if and only if f is an immersion.

In [5], J. N. Mather introduced the notion of $\mathscr{C}$-equivalence of map-germs. The ideal $\mathscr{C}_{f,x} = f_x^*(m_{\Omega,f(x)}).\mathscr{E}_x$ represents the contact of graph(f) with $\Omega \times \{f(x)\}$ at $(x,f(x))$ in $\Omega \times \Omega'$.

Our problem is closely related to the problem of composite differentiable functions originated by G. Gleaser and treated by J.-Cl. Tougeron, J. Merrien, J. J. Risler, E. Bierstone-P. D. Milman and many authors. In fact, an interesting application of a work of J. Merrien [6] appears in § 2. However notice that our treatment of functions and mappings is local not in targets but in sources.

In § 1, we exactly formulate our result – Theorem 1.7. This theorem shows that a partial investigation of $\mathscr{D}_f$ is reduced to the structure of certain sheaf of ideals $\mathscr{I}_f$ of $\mathscr{E}_\Omega$ or $\mathscr{I}_f^\omega$ of $\mathscr{O}_\Omega$. Combined with the deep investigations on sheaves of ideals of $\mathscr{E}_\Omega$ by H. Whitney, B. Malgrange, J. Cl. Tougeron and many authors (cf. [4], [9]), a merit of Theorem 1.7 would come out.

We treat, in § 2, a key proposition (2.2) for the proof of Theorem 1.7. Proposition 2.2 seems to be interesting in itself.

We prove Theorem 1.7 in § 3.

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1. Main result.

Let Ω be an open subset of $\mathbf{R}^n$. We denote $\mathscr{E} = \mathscr{E}_\Omega$ (resp. $\mathscr{O} = \mathscr{O}_\Omega$) the sheaf of germs of $\mathbf{R}$-valued C^∞ (resp. analytic) functions on Ω. Each stalk $\mathscr{E}_x$ of $\mathscr{E}$ is an $\mathbf{R}$-algebra with the unique maximal ideal m_x, that is, germs with zero target. We write m_x^∞ the ideal $\bigcap_{\ell \in \mathbf{N}} m_x^\ell$ of ∞-flat germs. For each open subset U of Ω and each $x \in U$, $\pi_{U,x}$ means the canonical mapping $\mathscr{E}(U) \to \mathscr{E}_x$ defined by taking of germs of functions at x.

Let T be a subset of $\mathscr{E}_{\Omega,x}$ for a point $x \in \Omega$. We denote $\langle T \rangle$ the set of compositions $\tau \circ k$ of a C^∞ function τ on $\mathbf{R}^r$ with a germ $k : \Omega, x \to \mathbf{R}^r$ of a C^∞ mapping with components $k_i \in T$ $(1 \leqslant i \leqslant r)$.

DEFINITION 1.1. – *A subset* T *of* $\mathscr{E}_{\Omega,x}$ *is called a sub* C^∞*-ring of* $\mathscr{E}_{\Omega,x}$ *if* $\langle T \rangle = T$. *A sub* C^∞*-ring* T *of* $\mathscr{E}_{\Omega,x}$ *is finitely generated* (*resp. formally* *finitely generated) if* $\mathrm{T} = \langle k_1, \ldots, k_e \rangle$ *(resp.* $\mathrm{T} \subseteq \langle k_1, \ldots, k_e \rangle + m_x^\infty$*) with a finite system of elements* $k_1, \ldots, k_e$ *of* T.

Remark 1.2. – Compare the definition of sheaves of sub C^∞-rings of $\mathscr{E}_\Omega$ in Introduction with that of sub C^∞-rings. Futhermore, we can define the category of C^∞-*rings* (see [1], for example). Typical examples of C^∞-rings are $\mathscr{E}_x$, $\mathscr{F}_x = \mathscr{E}_x/m_x^\infty$ and $\mathscr{E}(\mathrm{U})$ (U is an open subset of Ω). Naturally we can also define the notion of *sheaves of* C^∞-*rings*. A sheaf of sub C^∞-rings of $\mathscr{E}_\Omega$ is just a subobject of the sheaf of C^∞-rings $\mathscr{E}_\Omega$.

Let $\mathscr{D}$ be a sheaf of sub C^∞-ring of $\mathscr{E}_\Omega$.

DEFINITION 1.3. – *We call a subset* S *of* D(U) *for an open subset* U *of* Ω *a system of generators (resp. formal generators) of* D *over* U *if for each point* $\tilde{x}$ *of* U, $\mathscr{D}_{\tilde{x}} = \langle \pi_{\mathrm{U},\tilde{x}}(\mathrm{S}) \rangle$ *(resp.* $\mathscr{D}_{\tilde{x}} \subseteq \langle \pi_{\mathrm{U},\tilde{x}}(\mathrm{S}) \rangle + m_{\tilde{x}}^\infty$*), where* $\mathscr{D}_{\tilde{x}}$ *is the stalk of* $\mathscr{D}$ *over* $\tilde{x}$. *We say that* $\mathscr{D}$ *is finitely generated (resp. formally finitely generated) as a sheaf of sub* C^∞*-rings at a point* q *of* Ω *if* $\mathscr{D}$ *has a finite system of generators (resp. formal generators) over an open neighborhood of* q *in* Ω.

DEFINITION 1.4. – *The sheaf of relations of elements* $h_1, \ldots, h_e$ *of* $\mathscr{D}(\mathrm{U})$ *for an open subset* U *of* Ω *is the kernel of the canonical homomorphism* $h^*\mathscr{E}_{\mathbf{R}^e} \to \mathscr{D}_{|\mathrm{U}}$ *induced by the* C^∞ *mapping* $h = (h_1, \ldots, h_e) : \mathrm{U} \to \mathbf{R}^e$, *which is a sheaf of ideals of* $h^*\mathscr{E}_{\mathbf{R}^e}$. *We say that* $\mathscr{D}$ *is finitely presented as a sheaf of sub* C^∞*-rings at a point* q *of* Ω *if there exists a finite system of generators* $h_1, \ldots, h_e$ *over an open neighborhood* U *of* q *in* Ω *such that the sheaf of relations of* $h_1, \ldots, h_e$ *is of finite type at* q *as a sheaf of ideals of* $h^*\mathscr{E}_{\mathbf{R}^e}$.

Remark 1.5. – (1) The condition that $\mathscr{D}$ is finitely generated at q is different from that $\mathscr{D}_q$ is finitely generated.

(2) If $\mathscr{D}$ is finitely presented at q, then there is a natural isomorphism

$$h^*\mathscr{E}_{\mathbf{R}^e}/(g_1, \ldots, g_s) \, . \, h^*\mathscr{E}_{\mathbf{R}^e} \xrightarrow{\sim} \mathscr{D}_{|\mathrm{U}'}$$

(of sheaves of C^∞-rings) over an open neighborhood U′ of q, where $h = (h_1, \ldots, h_e) : \mathrm{U}' \to \mathbf{R}^e$, $h_i \in \mathscr{D}(\mathrm{U}')$ $(1 \leqslant i \leqslant e)$ and $g_i \in h^*\mathscr{E}_{\mathbf{R}^e}(\mathrm{U}')$ $(1 \leqslant i \leqslant s)$.

