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ETALE COVERINGS OF A MUMFORD CURVE

by Marius van der PUT

Introduction.

For a Riemann surface X over $\mathbf{C}$ of genus ≥ 2 the finite unramified coverings $Y \rightarrow X$ are easily obtained from the uniformization of X . Indeed, from the universal covering

$$\mathcal{H} = \{z \in \mathbf{C} \mid \text{Im}(z) > 0\} \rightarrow X$$

with group $\Gamma \cong \pi_1(X)$ one obtains all possibilities for Y by taking $\mathcal{H}/N$ where N is a subgroup of Γ of finite index.

For an algebraic curve X defined over a complete non-archimedean valued field K the situation is more complicated. In order to obtain "enough" unramified coverings $Y \rightarrow X$ one has to suppose that X is a Mumford curve. One further distinguishes between merely unramified (or étale) coverings and analytic coverings. This is done in section 1. In the next section the abelian étale coverings of a Mumford curve over an algebraically closed field are constructed. In section 3 the base field is a local field and the abelian unramified extensions of the function field of the curve X are calculated. The result of this section is due to G. Frey. We have presented here a rigid-analytic proof of this theorem. For general background concerning analytic spaces over K we refer to [1] and [3].

1. Analytic coverings and étale coverings.

The field K is supposed to be algebraically closed and to be complete with respect to a non-archimedean valuation. A morphism

$f: Y \longrightarrow X$ of analytic spaces over K is an *étale covering* if f is surjective and if for every point $x \in X$ there exists an affinoid subspace U of X containing x such that $f^{-1}(U)$ is a disjoint union of affinoid subspaces $V_i (i \in I)$ and such that each $f: V_i \longrightarrow U$ is an isomorphism.

Suppose that $f: Y \longrightarrow X$ is a finite morphism. This means that X has an admissible affinoid covering $(X_i)_{i \in I}$ such that each $f^{-1}(X_i)$ is a non-empty affinoid subset of Y and such that each $\mathcal{O}_X(X_i) \longrightarrow \mathcal{O}_Y(f^{-1}(X_i))$ is a finite injective map of affinoid algebras. In case that f is finite one has: f is an étale covering if and only if for each $y \in Y$ the map $\hat{f}_y^*: \hat{\mathcal{O}}_{Y,y} \longrightarrow \hat{\mathcal{O}}_{X,f(y)}$ is an isomorphism.

Indeed, $\hat{f}_y^*$ isomorphism implies that also $f_y^*: \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,f(y)}$ is an isomorphism and that there are affinoid sets V, U containing y and $f(y)$ such that $f: V \longrightarrow U$ is an isomorphism. Take $x \in X$ and put $f^{-1}(x) = \{y_1, \dots, y_n\}$. Choose affinoid neighbourhoods V_i of y_i and U of x such that every $V_i \longrightarrow U$ is an isomorphism. After shrinking U we may suppose that the V_i are disjoint and that every point $t \in U$ has n pre-images in Y . Then clearly $f^{-1}(U) = V_1 \cup \dots \cup V_n$, the V_i are disjoint and each $V_i \longrightarrow U$ is an isomorphism.

The morphism f is called an *analytic covering* if there exists an admissible affinoid covering $(X_i)_{i \in I}$ of X , an admissible covering $(Y_j)_{j \in J}$ of Y by affinoid subsets and a surjective map $\pi: J \longrightarrow I$ such that for all i :

- (i) $f^{-1}(X_i)$ is the disjoint union of the Y_j with $\pi(j) = i$
- (ii) $f: Y_j \longrightarrow X_i$ is an isomorphism for each j with $\pi(j) = i$.

An analytic covering is certainly an étale covering. The map $f: K^* \longrightarrow K^*$ given by $z \longmapsto z^n$ ($n > 1$ and n prime to $\text{char } K$) provides an example of an étale covering which is not an analytic covering. This is rather in contrast with the complex-analytic case where the corresponding notions coincide. In the sequel we will restrict ourselves to one-dimensional regular analytic spaces and especially to complete non-singular curves over K . It is clear however that many results will be correct for higher dimensional spaces.

LEMMA 1.1. — *Let $f: Y \longrightarrow X$ be an étale (resp. analytic) covering of non-singular complete irreducible algebraic curves. Then*

the minimal Galois extension $g: Z \rightarrow X$ is also an étale (resp. analytic) covering.

Proof. — For the function fields of X, Y and Z we have the inclusions $F(X) \subset F(Y) \subset F(Z)$ and $F(Z)$ is the minimal Galois-extension of $F(X)$ containing $F(Y)$. Let $Y_i \rightarrow X$ ($i = 1, \dots, s$) denote the morphisms corresponding to the subfields of $F(Z)$ which are conjugated with $F(Y)$. Since each $Y_i \rightarrow X$ is an étale (resp. analytic) covering the same holds for $Y_1 \times_X \dots \times_X Y_s \rightarrow X$. In particular $Y_1 \times_X \dots \times_X Y_s$ is non-singular and complete and every connected component is again an étale (resp. analytic) covering of X . The canonical map $Z \rightarrow Y_1 \times_X \dots \times_X Y_s$ induces an isomorphism of Z with a connected component.

