

ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 33, n° 2 (1983), p. 77-85

http://www.numdam.org/item?id=AIF_1983__33_2_77_0

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FINITELY GENERATED IDEALS IN $A(\Omega)$

by J. E. FORNÆSS and N. ØVRELID

1. Let $\Omega \subset\subset C^2(z,w)$ be a bounded pseudoconvex domain with smooth boundary containing the origin and let $A(\Omega)$ denote the set of continuous functions on $\bar{\Omega}$ which are holomorphic in Ω . In the special case when Ω is the unit ball, A. Gleason [4] asked the following :

The Gleason Problem : If $f \in A(\Omega)$ and $f(0,0) = 0$, does there exist $g, h \in A(\Omega)$ such that $f = zg + wh$?

This was solved affirmatively by Leibenzon, see [5], in the ball case and by Henkin [5], Kerzman-Nagel [6], Lieb [9] and Øvrelid [12] in the strongly pseudoconvex case. Beatrous [1] solved the problem for weakly pseudoconvex domains under the extra hypothesis that there exists a complex line through 0 which intersects the boundary of Ω only in strongly pseudoconvex points. In this paper we discuss the real analytic case.

MAIN THEOREM. — *Let $0 \in \Omega \subset\subset C^2(z,w)$ be a pseudoconvex domain with real analytic boundary. If $f \in A(\Omega)$ and $f(0) = 0$, then there exist $g, h \in A(\Omega)$ such that $f = zg + wh$.*

The main difficulty is that the Levi flat boundary points, $w(\partial\Omega)$, can be two-dimensional. This means that the projection of $w(\partial\Omega)$ into the space of complex lines through 0 (a P^1) can be onto. Thus no such complex line avoids $w(\partial\Omega)$ and therefore Beatrous' theorem does not apply. (On the other hand, if $w(\partial\Omega)$ is one-dimensional, then of course the Main Theorem is a direct consequence of Beatrous' result.)

To handle this difficulty we study the structure of $w(\partial\Omega)$. We show (Proposition 3) that except for a one-dimensional subset, $w(\partial\Omega)$ consists

of R -points. The R -points were first studied by Range [11] who proved sup norm estimates for $\bar{\partial}$ at such points. We give a precise definition of R -points in the next section. Their main property is that they allow holomorphic separating functions. In particular we thus show in this paper that the Kohn-Nirenberg points [8] constitute an at most one-dimensional subset of $\partial\Omega$. Next we choose a complex line through 0 intersecting $w(\partial\Omega)$ only in R -points. Then, one has good enough $\bar{\partial}$ -results to complete the proof along the same line as Beatrous.

The Main Theorem can still be proved if we replace $A(\Omega)$ by various holomorphic Hölder- and Lipschitz-spaces and if we replace z and w by arbitrary generators of the maximal ideal at 0 in these spaces. This requires several hard $\bar{\partial}$ -estimates. Therefore, in order to keep the length of this paper down, the authors have decided to postpone these generalizations to a later paper. We will then also show how these techniques can be used to prove that bounded pseudoconvex domains with real analytic boundary in C^2 have the Mergelyan property (see [3]).

2. We will make a detailed discussion of the weakly pseudoconvex boundary points $W = w(\partial\Omega)$ of a bounded pseudoconvex domain Ω with smooth real analytic boundary in C^2 . First we need a stratification of W into totally real manifolds.

LEMMA 1. — *There exist pairwise disjoint real analytic manifolds $S_0, S_1, S_2 \subset \partial\Omega$ with the following properties :*

- (i) *Each S_j consists of finitely many j -dimensional totally real real analytic manifolds,*
- (ii) $W = S_0 \cup S_1 \cup S_2$,
- (iii) S_1 *is closed in $\partial\Omega - S_0$; S_2 is closed in $\partial\Omega - (S_0 \cup S_1)$ and*
- (iv) *Each connected component of S_2 consists of points of the same finite type only.*

Here finite type is in the sense of Kohn [7].

The sets S_0 , S_1 and S_2 are actually semi analytic. During the proof we will use repeatedly standard facts about semi-analytic sets. The reader can consult Łojasiewicz [10] for details.