Let Ω (resp. Ω') be an open subset of $\mathbf{R}^n$ (resp. $\mathbf{R}^p$) and $f : \Omega \to \Omega'$ a finite C^∞ mapping (f is finite if, for any point $x \in \Omega$, $\mathscr{E}_{\Omega,x}$ is a finite $\mathscr{E}_{\Omega',f(x)}$-module via $f_x^* : \mathscr{E}_{\Omega',f(x)} \to \mathscr{E}_{\Omega,x}$).

In order to formulate Theorem 1.7, we define an auxiliary sheaf of ideals $\mathscr{I}_f$ of $\mathscr{E}_\Omega$ from the family of ideals $(\mathscr{C}_{f,x})_{x\in\Omega}$ (see Introduction) by

$$\mathscr{I}_f(\mathrm{U}) = \bigcap_{x\in \mathrm{U}} \pi_{\mathrm{U},x}{}^{-1}(\Delta_n \mathscr{C}_{f,x}), \tag{1.6}$$

for each open subset U of Ω, where $\Delta_r \mathrm{I} (0 \leqslant r \leqslant n)$ is the r-th Jacobian extension of an ideal I of $\mathscr{E}_x$:

$$\Delta_r \mathrm{I} = \mathrm{I} + \left(\det \frac{(g_1, \ldots, g_r)}{(x_{i_1}, \ldots, x_{i_r})};\ g_i \in \mathrm{I},\ 1 \leqslant i_1 < \cdots < i_r \leqslant n \right) . \mathscr{E}_x .$$

We can regard $\mathscr{I}_f$ as an infinitesimal version of $\mathscr{D}_f$. Further we define a sheaf of ideals $\mathscr{I}_f^\omega$ of $\mathscr{O}_\Omega$ by

$$\mathscr{I}_f^\omega(\mathrm{U}) = \mathscr{I}_f(\mathrm{U}) \cap \mathscr{O}_\Omega(\mathrm{U}),$$

for each open subset U of Ω.

THEOREM 1.7. – *Assume that f is analytic. Then, for a point q of Ω with the rank of Jacobian $df(q) \geqslant n-1$, the following conditions are equivalent to each other :*

(A) *$\mathscr{I}_f$ is formally generated by a finite number of analytic sections over an open neighborhood of q (as a sheaf of ideals of $\mathscr{E}_\Omega$).*

(B) *$\mathscr{D}_f$ is finitely generated by analytic sections at q (as a sheaf of sub C^∞-rings of $\mathscr{E}_\Omega$).*

(C) *$\mathscr{D}_f$ is finitely presented by analytic sections at q.*

Furthermore, the condition (A) *is also equivalent to one of the followings :*

(A′) *$\mathscr{I}_f^\omega$ is of finite type at q (as a sheaf of ideals of $\mathscr{O}_\Omega$).*

(A″) *For each point $\tilde{q}$ in a neighborhood of q, $\mathscr{I}_{f,\tilde{q}}$ is finitely generated as an ideal of $\mathscr{E}_{\tilde{q}}$.*

In Theorem 1.7, the restriction on the rank of the Jacobian of f at q is essential. Without this restriction, the relation between $\mathscr{I}_f$ and $\mathscr{D}_f$ seems to disappear.

2. Inverse images of sheaves of ideals by a non-singular vector field.

Let Ω be an open subset of $\mathbf{R}^n$, D a C^∞ vector field, $\mathscr{I}$ a sheaf of ideals of $\mathscr{E}_\Omega$ and q a point in Ω. Throughout this section, we use these notations.

An element g of $\mathscr{E}_{\Omega,q}$ is called D_q-*regular* if $(D_q^k g)(q) \neq 0$ for a non-negative integer k, where $D_q^k g$ is the element of $\mathscr{E}_{\Omega,q}$ obtained by operating k times the derivation D_q to g.

We put, for each open subset U of Ω,

$$(2.1) \qquad (D^{-1}\mathscr{I})(U) = D^{-1}(\mathscr{I}(U)) \quad (\subseteq \mathscr{E}_\Omega(U)).$$

Notice that the presheaf $D^{-1}\mathscr{I}$ is a sheaf of sub C^∞-rings of $\mathscr{E}_\Omega$.

The following proposition treats of structure of $D^{-1}\mathscr{I}$.

PROPOSITION 2.2. — *Suppose that* D *is non-singular at* q, *that* $\mathscr{I}$ *is formally generated by a finite number of analytic sections over an open neighborhood of* q *and that* $\mathscr{I}_q$ *contains an analytic* D_q-*regular element. Then there exists a finite system of generators* $h_1, \ldots, h_e \in (D^{-1}\mathscr{I})(U) \cap \mathscr{O}_\Omega(U)$ *of* $D^{-1}\mathscr{I}$ *over an open neighborhood* U *of* q *with the following properties :*

(1) *For each open subset* $U' \subseteq U$ *and each* $\xi \in D^{-1}\mathscr{I}(U')$, *there exists a* C^∞ *function* τ *on an open neighborhood of* $h(U')$ *such that* $\xi = \tau \circ h$ *in* U.

(2) *The sheaf of relations of* $h_1, \ldots, h_e$ *is of finite type.*

This proposition will be used in the proof of (A) $\Rightarrow$ (C) of Theorem 1.7.

To make the assertion of Proposition 2.2 more clear, we mention several observations.

Remark 2.3. — Suppose that D is *non-singular* at q.

(1) If $D^{-1}\mathscr{I}$ is finitely generated at q as a sheaf of sub C^∞-rings, then $\mathscr{I}$ is of finite type at q.

(2) If $(D^{-1}\mathscr{I})_q$ is finitely generated, then $\mathscr{I}_q$ is finitely generated.

(3) Suppose that $\mathscr{I}_q \nsubseteq m_q^\infty$. Then $(D^{-1}\mathscr{I})_q$ is formally finitely generated if and only if $\mathscr{I}_q$ contains a D_q-regular element.

(4) If $\mathscr{I}_q \subseteq m_q^\infty$ and $\mathscr{I}_q \neq \{0\}$, then $(D^{-1}\mathscr{I})_q$ is formally finitely generated, but is never finitely generated.

(5) $D^{-1}\{0\}$ is finitely generated.

(6) Even if $\mathscr{I}_q$ contains an analytic D_q-regular element, the C^∞-ring $(D^{-1}\mathscr{I})_q$ is not necessarily finitely generated (see Example 3.7).

The rest of this section is devoted to the proof of Proposition 2.2. Firstly we prepare several necessary lemmas.