This proves the lemma.

LEMMA 1.2. — Let $f: Y \rightarrow X$ be a non-constant morphism between (non-singular, irreducible, complete) curves. There exists a unique maximal decomposition $Y \xrightarrow{f} X = Y \xrightarrow{g_1} Y_1 \xrightarrow{f_1} X$ where Y_1 is a curve and f_1 is an étale covering. There exists a unique maximal decomposition $Y \xrightarrow{f} X = Y \xrightarrow{g_0} Y_0 \xrightarrow{f_0} X$ with Y_0 a curve and f_0 an analytic covering. Moreover $Y_1 \xrightarrow{f_1} X$ factors as $Y_1 \rightarrow Y_0 \xrightarrow{f_0} X$. If $Y \rightarrow X$ is Galois then also $Y_1 \rightarrow X$ and $Y_0 \rightarrow X$ are Galois.

Proof. — One has to consider subextensions of $F(X) \subset F(Y)$. For subextensions $F(Z_1)$ and $F(Z_2)$ let $F(Z_3)$ denote the least subfield containing $F(X_1)$ and $F(X_2)$. Then $Z_3 \rightarrow X$ is an étale (resp. analytic) covering if and only if $Z_1 \rightarrow X$ and $Z_2 \rightarrow X$ are étale (resp. analytic) coverings.

1.3 Let now X denote the Mumford curve Ω/Γ ; Γ a Schottky group with Ω as set of ordinary points in $\mathbb{P}^1$. It is known that $\Omega \rightarrow X$ is the universal analytic covering of X . In particular every finite analytic covering $Y \rightarrow X$ has uniquely the form $\Omega/\Gamma_0 \rightarrow X$ where Γ_0 is a subgroup of Γ of finite index. The étale coverings of X are hidden in Ω . We introduce the following notion: $c: \Omega_* \rightarrow \Omega$ is a Γ -equivariant covering if:

- (i) $c: \Omega_* \rightarrow \Omega$ is a finite, connected, Galois, étale covering with group H .

- (ii) Every automorphism $\gamma \in \Gamma$ of Ω lifts to an automorphism δ of Ω_* . (i.e. $c\delta = \gamma c$).

Let G denote the group of analytic automorphisms δ of Ω_* such that $c\delta = \gamma c$ holds for some $\gamma \in \Gamma$.

From the definitions one obtains a canonical exact sequence of groups $1 \longrightarrow H \xrightarrow{\pi} G \longrightarrow \Gamma \longrightarrow 1$. Let N denote a normal subgroup of G of finite index such that $N \cap H = \{1\}$. With the notations we can formulate the following results.

THEOREM 1.4. —

1) Ω_*/N is a non-singular, irreducible, complete curve over K . The map $\Omega_*/N \longrightarrow \Omega/\Gamma = X$ is a Galois, étale-covering with Galois group G/N . This map decomposes uniquely into

$$\Omega_*/N \longrightarrow \Omega/\pi(N) \longrightarrow X \text{ where } \Omega/\pi(N) \longrightarrow X$$

is the maximal analytic subcovering.

2) Let Y be an irreducible non-singular complete curve and let $f: Y \longrightarrow X$ be a Galois, étale-covering. There exists a pair (Ω_*, N) (unique up to isomorphism) and an isomorphism $g: Y \longrightarrow \Omega_*/N$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & X \\ \downarrow & \nearrow & \\ \Omega_*/N & & \end{array}$$

is commutative.

Proof. —

1) The construction of Ω_*/N as a 1-dimensional regular analytic space over K is very similar to the construction in [3] p. 105. One can make this construction explicit by a choice of a fundamental domain. Let $F \in \Omega$ be a good fundamental domain for the group $\pi(N)$ ([3] p. 28). Then F has the form $\mathbb{P}^1 - B_1 \cup \dots \cup B_{2a}$ where $\pi(N) = \langle \gamma_1, \dots, \gamma_a \rangle$ and $B_1, \dots, B_{2a}$ are open discs such that the corresponding discs B_i^+ are still disjoint and such that γ_i is an isomorphism of $B_i^+ - B_i$ with $B_{i+1}^+ - B_{i+a}$ ($i = 1, \dots, a$).

Let $\tilde{B}_i \supset B_i^+$ denote open discs such that the closed discs $\tilde{B}_i^+$ are still disjoint. Put $G = \mathbb{P}^1 - \tilde{B}_1 \cup \dots \cup \tilde{B}_{2a}$. Then $\Omega/\pi(N)$ can be constructed by glueing the affinoid pieces G , $\tilde{B}_1^+ - B_1, \dots, \tilde{B}_{2a}^+ - B_{2a}$ according to

- (i) $\tilde{B}_i^+ - B_i$ is glued to G over the subset $\tilde{B}_i^+ - \tilde{B}_i$.