Proof. — Let r be a real analytic defining function for Ω . (For example, one can choose r to be the Euclidean distance to $\partial\Omega$ outside Ω ,

but close to $\partial\Omega$, and the negative of the Euclidean distance in $\bar{\Omega}$ close to $\partial\Omega$.) Also let s be a real valued real analytic function defined on a neighbourhood of $\partial\Omega$ vanishing at a $p \in \partial\Omega$ if and only if p is a weakly pseudoconvex boundary point. (One can for example let

$$s(z,w) = \partial^2 r / \partial z \partial \bar{z} \cdot |\partial r / \partial w|^2 - 2 \operatorname{Re} \partial^2 r / \partial z \partial \bar{w} \cdot \partial r / \partial w \cdot \partial r / \partial \bar{z} + \partial^2 r / \partial w \partial \bar{w} \cdot |\partial r / \partial z|^2).$$

Hence the weakly pseudoconvex boundary points, W , is the common zero set $\{r=s=0\}$ of global real analytic functions.

Using real coordinates, $x + iy = z$, $u + iv = w$, we can identify as usual $C^2(z,w)$ with $R^4(x,y,u,v)$ with complex coordinates

$$X = x + ix', \quad Y = y + iy', \quad U = u + iu', \quad V = v + iv'.$$

Then r, s have unique extensions to holomorphic functions $R(X,Y,U,V)$ and $S(X,Y,U,V)$ respectively. The complexification M of $\partial\Omega$ is then given by $\{R=0\}$ which is a complex manifold since $dr \neq 0$. From now on we will consider only points of M . In $M, \Sigma := \{S=0\} \cap M$ is a complex hypersurface, hence has (complex) dimension 2.

Let p be any point in $W \subset \Sigma$. Since Σ and M are closed under complex conjugation, there exists a holomorphic function $h = h_p(X,Y,U,V)$ defined in a neighbourhood of p in C^4 which, when restricted to M , generates the ideal of Σ at every point of Σ in that neighbourhood, and such that h is real valued at points in $C^2 = R^4$. The function h has a nonvanishing gradient (on M) at regular points of Σ . Since $\operatorname{Im} h \equiv 0$ on $\partial\Omega$ it follows that W is given by $\{r = \operatorname{Re} h = 0\}$ near such regular points of Σ and that $\partial\Omega \cap \operatorname{reg} \Sigma$ is a pure 2-dimensional real analytic manifold. By Diederich-Fornæss [2] $\partial\Omega$ cannot contain a complex manifold. This implies that $\partial\Omega \cap \operatorname{reg} \Sigma$ is totally real at a (relatively) dense set of points. A point in $\partial\Omega \cap \operatorname{reg} \Sigma$ is totally real if and only if $\lambda := (\partial r)_{(z,w)} \wedge \partial(\operatorname{Re} h_p)_{(z,w)} \neq 0$ there. Here derivatives are taken in C^2 . This condition does not depend on p since different $(\operatorname{Re} h_p)'$'s only differ by real multiples on $\partial\Omega$.

Let $S' \subset W$ be the (at most) one-dimensional closed real analytic set consisting of $\partial\Omega \cap \operatorname{sing} \Sigma$ and the zeroes in W of the coefficient of λ . By Łojasiewicz [10], $W - S'$ consists of finitely many connected, pairwise disjoint semi-analytic sets, C_1, \dots, C_ℓ . Each C_j is a two dimensional

totally real real analytic manifold whose closure \bar{C}_j is also a semi analytic set, and $\bar{C}_j - C_j \subset S'$.