LEMMA 2.4. – *Let* U' *(resp.* V'*) be an open subset of* $\mathbf{R}^n$ *(resp.* $\mathbf{R}^e$*), and* $H' : U' \to V'$ *be a proper finite analytic mapping. Then we have*

$$H'^*(\mathscr{E}(V')) = H'^*(\mathscr{E}(V'))^{\wedge},$$

where $H'^*(\mathscr{E}(V'))^{\wedge}$ *is the set of elements* $h \in \mathscr{E}(U')$ *such that, for any* $y \in V'$, $h - H'^*(k)$ *is* ∞*-flat at* $H'^{-1}(y)$ *for some* $k \in \mathscr{E}(V')$ *depending on* y.

For the proof of Lemma 2.4, see [6] and [8].

Let A denotes $\mathscr{E}_{\mathbf{R}^{n-1},\tilde{q}'}$ for a point $\tilde{q}' \in \mathbf{R}^{n-1}$.

LEMMA 2.5. – *Let* $P = \sum_{j=0}^{s} b_j t^{s-j}$ *be a polynomial with coefficients* $b_j \in A$ $(0 \leqslant j \leqslant s)$ *and with a variable* t. *We define* $F_k \in A[t]$ $(k=0,1,2,\ldots)$ *by* $\partial F_k/\partial t = t^k.P$ *and* $F_{k|\{t=0\}} = 0$:

$$(1) \qquad F_k = \sum_{i=0}^{s} \frac{1}{s+k+1-i} b_i t^{s+k+1-i}.$$

If $b_0(q') \neq 0$, *then there exists* $\tau_k \in \mathscr{E}_{\mathbf{R}^{n-1+(3s+1)},0}$ $(k=0,1,2,\ldots)$ *such that*

(2) $F_k = \tau_k \circ (x_1 - x_1(\tilde{q}'), \ldots, x_{n-1} - x_{n-1}(\tilde{q}'), F_0, F_1, \ldots, F_{3s})$,

(3) $\mathrm{ord}_0 \tau_k \to +\infty$ *(as* $k \to +\infty$*)*,

where $\mathrm{ord}_0 \tau_k$ *means the supremum of* r *such that* $\tau_{k,0} \in m_0^r$.

LEMMA 2.6. – *Let* F_k *be as above. Then we have a formula*

$$(1) \quad F_k F_u = \sum_{i=1}^{s+1} \left(\frac{1}{k+i} + \frac{1}{u+i}\right) b_{s+1-i} F_{k+u+i}, \qquad (k, u \geqslant 0, k+u \geqslant s+1).$$

Further there exist rational numbers $\beta_{k,u}$, $(k, u \geqslant 0, k+u=m)$, *for any* m, *such that*

$$(2) \quad b_0 F_{m+s+1} = \sum_{k,u \geqslant 0, k+u=m} \beta_{k,u} F_k F_u, \qquad (m \geqslant 2s).$$

Proof. — It is clear that

$$\begin{aligned}\partial(\mathrm{F}_k\mathrm{F}_u)/\partial t &= \mathrm{F}_k.t^u.\mathrm{P} + t^k.\mathrm{P}.\mathrm{F}_u = (\mathrm{F}_k.t^u + \mathrm{F}_u.t^k).\mathrm{P} \\ &= \sum_{i=0}^{s}\left(\frac{1}{s+k+1-i} + \frac{1}{s+u+1-i}\right) b_i t^{s+k+u+1-i}\mathrm{P}.\end{aligned}$$

Thus, integrating with t, we have

$$\mathrm{F}_k\mathrm{F}_u = \sum_{i=0}^{s}\left(\frac{1}{s+k+1-i} + \frac{1}{s+u+1-i}\right) b_i \mathrm{F}_{s+k+u+1-i},$$

which is just (1). Let $\beta_{k,u}$ be indeterminants. From (1),

$$\sum_{k+u=m} \beta_{k,u}\mathrm{F}_k\mathrm{F}_u = \sum_{i=1}^{s+1}\left[\sum_{k+u=m}\left(\frac{1}{k+i} + \frac{1}{u+i}\right)\beta_{k,u}\right] b_{s+1-i}\mathrm{F}_{m+i}.$$

Now, if the system of equations of $(\beta_{k,u})$

$$\sum_{k+u=m}\left(\frac{1}{k+i} + \frac{1}{u+i}\right)\beta_{k,u} = 0, \qquad (1 \leqslant i \leqslant s),$$

$$\sum_{k+u=m}\left(\frac{1}{k+s+1} + \frac{1}{u+s+1}\right)\beta_{k,u} = 1,$$

has a solution $(\beta_{k,u})$ $(\beta_{k,u} \in \mathbf{Q})$, then we have the equality (2) for this solution. Hence it is sufficient to show that the $(s+1) \times (s+1)$-matrix M, (k,i) component of which is $1/(k+i-1) + 1/(m-k+i+1)$, belongs to $\mathrm{GL}(s+1,\mathbf{Q})$ when $m \geqslant 2s$. In fact,

$$\det \mathrm{M} = \prod_{t=0}^{s}\left[\frac{(t!)^4(m+2t+2)!(m-t)!}{(2t)!(2t+1)!(m+t+1)!^2}\prod_{j=t}^{2t-1}(m-j)\right] > 0.$$

Proof of Lemma 2.5. — As b_0 is invertible in A, we have, by Lemma 2.7(2),

$$(*) \qquad \mathrm{F}_{m+s+1} = \sum_{k,u \geqslant 0,\, k+u=m} b_0^{-1}\beta_{k,u}\mathrm{F}_k\mathrm{F}_u, \qquad (m \geqslant 2s),$$

for some $\beta_{k,u} \in \mathbf{Q}$. Now, for k and u with $k, u \geqslant 0, k+u = m$ for sufficiently large m, F_k or F_u can be represented as a quadratic homogeneous polynomial of F_i's with smaller indices i with coefficients in A using (*) again. Substitute this for F_k or F_u in (*). If we iterate this operation, we have polynomials T_m $(m=0,1,2,\ldots)$ such that the orders of them increase to infinity when $m \to \infty$, and that

$$F_{m+s+1} = T_m(F_0, F_1, \dots, F_{3s}), \qquad (m \geqslant 2s).$$

Thus if we take T_m as τ_{m+s+1} regarding as an element of $\mathscr{E}_{\mathbf{R}^{n-1+(3s+1)},0}$, then the conditions (2) and (3) of Lemma 2.5 are satisfied.

To show that a certain mapping is injective, we prepare the following lemma which is not hard to see.

LEMMA 2.7. – *Let* $P \in \mathbf{C}[z]$ *be a polynomial of degree* s. *Then the mapping* $\varphi : \mathbf{C} \to \mathbf{C}^{s+1}$ *defined by*

$$\varphi(z) = \left(\int_0^z P(t)\,dt, \int_0^z tP(t)\,dt, \dots, \int_0^z t^s P(t)\,dt\right)$$

is injective.

We need a lemma on the sheaf of relations of a mapping of a certain type.

LEMMA 2.8. – *Let* U (*resp.* V) *be an open subset of* $\mathbf{R}^n$ (*resp.* $\mathbf{R}^p$). *Let* $H : U \to V$ *be an analytic mapping which have a proper injective complexification* $H_C : \tilde{U} \to \tilde{V}$. *Then the kernel of the natural homomorphism* $H^*\mathscr{E}_V \to \mathscr{E}_U$ (*of sheaves of* C^∞*-rings*) *induced by* H *is of finite type as a sheaf of ideals of* $H^*\mathscr{E}_V$.