- (ii) for $1 \leq i \leq a$, $\tilde{B}_i^+ - B_i$ is glued to $\tilde{B}_{i+a}^+ - B_{i+a}$ by using the isomorphism $\gamma_i: B_i^+ - B_i \xrightarrow{\sim} B_{i+a}^+ - B_{i+a}$.

To obtain Ω_*/N we replace in the construction above the affinoid sets G , $\tilde{B}_i^+ - B_i$, $B_i^+ - B_i$ by the subsets $c^{-1}(G)$, $c^{-1}(\tilde{B}_i^+ - B_i)$, $c^{-1}(B_i^+ - B_i)$ of Ω_* and γ_i by the unique element $\tilde{\gamma}_i \in N$ with $\pi(\tilde{\gamma}_i) = \gamma_i$.

The only thing that one has to verify is that $c^{-1}(G)$ etc are affinoid subsets. Indeed, one can easily verify the more general statement: "Let $U \rightarrow V$ be a finite morphism of analytic spaces over K . If V is affinoid then U is also affinoid."

Using this construction of Ω_*/N and the given affinoid covering of Ω_*/N one can calculate that $\dim_K H^1(\Omega_*/N, \mathcal{O}) < \infty$ and finally prove that Ω_*/N is actually a complete, irreducible, non-singular algebraic curve over K . (See [3] p. 106-107). The only statement that we still have to verify is the maximality of the analytic subextension $\Omega/\pi(N) \rightarrow X$. The normal subextensions correspond to normal subgroups M of G containing N . We have to show that $\Omega_*/M \rightarrow \Omega/\Gamma$ is an analytic covering if and only if $M \supseteq H$.

Put $M \cap H = H_1$. We replace

$$\Omega_* \xrightarrow{c} \Omega \text{ by } \Omega'_* = \Omega_*/H_1 \xrightarrow{c'} \Omega$$

and H by $H' = H/H_1$; G by $G' = G/H_1$ and M by $M' = M/H_1$. Again we have an exact sequence $1 \rightarrow H' \rightarrow G' \rightarrow \Gamma \rightarrow 1$ and now $M' \cap H' = \{1\}$. We have to show $\Omega'_* = \Omega$ if $\Omega'_*/M' \rightarrow \Omega/\Gamma$ is an analytic covering. The hypothesis implies easily that $\Omega'_* \rightarrow \Omega$ is a connected analytic covering. According to [3] p. 151, (3.4), one has $\Omega'_* \xrightarrow{\sim} \Omega$.

2) We consider the commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\pi} & \Omega \\ f \uparrow & & \uparrow f' \\ Y & \xleftarrow{\pi'} & Y \times_X \Omega = \Omega' \end{array}$$

The fibre product Ω' is as a set of points equal to

$$\{(y, \omega) \in Y \times \Omega \mid f(y) = \pi(\omega)\}.$$

One can easily give Ω' the structure of an analytic space over K since

π is an analytic covering. We denote by G_0 the Galois group of $Y|X$. The group $G_0 \times \Gamma$ acts as group of analytic automorphisms on Ω' in the following way: $(\sigma, \gamma)(y, \omega) = (\sigma(y), \gamma(\omega))$. Easy arguments will prove the following statements:

- a) f' is an étale covering with group G_0 ; possibly not connected.
- b) π' is an analytic covering with group Γ ; possibly not connected.

c) $\Omega'/\Gamma = Y$ and $\Omega'/G_0 = \Omega$.

d) for every connected affinoid $U \subset \Omega$, the set $(f')^{-1}(U)$ is affinoid. G_0 acts transitively on the connected components and each of them is mapped surjectively to U .

e) After applying d) to a sequence $U_1 \subset U_2 \subset U_3 \subset \dots$ of connected affinoid subsets of Ω which defines the holomorphic structure on Ω , one finds that Ω' has finitely many components $\Omega'_1, \dots, \Omega'_s$. Each component is mapped surjectively to Ω and G_0 acts transitively on the components.

f) From $\Omega'/\Gamma = Y$ it follows that Γ acts transitively on the components and that $\Omega'_1/\Gamma = Y$ where

$$N = \{(1, \gamma) \in G_0 \times \Gamma \mid \gamma(\Omega'_1) = \Omega'_1\}.$$

Put $\Omega_* = \Omega'_1$ and let $c: \Omega_* \longrightarrow \Omega$ denote the restriction of f' to Ω_* . We make the following definitions:

$$G = \{(\sigma, \gamma) \in G_0 \times \Gamma \mid (\sigma, \gamma)\Omega_* = \Omega_*\}$$

$$H = \{(\sigma, 1) \in G_0 \times \Gamma \mid (\sigma, 1)\Omega_* = \Omega_*\}$$

$$N = \{(1, \gamma) \in G_0 \times \Gamma \mid (1, \gamma)\Omega_* = \Omega_*\}.$$

From c) $\Omega'/G_0 = \Omega$ it follows that $\Omega_*/H = \Omega$ and that $c: \Omega_* \longrightarrow \Omega$ is a Galois étale covering, connected, and with group H .