Locally, there exists a holomorphic vector field

$$L = a \partial/\partial z + b \partial/\partial w \neq 0$$

with real analytic coefficients tangent to the boundary, i.e. $L(r) = 0$ on $\partial\Omega$. The type of a point $p \in \partial\Omega$ is then given as the smallest integer $2k$ for which $(\partial r, L^{k-1}[L, L](r))(p) \neq 0$. This number is independent of the choices of r and L . Let n_j be the maximum type of points in C_j , and let T_j consist of all boundary points of type $> n_j$. Then T_j is a real analytic set. In particular, $\bar{C}_j \cap T_j$ is a semi analytic set of dimension at most one. Then $S_2 := \cup C_j - T_j$ is a pure 2-dimensional totally real real analytic manifold with finitely many connected components on each of which the type is constant. Also, $W - S_2$ is a closed semi analytic set in C^2 of dimension at most one, and can hence be written as $S_0 \cup S_1$ where S_0 is a finite set of points and S_1 is a relatively closed 1-dimensional real analytic manifold in $W - S_0$ with finitely many connected components. This completes the proof of Lemma 1.

Range [11] introduced a convexity condition which is satisfied by many weakly pseudoconvex boundary points.

DEFINITION 2. — Let $D = \{\rho < 0\} \subset \subset C^n$ be a domain with C^∞ boundary. A point $p \in \partial D$ is an R-point (of order m) if there exists a neighbourhood U of p and a C^∞ function

$$F(\zeta, z) : (\partial D \cap U)(\zeta) \times U(z) \rightarrow C$$

such that

- (i) F is holomorphic in z ,
- (ii) $F(\zeta, \zeta) \equiv 0$ and $d_z F \neq 0$ and
- (iii) $\rho(z) \geq \varepsilon |z - \zeta|^m$ whenever $F(\zeta, z) = 0$, $\varepsilon > 0$ some constant.

Using the Levi polynomial

$$F(\zeta, z) = \sum_{j=1}^n \frac{\partial \rho}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \frac{1}{2} \sum_{i, j=1}^n \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(\zeta_i - z_i)(\zeta_j - z_j)$$

one immediately obtains that strongly pseudoconvex boundary points are R-points of order 2.

PROPOSITION 3. — *Every point in S_2 is an R-point.*

In the proof of the proposition we will need two elementary inequalities.

LEMMA 4. — *Let $p_k(s,t) := (s+t)^{2k} - s^{2k} - 2kts^{2k-1}$ for $s, t \in \mathbb{R}$, $k \in \{1, 2, \dots\}$. Then there exists a constant $c_k > 0$ such that*

$$p_k(s,t) \geq c_k(s^{2k-2}t^2 + t^{2k}) \text{ for all } s, t.$$

Proof. — For each fixed s , $q_s(t) = (s+t)^{2k}$ is a convex function of t and $T_s(t) = s^{2k} + 2ks^{2k-1}t$ is an equation for the tangentline through $(0, s^{2k})$. Hence,

$$p_k(s,t) = q_s(t) - T_s(t) > 0$$

whenever $t \neq 0$. Since

$$p_k(s,t) = t^2 \left[\binom{2k}{2} s^{2k-2} + O(t) \right] \text{ and } s^{2k-2}t^2 + t^{2k} = t^2[s^{2k} + O(t)]$$

it follows that there exists a $c_k > 0$ such that

$$p_k(s,t) \geq c_k(s^{2k}t^2 + t^{2k})$$

for all (s,t) on the unit circle and hence by homogeneity for all (s,t) .

LEMMA 5. — *Let $k \in \{1, 2, \dots\}$ and $\delta > 0$, $\delta < 4^{-k}$ be given. Then $y^{2k} + \delta \operatorname{Re}(z^{2k}) \geq 2^{-k} \delta |z|^{2k}$ for every complex number $z = x + iy$.*

Proof. — Expanding $\operatorname{Re} z^{2k}$, we get

$$y^{2k} + \delta \operatorname{Re}(z^{2k}) \geq y^{2k} + \delta x^{2k} - R(z)$$

with $R(z) = 2^{2k-1} \delta y^2 \max(|x|, |y|)^{2k-2}$. Elementary computation gives $y^{2k} \geq 2R(z)$ when $|x| \leq 2^k|y|$, while $\delta x^{2k} \geq 2R(z)$ otherwise. In any case,

$$y^{2k} + \delta \operatorname{Re}(z^{2k}) \geq \frac{\delta}{2} (x^{2k} + y^{2k}) \geq 2^{-k} \delta (x^2 + y^2)^k,$$

so the lemma follows.