Proof. – Since H_C is proper, the direct image $(H_C)_*(\mathcal{O}_{\tilde{U}})$ of $\mathcal{O}_{\tilde{U}}$ is a coherent $\mathcal{O}_{\tilde{V}}$-module. Furthermore, since the inverse images by H and H_C of a point in $H(U)$ are coincident, it is easy to verify that $H_*(\mathcal{O}_U)$ is a coherent $\mathcal{O}_V$-module, taking real and imaginary parts of generators of $(H_C)_*(\mathcal{O}_{\tilde{U}})$. We take the $\mathcal{O}_V$-homomorphism $\alpha : \mathcal{O}_V \to H_*(\mathcal{O}_U)$ and the $\mathscr{E}_V$-homomorphism $\beta : \mathscr{E}_V \to H_*(\mathscr{E}_U)$ induced by H. Now $\ker\alpha$, which is the sheaf of ideals of germs of analytic functions vanishing on $H(U)$, is coherent. Thus, by a theorem of Malgrange ([4], Theorem VI.3.10), $\ker\beta$ is an $\mathscr{E}_V$-module of finite type. We consider the following commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathscr{R} & \longrightarrow & H^*(\mathscr{E}_V) & \longrightarrow & \mathscr{E}_U \\
& \downarrow & & \wr\downarrow & & \wr\downarrow \\
0 \longrightarrow & \ker\beta_{|H(U)} & \longrightarrow & \mathscr{E}_{V|H(U)} & \longrightarrow & H_*(\mathscr{E}_U)_{|H(U)}
\end{array}$$

where $\mathscr{R}$ is the sheaf of relations of the components of H, and the second and third columns are isomorphism of ringed-spaces over the homeomorphism $H : U \to H(U)$. Hence $\mathscr{R}$ is a $H^*(\mathscr{E}_V)$-module of finite type.

Proof of Proposition 2.2. – There is an open neighborhood U of q such that $D = \partial/\partial x_n$ for a system of analytic coordinates $x_1, \ldots, x_n$ over U centered at q. From assumption, shrinking U if necessary, $\mathscr{I}$ is formally generated over U by some elements $g_1, \ldots, g_m \in \mathscr{I}(U) \cap \mathscr{O}_\Omega(U)$. Let g_0 be an analytic D_q-regular element of $\mathscr{I}_q$. By the preparation theorem, we may assume that g_0 is a monic pseudopolynomial with coefficients in $\mathscr{O}(\pi(U))$ of degree s in x_n after we take U smaller, where $\pi : U \to \mathbf{R}^{n-1}$ is the projection to the first $(n-1)$-components. By the division theorem,

$$g_i = Q_i g_0 + \sum_{j=1}^{s} A_{ij} x_n^{s-j}, \qquad (1 \leqslant i \leqslant m),$$

for $Q_i \in \mathscr{O}(U)$ and $A_{ij} \in \mathscr{O}(\pi(U))$, $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant s)$, taking U smaller again if necessary. If we put

$$P_0 = g_0, \qquad P_i = g_0 + \sum_{j=1}^{s} A_{ij} x_n^{s-j}, \qquad (1 \leqslant i \leqslant m),$$

then $P_0, P_1, \ldots, P_m \in \mathscr{I}(U) \cap \mathscr{O}(U)$ also generate $\mathscr{I}$ over U formally, and each P_i is of the following type :

$$\text{(1)} \qquad P_i = x_n^s + \sum_{j=1}^{s} B_{ij} x_n^{s-j}, \qquad (0 \leqslant i \leqslant m)$$

where $B_{ij} \in \mathscr{O}(\pi(U))$.

Now we define an element $F_{ik}^{\tilde{q}}$ $(0 \leqslant i \leqslant m, k = 0,1,2,\ldots)$ of $(D^{-1}\mathscr{I})(U)$, for each point $\tilde{q}$ of U, by

$$\text{(2)} \qquad \begin{cases} D(F_{ik}^{\tilde{q}}) = (x_n - x_n(\tilde{q}))^k . P_i, \\ F^{\tilde{q}}_{ik|\{x_n = x_n(\tilde{q})\}} = 0 . \end{cases}$$

As $D = \partial/\partial x_n$, $F_{ik}^{\tilde{q}}$ is well defined. Further we put

$$H^{\tilde{q}} = (x_1 - x_1(\tilde{q}), \ldots, x_{n-1} - x_{n-1}(\tilde{q}), \quad F_{ik}^{\tilde{q}} \, (0 \leqslant i \leqslant m, 0 \leqslant k \leqslant 3s)) : U \to \mathbf{R}^e$$

where $e = n - 1 + (m+1)(3s+1)$, and for $\tilde{q} = q$, $H = H^q$. Then we shall show that

(3) : If one takes the open neighborhood U of q smaller if necessary, then for each open subset U′ of U,

$$(\mathrm{H}_{|\mathrm{U}'})^*(\mathscr{E}(\mathbf{R}^e))^{\wedge} \supseteq \mathrm{D}^{-1}\mathscr{I}(\mathrm{U}').$$

For the definition of $(\mathrm{H}_{|\mathrm{U}'})^*(\mathscr{E}(\mathbf{R}^e))^{\wedge}$, see Lemma 2.4.

We will prove Proposition 2.2 supposing (3). At first we see that

(4) H is a finite analytic mapping.

This is obvious because already $(x_1, \ldots, x_{n-1}, \mathrm{F}_{0,0})$ is finite. Next we see, for a complexification $\mathrm{Hc} : \tilde{\mathrm{U}} \to \mathbf{C}^e$ of H, that

(5) $\mathrm{H_C}$ is injective on a neighborhood of q in $\tilde{\mathrm{U}}$.

In fact, for

$$\Phi = (x_1, \ldots, x_{n-1}, \mathrm{F}_{0,0}, \mathrm{F}_{0,1}, \ldots, \mathrm{F}_{0,s}),$$

$\Phi_{\mathrm{C},q} : \mathbf{C}^n, q \to \mathbf{C}^{n+s}$ has an injective representative by Lemma 2.7. Moreover since $\mathrm{H_C}$ is finite, there exists an open neighborhood $\tilde{\mathrm{V}}$ of $\mathrm{H_C}(\tilde{\mathrm{U}})$ in $\mathbf{C}^e$ such that

(6) $\mathrm{H_C} : \tilde{\mathrm{U}} \to \tilde{\mathrm{V}}$ is proper,

if we take $\tilde{\mathrm{U}}$ smaller if necessary. Further for each open subset U′ of U, there exists an open neighborhood V′ of H(U′) in $\mathbf{R}^e$ such that $\mathrm{H}_{|\mathrm{U}'} : \mathrm{U}' \to \mathrm{V}'$ is proper. Thus we can apply Lemma 2.4 to this case, and we have

$$(\mathrm{H}_{|\mathrm{U}'})^*(\mathscr{E}(\mathrm{V}')) = (\mathrm{H}_{|\mathrm{U}'})^*(\mathscr{E}(\mathrm{V}'))^{\wedge}.$$