The sequence $1 \longrightarrow H \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$ is exact since for every $\gamma \in \Gamma$ there exists a $\sigma \in G_0$ such that $\sigma(\Omega_*) = \gamma(\Omega_*)$. So $(\sigma^{-1}, \gamma) \in G$ and this element maps to γ . The group N is clearly a normal subgroup of finite index in G and $N \cap H = \{1\}$. Finally, according to f) we have $\Omega_*/N \cong Y$.

Similar methods will easily give the uniqueness (up to isomorphism) of the pair (Ω_*, N) .

PROPOSITION 1.5. — *Let Y be a complete, non-singular, irreducible curve over K or a 1-dimensional, regular, connected affinoid space. Then Y has a universal analytic covering. The Galois group of this universal analytic covering is a finitely generated free (non-commutative) group.*

Proof of 1.5. — The analytic space Y has a reduction $r: Y \rightarrow Z$ which is pre-stable and such that every component of Z is non-singular. (This is proved in [4].) The graph G of Z , i.e. the vertices of G are the components of Z and the edges of G are the double points of Z , is in general no a tree. Let $T \rightarrow G$ be the universal covering of the graph. Then T is a tree and on it operates a group $\Gamma \cong \pi_1(G)$ which is a finitely generated free group such that $T/\Gamma \simeq G$. As in [3] p. 149 (3.2), one can lift the construction of T and Γ to obtain an analytic space Ω and an analytic covering $u: \Omega \rightarrow Y$ with group Γ , such that Ω has a reduction $\bar{\Omega}$ and an induced map $\bar{u}: \bar{\Omega} \rightarrow Z$ which is for the Zariski-topology the universal covering and such that the graph associated with $\bar{\Omega}$ is T and $\bar{u}: T \rightarrow G$ is the universal covering of the graph mentioned above. The proposition will follow now if we can show that Ω admits only trivial analytic coverings. It suffices to show that an affinoid space U such that its canonical reduction $\bar{U}$ consists of non-singular affine curves intersecting normally has only trivial analytic coverings. Indeed Ω is build up out of such affinoid spaces U in an acyclic way.

Let now $\varphi: V \rightarrow U$ be an analytic covering. According to the definition $U = U_1 \cup \dots \cup U_n$ where the U_i are affinoid subspaces of U and such that $\varphi^{-1}(U_i)$ is the disjoint union of affinoid subsets of V , each of them mapped isomorphically to U_i . After refining the covering $\{U_1, \dots, U_n\}$ of U we may suppose that it is a pure covering such that the corresponding reduction $\bar{U}$ of U is prestable and has non-singular components (see [4]). The reduction $\bar{V}$ of V with respect to $\{\varphi^{-1}(U_1), \dots, \varphi^{-1}(U_n)\}$ is also prestable and the induced map $\bar{V} \rightarrow \bar{U}$ is a covering for the Zariski-topology. One knows that $\bar{U}$ is obtained from $\bar{U}$ by a finite number of steps. In each step a point is replaced by a projective line over $\bar{K}$. This shows that $\bar{U}$ has only trivial coverings for the Zariski-topology. If we assume that V is connected then also $\bar{V}$ is connected. Hence $\bar{V} = \bar{U}$ and so $V = U$. This shows finally

the existence of the universal analytic covering $u: \Omega \rightarrow Y$. We want to show that Ω has the usual property:

"Given a morphism $f: \hat{S} \rightarrow Y$, where S is a connected analytic space which has only trivial analytic coverings, and given points $s \in S$ and $\omega \in \Omega$ with $u(\omega) = f(s)$, then there exists a unique lift $f': S \rightarrow \Omega$ with $u f' = f$ and $f'(s) = \omega$."

We consider the fibre-product $\Omega' = \Omega \times_Y S \rightarrow S$. This is an analytic covering S . By assumption, every component of Ω' maps isomorphically to S . Taking the component of Ω' which contains the point (ω, s) one finds f' and one shows that f' is unique.

COROLLARY 1.6. — *Let Y, N, Ω_* be as in (1.4) and let $\Omega(Y)$ denote the universal analytic covering of Y which has group $\Gamma(Y)$. There exists a normal subgroup Γ_0 of $\Gamma(Y)$ such that $\Omega_* \cong \Omega(Y)/\Gamma_0$ and $\Gamma(Y)/\Gamma_0 \cong N$.*

Proof. — Easy consequence of (1.4) and (1.5).

Remark. — In general, Ω_* is not the universal analytic covering of Y . In section 2 we will discuss examples. The reason is that a connected, Galois, étale covering $e: \Omega_* \rightarrow \Omega$, admits itself in general non-trivial analytic coverings.