To simplify our computations it is convenient to change coordinates locally so that S_2 becomes a plane.

LEMMA 6. — Let $p_0 \in S_2$. There exist local holomorphic coordinates $z = x + iy$, $w = u + iv$ in a neighbourhood U of p_0 , such that in U ,

(i) S_2 is given by $y = v = 0$, and

(ii) $\partial\Omega$ is tangent to the plane $v = 0$ along S_2 .

As a consequence $T_p^c \partial\Omega$ is given by $w = 0$ along S_2 .

Proof. — Local coordinates satisfying (i) are constructed by choosing a real analytic parametrization $F: W \rightarrow S_2$ near p_0 , with W open in \mathbb{R}^2 . Since S_2 is totally real, the prolongation \tilde{F} of F to complex arguments is invertible near p_0 , and we set $(z(p), w(p)) = \tilde{F}^{-1}(p)$. Then (ii) means that the vector field $\frac{\partial}{\partial y} = J \frac{\partial}{\partial x}$ is tangential to $\partial\Omega$ on S_2 , i.e. $\left(\frac{\partial}{\partial x}\right)_p \in T_p^c \partial\Omega$ when $p \in S_2$. Now $L = TS_2 \cap T^c \partial\Omega$ is a real analytic line field on S_2 , and we just have to choose a parametrization F where the curves $u = \text{const.}$ are integral curves of L to complete the proof.

When $v = -V(x, y, u)$ is a local parametrization of $\partial\Omega$, Ω is given near p_0 by $\rho = v + V(x, y, u) < 0$, provided $\partial/\partial v$ points out of Ω . We may write

$$\rho = v + g(x, y, u) = v + \sum_{\ell=2k}^{\infty} a_{\ell}(x, u)y^{\ell}$$

for some $k > 1$ and $a_{2k} > 0$, since Ω is weakly pseudoconvex of constant type on S_2 .

After these preliminary remarks we can prove Proposition 3. To show that $p_0 \in S_2$ is an R -point, choose at first a neighbourhood $U = U(p_0)$ of p_0 on which $a_{2k}(x, u) > a > 0$. We will shrink U whenever necessary without saying so each time.

For $\zeta = (z_0, w_0) \in U \cap \partial\Omega$, we write $z = z_0 + z'$, $w = w_0 + w'$, $w' = u' + iv'$ etc., and Taylor-expand ρ around ζ . Since $\rho(\zeta) = 0$ we get

$$\rho = v' + g_x(\zeta)x' + g_y(\zeta)y' + g_u(\zeta)u' + a_{2k}(x_0, u_0)p_k(y_0, y') + R$$

where the remainder R satisfies an estimate

$$|R| \leq C(|z'| + |w'|)^2(|y_0| + |z'| + |w'|)^{2k-1}$$

in U with C independent of ζ .

The linear function $\tilde{w} = (g_y(\zeta) + ig_x(\zeta))z' + (1 + ig_u(\zeta))w'$ has imaginary part \tilde{v} equal to the linear part of ρ , so by Lemma 4 $\rho \geq \tilde{v} + ac_k(y_0^{2k+2}y'^2 + y'^{2k}) - |R|$ in U .

Set $F_\zeta(z, w) = i\tilde{w} + \varepsilon(y_0^{2k-2}z'^2 + z'^{2k})$, with $0 < \varepsilon < 4^{-k^2}c_k a$. On the zero set of F_ζ

$$(1) \quad \begin{aligned} \tilde{w} &= i\varepsilon(y_0^{2k-2}z'^2 + z'^{2k}), \text{ and in particular} \\ \tilde{v} &= \varepsilon(y_0^{2k-2}\text{Re}(z'^2) + \text{Re}(z'^{2k})). \end{aligned}$$

Applying Lemma 5 this gives $\rho \geq 2^{-k}\varepsilon(y_0^{2k-2}|y'|^2 + |z'|^{2k}) - |R|$.