By definition,

$$(\mathrm{H}_{|\mathrm{U}'})^*(\mathscr{E}(\mathbf{R}^e))^{\wedge} = (\mathrm{H}_{|\mathrm{U}'})^*(\mathscr{E}(\mathrm{V}'))^{\wedge}.$$

From (3),

$$(\mathrm{H}_{|\mathrm{U}'})^*(\mathscr{E}(\mathbf{R}^e))^{\wedge} \supseteq (\mathrm{D}^{-1}\mathscr{I})(\mathrm{U}'),$$

and since each component of H belongs to $(\mathrm{D}^{-1}\mathscr{I})(\mathrm{U})$, we have

$$(\mathrm{D}^{-1}\mathscr{I})(\mathrm{U}') \supseteq (\mathrm{H}_{|\mathrm{U}'})^*(\mathscr{E}(\mathrm{V}')).$$

Hence we know that

$$(\mathrm{D}^{-1}\mathscr{I})(\mathrm{U}') = (\mathrm{H}_{|\mathrm{U}'})^*(\mathscr{E}(\mathrm{V}')).$$

If we put all components of H as $h_1, \ldots, h_e$, then we have the first part of Proposition 2.2.

From (5) and (6), we can apply Lemma 2.8 to H . Therefore, we have the rest of Proposition 2.2.

There remains to show the assertion (3). From (5), it is sufficient to prove that

$$(7) \quad \begin{cases} \text{for any open subset } U' \subseteq U, \text{ each element } \xi \text{ of } D^{-1}\mathscr{I}(U') \\ \text{satisfies, for each } \tilde{q} \in U', \text{ that } \xi \equiv \tau \circ H \text{ at } \tilde{q} \text{ modulo } m_{\tilde{q}}^{\infty} \\ \text{with a } C^{\infty} \text{ function } \tau \text{ on } \mathbf{R}^e. \end{cases}$$

Let $\tilde{q} \in U'$. Then $(D\xi)_{\tilde{q}} \in \mathscr{I}_{\tilde{q}}$. Thus there exists a system of elements $c_1, \ldots, c_m$ of $\mathscr{E}_{\tilde{q}}$ such that

$$(D\xi)_{\tilde{q}} \equiv \sum_{i=0}^{m} c_i P_i \text{ at } \tilde{q}, \text{ mod. } m_{\tilde{q}}^{\infty}.$$

We denote by $\hat{c}_i \in \mathscr{F}_{\tilde{q}}$ $(=\mathscr{E}_{\tilde{q}}/m_{\tilde{q}}^{\infty})$ the Taylor series of c_i, and put

$$\hat{c}_i = \sum_{k=0}^{\infty} c_{ik}(x'-x'(\tilde{q})).(x_n-x_n(\tilde{q}))^k,$$

where $x' = (x_1, \ldots, x_{n-1})$ and $c_{ik}(x'-x'(\tilde{q})) \in \mathscr{F}_{\mathbf{R}^{n-1}, \pi(\tilde{q})}$. We have an equality

$$\widehat{(D\xi)}_{\tilde{q}} = \sum_{i=0}^{m} \sum_{k=0}^{\infty} c_{ik}(x'-x'(\tilde{q})).(x_n-x_n(\tilde{q}))^k.\hat{P}_i$$

in $\mathscr{F}_{\mathbf{R}^n, \tilde{q}}$. We expand P_i for $x_n - x_n(\tilde{q})$ and apply Lemma 2.5, for $\tilde{q}' = \pi(\tilde{q})$. Then there exists a system of C^{∞} functions τ_{ik} $(0 \leqslant i \leqslant m, k=0,1,2, \ldots)$ on $\mathbf{R}^e$ such that

$$F_{ik}^{\tilde{q}} = \tau_{ik} \circ H^{\tilde{q}} \text{ at } \tilde{q},$$
$$\operatorname{ord}_0 \tau_{ik} \to \infty \ (k \to \infty).$$

The sum $\sum_{i=0}^{m} \sum_{k=0}^{\infty} c_{ik}.\tau_{ik}$ has a meaning as an element of $\mathscr{F}_{\mathbf{R}^e, 0}$. Denote this $\hat{\tau}_1$, and take $\tau_1 \in \mathscr{E}(\mathbf{R}^e)$ with $\widehat{(\tau_1)_0} = \hat{\tau}_1$.

We see that there exists a C^∞ mapping $\tau_2 : \mathbf{R}^e \to \mathbf{R}^e$ such that

$$H^{\tilde{q}} = \tau_2 \circ H \quad \text{at} \quad \tilde{q}.$$

In fact,

$$D_{\tilde{q}}(F_{ik}^{\tilde{q}}) = (x_n - x_n(\tilde{q}))^k . P_i = \sum_{j=0}^{k} \binom{k}{j} (-x_n(\tilde{q}))^{k-j} x_n^j P_i,$$

thus we have that

$$F_{ik}^{\tilde{q}} \equiv \sum_{j=0}^{k} \binom{k}{j} (-x_n(\tilde{q}))^{k-j} F_{ij}^{q} \quad \text{mod. ker } D_{\tilde{q}}.$$

As $\ker D_{\tilde{q}} = \mathscr{E}_{\mathbf{R}^{n-1}, \tilde{q}'} \subset \mathscr{E}_{\mathbf{R}^n, \tilde{q}}$, there exists a function $\eta \in \mathscr{E}(\mathbf{R}^e)$ such that

$$F_{ik}^{\tilde{q}} = \eta \circ (x_1, \ldots, x_{n-1}, F_{ij}(0 \leqslant j \leqslant k)).$$

This shows the existence of τ_2.

By the above arguments,

$$\begin{aligned} \widehat{\xi}_{\tilde{q}} &\equiv \sum_{i=0}^{m} \sum_{k=0}^{\infty} c_{ik}(x' - x'(\tilde{q})) . \int (x_n - x_n(\tilde{q}))^k \widehat{P}_i \, dx_n, \text{ mod. ker } D_{\tilde{q}} \\ &= \left(\sum_{i=0}^{m} \sum_{k=0}^{\infty} c_{ik} \hat{\tau}_{ik} \right) . H^{\tilde{q}}. \end{aligned}$$

Hence

$$\begin{aligned} \xi &\equiv \tau_1 \circ H^{\tilde{q}} \quad \text{at} \quad \tilde{q}, \quad \text{mod. } [(\ker D_{\tilde{q}}) + m_{\tilde{q}}^{\infty}] \\ &= \tau_1 \circ \tau_2 \circ H. \end{aligned}$$

Thus there exists a C^∞ function τ on $\mathbf{R}^e$ such that

$$\xi \equiv \tau \circ H \quad \text{at} \quad \tilde{q}, \text{ mod. } m_{\tilde{q}}^{\infty}.$$

This proves (7), therefore (3). This completes the proof of Proposition 2.2.