Example 1.7. — Take

$$\Omega = \mathbf{P}^1 - \{0, \pi, 1, \infty\} \text{ where } 0 < |\pi| < 1.$$

And let $\Omega_* = \{(x, y) \in \Omega \times K \mid y^2 = x(x - \pi)(x - 1)\}$. Assuming that the characteristic of K is unequal to two, one finds that $c: \Omega_* \rightarrow \Omega$ is a connected étale covering with Galois group $\mathbf{Z}/2$. The elliptic curve, corresponding to the equation $y^2 = x(x - \pi)(x - 1)$ is the Tate curve $K^*/\langle q \rangle$ for a suitable q , $0 < |q| < 1$. Further $\Omega_* = K^*/\langle q \rangle - \{\pm 1, \pm q^{1/2}\}$. The Tate curve has the universal analytic covering $K^* \rightarrow K^*/\langle q \rangle$. This easily implies that the universal analytic covering of Ω_* must be $U = K^* - \{\pm q^{n/2} \mid n \in \mathbf{Z}\}$. The resulting connected étale covering $U \rightarrow \Omega$ is in this case Galois. Its group is generated by two elements γ, δ , defined as automorphisms of U by $\gamma(z) = qz$ and $\delta(z) = z^{-1}$. The only relations are $\delta^2 = 1$ and $\delta\gamma = \gamma^{-1}\delta$.

More examples 1.8. — Let Γ denote a finitely generated discontinuous subgroup of $\mathrm{PGL}(2, K)$. Suppose that $\Gamma/[\Gamma, \Gamma]$ is a finite group. Let Ω denote the set of ordinary points for Γ . It is known that $\Omega/\Gamma \cong \mathbf{P}^1$ (see [3] Ch. VIII, (4.3)). There exists a normal subgroup $\Gamma_0 \subset \Gamma$ of finite index, which is a Schottky group. That implies that $c: \Omega \longrightarrow \Omega/\Gamma = \mathbf{P}^1$ is only ramified above a finite subset S of $\mathbf{P}^1$. Then $\Omega - c^{-1}(S) \longrightarrow \mathbf{P}^1 - S$ is a Galois étale map with group Γ . Special cases of such groups Γ are provided by Whittaker groups or by cyclic extensions of $\mathbf{P}^1$ (see [3, 6]).

Remark 1.9. — Let the Schottky group Γ and its space of ordinary points $\Omega \subset \mathbf{P}^1$ be given. It is rather difficult to construct equivariant étale coverings $\Omega_* \longrightarrow \Omega$. In the next section we will restrict our attention to abelian extensions $\Omega_* \longrightarrow \Omega$.

2. Construction of the abelian étale coverings.

We assume in this section that X is a Mumford curve over K of genus g and we fix a presentation $X = \Omega/\Gamma$ with Γ a Schottky group on g generators and in which $\Omega \subset \mathbf{P}^1$ is the subspace of ordinary points of Γ . According to (1.4) we have to construct the abelian Γ -equivariant étale morphisms $c: \Omega_* \longrightarrow \Omega$ such that in the notation of (1.3), one has $[G, G] \cap H = \{1\}$. Indeed, there must exist a normal subgroup N , of finite index, in G with abelian factor group and $N \cap H = \{1\}$. We call an abelian Γ -equivariant étale map $c: \Omega_* \longrightarrow \Omega$ *strongly abelian* if $[G, G] \cap H = \{1\}$. This condition is clearly equivalent to “ G is the direct product of H and Γ ”. Let Θ denote the group of invertible holomorphic functions f on Ω satisfying $f(\gamma\omega)/f(\omega)$ is a constant for every $\gamma \in \Gamma$. According to [3] Ch. II, the group Θ/K^* is isomorphic to $\mathbf{Z}^g$. Elements $\theta_1, \dots, \theta_g$ in Θ are called a basis if their images in $\mathbf{Z}^g$ form a $\mathbf{Z}$ -basis. The main result of this section states that every Γ -equivariant strongly abelian covering of Ω has the form

$$\Omega_* = \{(\omega, \lambda_1, \dots, \lambda_g) \in \Omega \times (K^*)^g \mid \lambda_i^{n_i} = \theta_i(\omega) \text{ for } i = 1, \dots, g\}$$

where we have chosen a basis $\theta_1, \dots, \theta_g$ of Θ and where $n_1, \dots, n_g$ are positive integers, not divisible by $\mathrm{char} K$. We start the proof by giving Ω_* the structure of an analytic space over K . Let $\{\Omega_n\}$

denote a sequence of connected affinoid subsets of Ω such that (i) $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \dots$ and (ii) every affinoid subset of Ω is contained in some Ω_n . For each n we consider the affinoid space Ω_{*n} corresponding to the affinoid algebra

$$\Theta(\Omega_n) [X_1, \dots, X_g] / (X_1^{n_1} - \theta_1, \dots, X_g^{n_g} - \theta_g).$$

As a point set Ω_{*n} is equal to $\{(\omega, \lambda_1, \dots, \lambda_g) \in \Omega_* \mid \omega \in \Omega_n\}$.