Since g_x, g_y and g_u are small near the origin, it follows from (1) and the definition of \tilde{w} that $|w'| < |z'|$ on $\{F_\zeta=0\} \cap U$ whenever $\zeta \in U$. Thus

$$\begin{aligned} \rho &\geq 2^{-k}\varepsilon(y_0^{2k-2}|z'|^2 + |z'|^{2k}) - c'|z'|^2(|y_0| + |z'|)^{2k-1} \\ &\geq \tilde{\varepsilon}(y_0^{2k-2}|z'|^2 + |z'|^{2k}) \\ &\geq 2^{-k}\tilde{\varepsilon}|(z, w) - \zeta|^{2k}. \end{aligned}$$

It follows that $F(\zeta, (z, w)) := F_\zeta(z, w)$ satisfies Range's condition in Definition 2 with order $m = 2k$. This completes the proof of Proposition 3.

3. We can now prove the Main Theorem. Let Ω be a bounded pseudoconvex domain in C^2 with real analytic boundary: By Lemma 1 the weakly pseudoconvex points $w(\partial\Omega)$ can be stratified by real analytic sets S_0, S_1 and S_2 where S_j has dimension $j, j = 0, 1, 2$. Proposition 3 gives that S_2 consists only of R-points. We need the following $\bar{\partial}$ -result by Range [11].

THEOREM 7. — *Let $D \subset\subset C^2$ be a pseudoconvex domain with C^∞ boundary. Assume that \bar{D} has a Stein neighbourhood basis. If λ is a $\bar{\partial}$ -closed $(0, 1)$ -form with uniformly bounded coefficients on D whose support clusters on ∂D only at R-points, then there exists a continuous function g on \bar{D} with $\bar{\partial}g = \lambda$ on D .*

This theorem applies as it is shown in [2] that $\bar{\Omega}$ has a Stein neighbourhood basis.

By rotation of the axis we may assume that the z -axis does not intersect $S_0 \cup S_1$. In particular, if $\varepsilon > 0$ is small enough, $F_\varepsilon := \{(z, w) \in \partial\Omega; \varepsilon/2 \leq |w| \leq \varepsilon\}$ consists only of R-points.

Following Beatrous [1], if $f \in A(\Omega)$ and $f(0) = 0$, we can write $f = zg^1 + wh^1$ in a small neighbourhood of 0. On the set $\{(z,w) \in \bar{\Omega}; |z| > \varepsilon\}$ we can write $f = zg^2 + wh^2$ with $g^2 = f/z$ and $h = 0$, ε arbitrarily small. Solving an additive Cousin problem we obtain the decomposition $f = zg^3 + wh^3$ on the set:

$$\bar{\Omega}_1 = \{(z,w) \in \bar{\Omega}; |w| < \varepsilon\},$$

with g^3, h^3 holomorphic and continuous up to the boundary. On the set

$$\bar{\Omega}_2 = \{(z,w) \in \bar{\Omega}; |w| > \varepsilon/2\}$$

we have the decomposition $f = zg^4 + wh^4$ where $g^4 = 0$ and $h^4 = f/w$. Where the two sets overlap, we get the equation

$$G := (g^3 - g^4)/w = (h^4 - h^3)/z.$$

We need holomorphic functions G_1, G_2 with continuous boundary values on $\bar{\Omega}_1, \bar{\Omega}_2$ respectively so that $G = G_1 - G_2$ on the intersection. This reduces in a standard way to solving a $\bar{\partial}$ -problem for a form with support in $\bar{\Omega}_1 \cap \bar{\Omega}_2$. Hence Theorem 7 shows that such G_1, G_2 exist.

We then obtain the decomposition $f = zg + wh$, $g, h \in A(\Omega)$ by letting

$$g = \begin{cases} g^3 - wG_1 & \text{on } \bar{\Omega}_1 \\ g^4 - wG_2 & \text{on } \bar{\Omega}_2 \end{cases}, \quad h = \begin{cases} h^3 + zG_1 & \text{on } \bar{\Omega}_1 \\ h^4 - zG_2 & \text{on } \bar{\Omega}_2 \end{cases}.$$

This completes the proof of the Main Theorem.

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Manuscrit reçu le 16 juillet 1982.

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