3. Proof of Theorem 1.7.

In this section we give the proof of Theorem 1.7 owing to Proposition 2.2, and mention several notes.

Firstly we remark that, from the naturality of notions defined in § 1, it is sufficient to prove Theorem 1.7 in a convenient system of coordinates.

The following observation has a key role in the proof of Theorem 1.7.

PROPOSITION 3.1. — *Let Ω be an open subset of $\mathbf{R}^n$, and $f:\Omega \to \mathbf{R}^p$ $(n \leqslant p)$ a finite C^∞ mapping of the following type :*

$$f = (x_1, \ldots, x_{n-1}, f_n, \ldots, f_p).$$

Then

$$\mathscr{D}_f = \left(\frac{\partial}{\partial x_n}\right)^{-1} \mathscr{I}_f.$$

Proof. — Consider the subsets of Ω

$$V_u = \{q \in \Omega; \quad (\partial^j f_i/\partial x_n^j)(q) = 0, \quad (1 \leqslant j \leqslant u, n \leqslant i \leqslant p)\},$$

$(u=0,1,2,\ldots)$ and a filtration of Ω :

$$\Omega = V_0 \supseteq V_1 \supseteq V_2 \supseteq \ldots \supseteq V_u \supseteq \cdots.$$

We see that $q \in V_u$ if and only if

$$\mathscr{C}_{f,q} \subseteq (x_1 - x_1(q), \ldots, x_{n-1} - x_{n-1}(q), (x_n - x_n(q))^{u+1}).\mathscr{E}_q$$

and that this condition is equivalent to that

$$\Delta_n \mathscr{C}_{f,q} \subseteq (x_1 - x_1(q), \ldots, x_{n-1} - x_{n-1}(q), (x_n - x_n(q))^{u+1}).\mathscr{E}_q.$$

Hence we have immediately explicite discriptions of $\mathscr{I}_f$ and $\mathscr{D}_f$:

(3.1.1.) $\mathscr{I}_f(U)$

$$= \left\{k \in \mathscr{E}_\Omega(U); \frac{\partial^u k}{\partial x_n^u} \text{ vanishes on } V_{u+1}, (u=0,1,2,\ldots).\right\},$$

(3.1.2) $\mathscr{D}_f(U)$

$$= \left\{h \in \mathscr{E}_\Omega(U); \frac{\partial^{u+1} h}{\partial x_n^{u+1}} \text{ vanishes on } V_{u+1}, (u=0,1,2,\ldots).\right\},$$

for each open subset U of Ω. Thus we have in particular that

$$\mathscr{D}_f = (\partial/\partial x_n)^{-1} \mathscr{I}_f.$$

Proof of the first half of Theorem 1.7. — Let $q \in \Omega$ with rank $df(q) \geqslant n-1$. There are systems of analytic coordinates on open neighborhoods of q and $f(q)$ respectively such that f is of type as in Proposition 3.1. Thus, it is sufficient to prove in the case of Proposition 3.1 and $q = 0$.

Put $\mathrm{D} = \partial/\partial x_n$. We see that $\mathscr{I}_{f,0}$ contains an analytic D_0-regular element. In fact, as f is finite, $f_i(0, x_n) - f_i(0,0)$ is not identically zero, for some i, $(n \leqslant i \leqslant p)$. For such i, $f_i \in \mathscr{D}_f(\Omega) = \mathrm{D}^{-1}\mathscr{I}_f(\Omega)$ by Lemma 3.1, and the element $(\mathrm{D}f_i)_0 \in \mathscr{I}_{f,0}$ is analytic and D_0-regular.

Thus we can apply Proposition 2.2 to the case $\mathscr{I} = \mathscr{I}_f$, and we have the implication (A) $\Rightarrow$ (C). Obviously (C) $\Rightarrow$ (B). For (B) $\Rightarrow$ (A), we refer to Remark 2.3 (1).

So as to complete the proof of Theorem 1.7, we proceed to consider the conditions (A), (A′) and (A″) of Theorem 1.7 in a rather general framework.

Let Ω be an open subset in $\mathbf{R}^n$. Let $\mathscr{M}$ be a subsheaf of $\mathscr{E}_\Omega$-modules of $\mathscr{E}^\ell_\Omega$. We define a subsheaf of $\mathscr{O}_\Omega$-modules $\mathscr{M}^\omega$ of $\mathscr{O}^\ell_\Omega$ by

$$\mathscr{M}^\omega(\mathrm{U}) = \mathscr{M}(\mathrm{U}) \cap \mathscr{O}_\Omega(\mathrm{U}),$$

for each open subset U of Ω.

Now we introduce the following seven conditions for $\mathscr{M}$:

(A1) For any point $x \in \Omega$, there exist sections $g_1, \ldots, g_r$ of $\mathscr{M}^\omega$ over an open neighborhood U of x such that $g : \mathscr{O}^r_\mathrm{U} \to \mathscr{M}^\omega|\mathrm{U}$ is surjective, that is, $\mathscr{M}^\omega$ is of finite type.

(A2) For any point $x \in \Omega$, $\mathscr{M}_x = \mathscr{M}^\omega_x . \mathscr{E}_{\Omega,x}$.

(A3) For any point $x \in \Omega$, $\mathscr{M}_x \subseteq \mathscr{M}^\omega_x . \mathscr{E}_{\Omega,x} + \mathscr{E}^\ell_{\Omega,x} . m^\infty_{\Omega,x}$.

(A4) For any point $x \in \Omega$, there exist sections $g_1, \ldots, g_r$ of $\mathscr{M}$ over an open neighborhood U of x such that $g : \mathscr{E}^r_\mathrm{U} \to \mathscr{M}_{|\mathrm{U}}$ is surjective, that is, $\mathscr{M}$ is of finite type.

(A5) For any point $x \in \Omega$, the $\mathscr{E}_{\Omega,x}$-module $\mathscr{M}_x$ is finitely generated.

(A6) For any open subset U of Ω, $\mathscr{M}(\mathrm{U})$ is closed in $\mathscr{E}^\ell_\Omega(\mathrm{U})$ with respect to the C^∞ topology.

(A) For any point $x \in \Omega$, there exist an open neighborhood U of x and a finite number of elements $g_1, \ldots, g_r$ of $\mathscr{M}^\omega(\mathrm{U})$ such that for any point $\tilde{x} \in \mathrm{U}$,

$$\mathscr{M}_{\tilde{x}} \subseteq (g_{1,\tilde{x}}, \ldots, g_{r,\tilde{x}}).\mathscr{E}_{\Omega,\tilde{x}} + \mathscr{E}'_{\Omega,\tilde{x}}.m^{\infty}_{\Omega,\tilde{x}}.$$

PROPOSITION 3.2. — *The following conditions for a subsheaf of $\mathscr{E}_{\Omega}$-modules $\mathscr{M}$ of $\mathscr{E}'_{\Omega}$ are equivalent to each other :*

(A), (A1) & (A3), (A1) & (A2), (A2) & (A6), (A2) & (A5) & (A6), (A3) & (A5) & (A6), (A2) & (A4).