The analytic space Ω_* is obtained by glueing together the affinoid spaces Ω_{*n} according to the natural inclusions $\Omega_{*n} \longrightarrow \Omega_{*m}$ (for $n \leq m$). The map $c: \Omega_* \longrightarrow \Omega$ is etale and finite of degree $n_1 \dots n_g$. The automorphisms of $\Omega_* \longrightarrow \Omega$ are of the form $(\omega, \lambda_1, \dots, \lambda_g) \longrightarrow (\omega, \xi_1^{\alpha_1} \lambda_1, \dots, \xi_g^{\alpha_g} \lambda_g)$ where ξ_i denote a primitive n_i -th root of unity and $0 \leq \alpha_i < n_i$. So $\Omega_* \longrightarrow \Omega$ is Galois with group $H = \mathbb{Z}/n_1 \oplus \dots \oplus \mathbb{Z}/n_g$. The function theory on Ω_* is not much more complicated than that of Ω . Indeed $\Theta(\Omega_*)$ equals $\varprojlim \Theta(\Omega_{*n})$ and turns out to be

$$\Theta(\Omega) [X_1, \dots, X_g] / (X_1^{n_1} - \theta_1, \dots, X_g^{n_g} - \theta_g).$$

As usual we write $\mathfrak{M}$ for the sheaf of meromorphic functions. For any affinoid U one has $\mathfrak{M}(U) =$ the total ring of fractions of $\Theta(U)$.

Again $\mathfrak{M}(\Omega_*) = \varprojlim \mathfrak{M}(\Omega_{*n})$ coincides with

$$\mathfrak{M}(\Omega) [X_1, \dots, X_g] / (X_1^{n_1} - \theta_1, \dots, X_g^{n_g} - \theta_g).$$

The space Ω_* is connected if and only if $\mathfrak{M}(\Omega_*)$ is a field. Let m denote the smallest common multiple of $n_1, \dots, n_g$. It suffices to verify that $\mathfrak{M}(\Omega) [Y_1, \dots, Y_g] / (Y_i^m - \theta_i; i = 1, \dots, g)$ is a field. By Kummer-theory this is translated into: the images of $\theta_1, \dots, \theta_g$ in $\mathfrak{M}(\Omega)^* / \mathfrak{M}(\Omega)^{*m}$ are independent over $\mathbb{Z}/m$.

Suppose now that $\theta_1^{\alpha_1} \dots \theta_g^{\alpha_g}$, with $0 \leq \alpha_i < m$, equals f^m for some $f \in \mathfrak{M}(\Omega)$. Then clearly $f \in \Theta(\Omega)^*$. Since $(f(\gamma\omega)/f(\omega))^m$ is constant for every $\gamma \in \Gamma$ and since Ω is connected, one finds that $f \in \Theta$. The independance of $\theta_1, \dots, \theta_g$ yields $\alpha_1 = \dots = \alpha_g = 0$. This finally shows that Ω_* is connected. Let further a_i denote the homomorphism of Γ in K^* satisfying $\theta_i(\gamma\omega) = a_i(\gamma) \theta_i(\omega)$. Let $b_i \in \text{Hom}(\Gamma, K^*)$ be chosen such that $b_i^{n_i} = a_i$. Then we can define a Γ -action on Ω_* by

$$\gamma(\omega, \lambda_1, \dots, \lambda_g) = (\gamma(\omega), \lambda_1 b_1(\gamma), \dots, \lambda_g b_g(\gamma)).$$

This action commutes with the H -action on Ω_* . Hence $\Omega_* \longrightarrow \Omega$ is a strongly abelian Γ -equivariant étale morphism with group H . Next we want to find a presentation of Ω_* which does not depend on the choice of $\theta_1, \dots, \theta_g, n_1, \dots, n_g$. This is done as follows. Let G be the group of automorphisms of Ω_* , as defined in (1.3). The group acts on $\mathfrak{M}(\Omega_*)$, $\Theta(\Omega_*)$ etc. We consider its action on $\Theta(\Omega_*)^*/K^*$. Let $x_1, \dots, x_g \in \Theta(\Omega_*)^*$ be given by

$$x_i(\omega, \lambda_1, \dots, \lambda_g) = \lambda_i.$$

A straightforward calculation shows that $H^0(G, \Theta(\Omega_*)^*/K^*)$ is the free $\mathbb{Z}$ -module generated by the images of $x_1, \dots, x_g$. And this group is a finite extension of $H^0(\Gamma, \Theta(\Omega)^*/K^*) = \Theta/K^*$. We obtain in this way a $\mathbb{Z}$ -lattice T in $\Theta/K^* \otimes \mathbb{Q}$ containing Θ/K^* . The lattice T is uniquely determined by Ω_* and determines Ω_* . We will write $\Omega_* = \Omega(T)$ in the sequel. The group of automorphisms of $\Omega(T) \longrightarrow \Omega$ is equal to the Pontryagin dual of the cokernel of $\Theta/K^* \longrightarrow T$. We can now formulate the main result of this section, using again the notation of (1.3). We consider only lattices T such that $\text{char}(K)$ does not divide the order of H .

THEOREM 2.1. — *For every strongly abelian Γ -equivariant map $\Omega_* \longrightarrow \Omega$ there exists a unique $\mathbb{Z}$ -lattice and an isomorphism $\Omega_* \xrightarrow{\sim} \Omega(T)$.*

COROLLARY 2.2. — *Every finite abelian étale-covering of $X = \Omega/\Gamma$ has uniquely the form $\Omega(T)/N$, where T is a $\mathbb{Z}$ -lattice and where N is a subgroup of G with $N \cap H = \{1\}$ and πN is a normal subgroup of Γ of finite index and with an abelian factor group.*

Proof of 2.2. — The corollary follows from (1.4), (2.1) and the fact that G is the direct product of H and Γ . A further consequence is:

COROLLARY 2.3. — *The Galois group Δ of the maximal unramified abelian extension of $\mathfrak{M}(X)$, the function field of $X = \Omega/\Gamma$, is isomorphic to:*

- a) $\hat{\mathbb{Z}}^{2g}$ if $\text{char } K = 0$
- b) $\hat{\mathbb{Z}}^g \times \prod_{\ell \neq p} \hat{\mathbb{Z}}_{\ell}^g$ if $\text{char } K = p \neq 0$.

There is further a canonical surjective homomorphism of Δ onto $\hat{\mathbb{Z}}^g$ = the Galois group of the maximal abelian analytic covering of X .

Proof of (2.1). — It suffices to show the following two statements:

a) if $\text{char } K = p \neq 0$ then there does not exist an equivariant $\Omega_* \longrightarrow \Omega$ with group $\mathbb{Z}/p$.

b) if $\Omega_* \longrightarrow \Omega$ is a cyclic equivariant étale covering with group $H = \mathbb{Z}/n$ such that $\text{char}(K)/n$ and $H \cap [G, G] = \{1\}$, then there is a suitable $\theta \in \Theta$ with $\Omega_* \simeq \{(\omega, \lambda) \in \Omega \times K^* \mid \lambda^n = \theta(\omega)\}$.

Proof of a).

The map $c: \Omega_* \longrightarrow \Omega$ induces a field extension $\mathfrak{N}(\Omega) \subset \mathfrak{N}(\Omega_*)$ which is supposed to be cyclic of degree p . By Schreier theory, $\mathfrak{N}(\Omega_*)$ is obtained from $\mathfrak{N}(\Omega)$ by adjoining a root of $X^p - X - f$. One can change the f in this equation by adding a meromorphic function of the form $g^p - g$ with $g \in \mathfrak{N}(\Omega)$. After a suitable change of this type we may suppose that every pole (if any) of f has order $< p$. In a pole $\omega_0 \in \Omega$ of f of order $< p$ the map $\Omega_* \longrightarrow \Omega$ is ramified. So we have shown that f can be supposed to belong to $\Theta(\Omega)$.

Consider the exact sequence

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \Theta(\Omega) \xrightarrow{\tau} \Theta(\Omega) \xrightarrow{\sigma} M \longrightarrow 0$$

where τ is given by $\tau(h) = h^p - h$. The extension $\mathfrak{N}(\Omega_*) \mid \mathfrak{N}(\Omega)$ determines uniquely the subgroup of M generated by $\tau(f)$. The action of Γ of $\mathfrak{N}(\Omega)$ extends to $\mathfrak{N}(\Omega_*)$. This implies that $\sigma(f \circ \gamma) = c(\gamma) \sigma(f)$ for a certain homomorphism $c: \Gamma \longrightarrow \mathbb{F}_p^*$. After replacing Γ by a subgroup of finite index, we may suppose that $\sigma(f)$ is invariant under Γ . We recall that $H^0(\Gamma, \Theta(\Omega)) = H^0(\Omega/\Gamma, \Theta_X)$ and $H^1(\Gamma, \Theta(\Omega)) = H^1(\Omega/\Gamma, \Theta_X)$ with $X = \Omega/\Gamma$. For the constant sheaf K_X on X with stalk K one also has $H^0(\Gamma, K) = H^0(X, K_X)$ and $H^1(\Gamma, K) = H^1(X, K_X)$. Further the canonical maps

$$H^i(X, K_X) \longrightarrow H^i(X, \Theta_X) \quad (i = 0, 1)$$

are bijective. Using the exact sequence of Γ -modules

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \Theta(\Omega) \longrightarrow \Theta(\Omega)/\mathbb{F}_p \longrightarrow 0$$

one finds

$$H^0(\Gamma, \Theta(\Omega)/\mathbb{F}_p) = K/\mathbb{F}_p \quad \text{and} \quad H^1(\Gamma, \Theta(\Omega)/\mathbb{F}_p) = \text{Hom}(\Gamma, K/\mathbb{F}_p).$$