Though this proposition is a collection of known results, we give an outline of the proof to assure ourself.

Proof. — As $\mathscr{F}_x$ is faithfully flat over $\mathcal{O}_x$, we have the implication (A) $\Rightarrow$ (A1) (cf. [4], III.4). Trivially (A) implies (A3). A theorem of Malgrange ([4] Theorem VI.1.1′) verifies (A1) & (A3) $\Rightarrow$ (A2) and (A1) & (A2) $\Rightarrow$ (A6). Obviouly (A2) implies (A3) and (A5). For (A3) & (A5) & (A6) $\Rightarrow$ (A2), we discuss by induction using the fact that, under the condition (A6), if $\mathscr{M}_x$ is generated by $g_1, \ldots, g_{r-1}, g_r$ over $\mathscr{E}_x$ and formally generated by $g_1, \ldots, g_{r-1}$, then $g_1, \ldots, g_{r-1}$ generates $\mathscr{M}_x$ over $\mathscr{E}_x$, according to the idea of [3] Lemma 27. The implication (A2) & (A5) & (A6) $\Rightarrow$ (A2) & (A6) is trivial. For (A2) & (A6) $\Rightarrow$ (A1), we use a method of Tougeron (see [4] Theorem VI.3.10, for example). Clearly

(A1) & (A2) $\Rightarrow$ (A1) & (A3) $\Rightarrow$ (A),
(A1) & (A2) $\Rightarrow$ (A4) and (A2) & (A4) $\Rightarrow$ (A).

Thus we have required equivalences of the seven conditions.

Remark 3.3. — (1) If f is finite (more generally, if the ideal $\Delta_n\mathscr{C}_{f,q}$ contains $m^{\infty}_{\Omega,q}$ for all $q \in \Omega$), then $\mathscr{I}_f(\mathrm{U})$ (see (1.6)) is closed in $\mathscr{E}_{\Omega}(\mathrm{U})$ for each open subset U of Ω. Thus in this case the condition (A6) is satisfied for $\mathscr{M} = \mathscr{I}_f$.

(2) The condition (A), (A′) and (A″) in Theorem 1.7 are (A), (A1) and (A5) respectively for $\mathscr{M} = \mathscr{I}_f$ in an open neighborhood of q.

Let us consider the condition (A3) for $\mathscr{I}_f$.

Let $\mathscr{J}$ be a coherent sheaf of ideals of $\mathcal{O}_{\Omega}$, and D an analytic vector field over Ω. We define a sheaf of ideals $\mathscr{J}_{\mathrm{D},r}$ $(r=0,1,2,\ldots)$ of $\mathscr{E}_{\Omega}$ by

$$\mathscr{J}_{\mathrm{D},r}(\mathrm{U}) = \left\{k \in \mathscr{E}_{\Omega}(\mathrm{U});\ \mathrm{V}(\mathrm{D}^i k) \supseteq \mathrm{V}\left(\sum_{j=0}^{i} \mathrm{D}^j\mathscr{J}\right),\ 0 \leqslant i \leqslant r\right\},$$

for each open subset U of Ω, where $\sum_{j=0}^{i} \mathrm{D}^j\mathscr{I}$ is a *coherent sheaf of ideals* defined by

$$\left(\sum_{j=0}^{i} \mathrm{D}^j\mathscr{I}\right)(\mathrm{U}) = \left(\sum_{j=0}^{i} \mathrm{D}^j(\mathscr{I}(\mathrm{U}))\right).\mathscr{E}(\mathrm{U}),$$

for each open subset U of Ω, and $\mathrm{V}(.)$ is the *zero locus*. Notice that $\mathscr{I}_{\mathrm{D},0}$, which is independent of D, is the sheaf of germs of C^∞ functions vanishing on the zero locus of $\mathscr{I}$. Further we put, for each open subset $\mathrm{U} \subseteq \Omega$,

$$\mathscr{I}^{\omega}_{\mathrm{D},r}(\mathrm{U}) = \mathscr{I}_{\mathrm{D},r}(\mathrm{U}) \cap \mathscr{O}_{\Omega}(\mathrm{U}).$$

LEMMA 3.4. — *With the same notation as above, we have*

$$\mathscr{I}_{\mathrm{D},r,x} \subseteq \mathscr{I}^{\omega}_{\mathrm{D},r,x}.\mathscr{E}_x + m_x^\infty, \qquad (r=0,1,2,\ldots),$$

for each point $x \in \Omega$.

Remark 3.5. — Let $f:\Omega \to \Omega'$ be a finite analytic mapping, and $q \in \Omega$ a point with rank $df(q) \geqslant n-1$. Then, by (3.1.1), we know that $\mathscr{I}_f = \mathscr{I}_{\mathrm{D},r}$, $\mathscr{I} = (\mathrm{D}f_1,\ldots,\mathrm{D}f_p).\mathscr{O}$, for some D and r locally at q. Thus, for such $\mathscr{I}_f$, the condition (A3) is satisfied.

Proof of Lemma 3.4. — We will show that $\mathscr{I}_{\mathrm{D},r,x}.\mathscr{F}_x = \mathscr{I}^{\omega}_{\mathrm{D},r,x}.\mathscr{F}_x$ by the induction on r, where $\mathscr{F}_x = \mathscr{E}_x/m_x^\infty$ and, by the natural homomorphisms $\mathscr{O}_x \to \mathscr{E}_x \to \mathscr{F}_x$, we regard $\mathscr{F}_x$ as $\mathscr{O}_x$-module and $\mathscr{E}_x$-module. In the case $r=0$, the assertion follows from a theorem of Malgrange ([4] Theorem VI.3.5.) which states that $\mathscr{I}^{\omega}_{\mathrm{D},0,x}.\mathscr{F}_x$ equals to the set of infinite jets of germs of C^∞ functions vanishing on the zero locus of $\mathscr{I}$.

Assume $r>0$. We consider an exact sequence

$$0 \to \mathscr{I}^{\omega}_{\mathrm{D},r,x} \to \mathscr{I}^{\omega}_{\mathrm{D},r-1,x} \xrightarrow{\psi} \mathscr{O}_x\Big/\left(\sum_{j=0}^{r} \mathrm{D}^j\mathscr{I}\right)^{\omega}_{\mathrm{D},0,x},$$

where $\psi(k)$ is defined to be $\mathrm{D}^r_x(k)$ modulo $\left(\sum_{j=0}^{r} \mathrm{D}^j\mathscr{I}\right)^{\omega}_{\mathrm{D},0,x}$ for each $k \in \mathscr{I}^{\omega}_{\mathrm{D},r-1,x}$. Note that ψ is certainly a homomorphism of $\mathscr{O}_x$-modules.

Now we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I}^{\omega}_{D,r,x} \otimes \mathscr{F}_x \arrow[r] \arrow[d, "\wr"] & \mathscr{I}^{\omega}_{D,r-1,x} \otimes \mathscr{F}_x \arrow[r, "\psi \otimes \mathscr{F}_x"] \arrow[d, "\wr"] & \left(\mathscr{O}_x \Big/ \left(\sum_{j=0}^{r} D^j \mathscr{I}\right)^{\omega}_{D,0,x}\right) \otimes \mathscr{F}_x \arrow[d, "\wr"] \\ 0 \arrow[r] & \mathscr{I}^{\omega}_{D,r,x} . \mathscr{F}_x \arrow[r] & \mathscr{I}^{\omega}_{D,r-1,x} . \mathscr{F}_x \arrow[r, "\hat{\psi}"] & \mathscr{F}_x \Big/ \left(\sum_{j=0}^{r} D^j \mathscr{I}\right)^{\omega}_{D,0,x} . \mathscr{F}_x \end{tikzcd}$$

where $\hat{\psi}$ is defined by $\hat{\psi}(k) = \hat{D}^r_x(k)$ modulo $\left(\sum_{j=0}^{r} D^j \mathscr{I}\right)^{\omega}_{D,0,x} . \mathscr{F}_x$ for each $k \in \mathscr{I}^{\omega}_{D,r-1,x} . \mathscr{F}_x (\subseteq \mathscr{F}_x)$. Since $\mathscr{F}_x$ is flat over $\mathscr{O}_x$, in the above diagram, the first row is exact and each column is isomorphic. Using the equality in the case $r = 0$, we have

$$\left(\sum_{j=0}^{r} D^j \mathscr{I}\right)^{\omega}_{D,0,x} . \mathscr{F}_x = \left(\sum_{j=0}^{r} D^j \mathscr{I}\right)_{D,0,x} . \mathscr{F}_x ,$$

and by the assumption of induction,

$$\mathscr{I}^{\omega}_{D,r-1,x} . \mathscr{F}_x = \mathscr{I}_{D,r-1,x} . \mathscr{F}_x .$$

Clearly $\mathscr{I}_{D,r,x} . \mathscr{F}_x$ is contained in the kernel of $\hat{\psi}$. Hence we have that

$$\mathscr{I}_{D,r,x} . \mathscr{F}_x \subseteq \mathscr{I}^{\omega}_{D,r,x} . \mathscr{F}_x ,$$

and the reverse inclusion is obvious. Now the inclusion of Lemma 3.4 follows immediately from the equality $\mathscr{I}_{D,r,x} . \mathscr{F}_x = \mathscr{I}^{\omega}_{D,r,x} . \mathscr{F}_x$.

Proof of the second half of Theorem 1.7. — Let f and q be as in Theorem 1.7. By Remark 3.3.(1) and Remark 3.5, the conditions (A6) and (A3) are satisfied for $\mathscr{M} = \mathscr{I}_f$ on an open neighborhood of q. Thus, by Proposition 3.2, the three conditions (A), (A1) and (A5) for $\mathscr{M} = \mathscr{I}_f$ in an neighborhood of q are equivalent to each other. Hence, by Remark 3.3.(2), we have that the three conditions (A), (A′) and (A″) in Theorem 1.7 are equivalent to each other.

Theorem 1.7 is now proved completely.

Here is a case where the equivalent conditions of Theorem 1.7 are satisfied.

COROLLARY 3.6. — *Let $f : \Omega \to \Omega'$ be a finite analytic mapping. If the* 1-*jet extension $j^1 f : \Omega \to J^1(\Omega, \Omega')$ of f is transverse to $\Sigma^1(\Omega, \Omega')$ except at* *isolated points and* $j^1f(\Omega) \subseteq \Sigma^0(\Omega,\Omega') \cup \Sigma^1(\Omega,\Omega')$, *then* $\mathscr{D}_f$ *is finitely presented (at any point in* Ω*).*

Proof. – According to Theorem 1.7, it is sufficient to show that the condition (A′) of Theorem 1.7 is satisfied at any point in Ω, that is, $\mathscr{I}^\omega_f$ is of finite type. We reduce f to the form $f = (x_1, \ldots, x_{n-1}, f_n, \ldots, f_p)$ on an open neighborhood U of each point $q \in \Omega$ by an analytic change of coordinates. Then we have

$$\mathscr{I}^\omega_{f|\mathrm{U}} = \left(\left(\frac{\partial f_n}{\partial x_n}, \frac{\partial f_{n+1}}{\partial x_n}, \ldots, \frac{\partial f_p}{\partial x_n}\right).\mathscr{O}_\mathrm{U}\right)_{\frac{\partial}{\partial x_n}, r}, \qquad (r \gg 0),$$

with the same notation as in Lemma 3.4. At each point q of the zero locus of $\partial f_n/\partial x_n, \ldots, \partial f_p/\partial x_n$, the transversality condition means that $\operatorname{grad}(\partial f_n/\partial x_n)(q), \ldots, \operatorname{grad}(\partial f_p/\partial x_n)(q)$ are linearly independent. Thus $\mathscr{I}^\omega_{f|\mathrm{U}} = (\partial f_n/\partial x_n, \ldots, \partial f_p/\partial x_n).\mathscr{O}_n$ outside isolated points. At each exceptional point $q \in \mathrm{U}$, we take a finite system of generators $g_1, \ldots, g_r$ of the ideal $\mathscr{I}^\omega_{f,q}$ and take analytic representatives $\tilde{g}_1, \ldots, \tilde{g}_r \in \mathscr{I}^\omega_f(\mathrm{U}')$ of $g_1, \ldots, g_r$ respectively over an open neighborhood U′ of q in U. Then we have

$$\mathscr{I}^\omega_{f|\mathrm{U}'} = \left(\frac{\partial f_n}{\partial x_n}, \ldots, \frac{\partial f_p}{\partial x_n}, g_1, \ldots, g_r\right).\mathscr{O}_{\mathrm{U}'}.$$

This completes the proof.

There is an example in which the conditions of Theorem 1.7 are not satisfied.

Example 3.7. – Let

$$f = (x_1, x_2, x_3^3 - 3x_1x_2^2x_3) : \mathbf{R}^3 \to \mathbf{R}^3.$$

The singular locus of this polynomial mapping f is the *Whitney's umbrella.* By (3.1.1), we have that $\mathscr{I}^\omega_f(\mathrm{U})$ is the set of functions $k \in \mathscr{O}_{\mathbf{R}^3}(\mathrm{U})$ such that $k(x) = 0$ if $x_3^2 - x_1x_2^2 = 0$ and $(\partial k/\partial x_3)(x) = 0$ if $x_3 = x_1x_2 = 0$, for each open subset U of $\mathbf{R}^3$. It is easy to see that $\mathscr{I}^\omega_f$ is not of finite type at 0. Thus $\mathscr{D}_f$ is not finitely generated. Furthermore the ideal $\mathscr{I}_{f,0}$ is not finitely generated and $\mathscr{D}_{f,0} = (\partial/\partial x_3)^{-1}\mathscr{I}_{f,0}$ is not finitely generated by Remark 2.3.(2) and Proposition 3.1.

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