The exact sequence of Γ -modules

$$0 \longrightarrow \Theta(\Omega)/\mathbb{F}_p \xrightarrow{\tau} \Theta(\Omega) \longrightarrow M \longrightarrow 0$$

induces the long exact sequence

$$\begin{aligned} 0 \longrightarrow K/\mathbb{F}_p \xrightarrow{\tau} K \longrightarrow H^0(\Gamma, M) \longrightarrow \text{Hom}(\Gamma, K/\mathbb{F}_p) \\ \xrightarrow{\tau} \text{Hom}(\Gamma, K) \longrightarrow \dots \end{aligned}$$

This implies that $H^0(\Gamma, M) = 0$. Hence $\tau(f) = 0$. This contradicts the assumption that the equation $X^p - X - f$ is irreducible.

Proof of b).

The map $c: \Omega_* \longrightarrow \Omega$ induces a field extension $\mathcal{N}(\Omega) \subset \mathcal{N}(\Omega_*)$ with cyclic group $\mathbb{Z}/n$ and irreducible equation $X^n - f$, for some $f \in \mathcal{N}(\Omega)$. Since $\Omega_* \longrightarrow \Omega$ is étale one may suppose that $f \in \mathcal{O}(\Omega)^*$. We consider the exact sequence

$$1 \longrightarrow \mathcal{O}(\Omega)^*/K^* \xrightarrow{\tau} \mathcal{O}(\Omega)^*/K^* \xrightarrow{\sigma} M \longrightarrow 0$$

where τ is defined by $\tau(g) = g^n$.

The subgroup of M generated by $g = \tau(f \bmod K^*)$ has $\mathbb{Z}/n$ elements and is uniquely determined by the extension $\mathcal{N}(\Omega) \subset \mathcal{N}(\Omega_*)$. The action of Γ on $\mathcal{N}(\Omega)$ extends to $\mathcal{N}(\Omega_*)$. This implies that $\gamma(g) = g^{a(\gamma)}$ where $a: \Gamma \longrightarrow (\mathbb{Z}/n)^*$ is some group homomorphism. This means that $f(\gamma\omega) = f(\omega)^{a(\gamma)} b_\gamma(\omega)^n$ holds for some $b_\gamma \in \mathcal{O}(\Omega)^*$. Let x denote an element of $\mathcal{N}(\Omega_*)$ with $x^n = f$. The action of γ on $\mathcal{N}(\Omega_*)$ must have the form $\gamma(x) = x^{a(\gamma)} b_\gamma$. This action must commute with the automorphism δ of $\mathcal{N}(\Omega_*) | \mathcal{N}(\Omega)$ given by $\delta(x) = \zeta x$ where ζ is a primitive n -th root of unity. Since $\delta\gamma(x) = \zeta^{a(\gamma)} x^{a(\gamma)} b_\gamma$ and $\gamma\delta(x) = \zeta x^{a(\gamma)} b_\gamma$, one finds that $a(\gamma) = 1$ for all $\gamma \in \Gamma$. The map $\gamma \longmapsto b_\gamma$ is a 1-cocycle with values in $\mathcal{O}(\Omega)^*$ and its n -th power is the trivial cocycle $\gamma \longmapsto \frac{f \circ \gamma}{f}$. In [5] one has derived an exact sequence

$$\dots \text{Hom}(\Gamma, K^*) \longrightarrow H^1(\Gamma, \mathcal{O}(\Omega)^*) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

This implies that the image of the 1-cocycle $\{\gamma \longmapsto b_\gamma\}$ in $\mathbb{Z}$ is zero. Hence b_γ has the form $d(\gamma) \cdot c \circ \gamma / c$ for some homomorphism $d: \Gamma \longrightarrow K^*$ and some functions $c \in \mathcal{O}(\Omega)^*$. Hence $\theta = c^{-n} f$ satisfies $\theta(\gamma\omega) = d(\gamma)^n \theta(\omega)$ and so θ belongs to Θ . The extension $\mathcal{N}(\Omega) \subset \mathcal{N}(\Omega_*)$ is then also described by the equation $X^n - \theta$. It follows easily that Ω_* is isomorphic to $\{(\omega, \lambda) \in \Omega \times K^* \mid \lambda^n = \theta(\omega)\}$. This finishes the proof of (2.1).

Example 2.4. — The special case of (2.1) and (2.2) where the genus of X is 1 is particularly simple. The statement reads:

Every finite abelian étale extension of $X = K^*/\langle q \rangle$ (where $0 < |q| < 1$) is of the form $K^*/\langle q' \rangle \xrightarrow{\varphi} K^*/\langle q \rangle$ where the map φ is induced by $z \longmapsto z^n$ from $K^* \longrightarrow K^*$ with n not divisible by $\text{char } K$ and where q' satisfies $(q')^n \in \langle q \rangle = q^{\mathbb{Z}}$.

PROPOSITION 2.5. — Let $\varphi: Y \longrightarrow X$ be a finite abelian étale of the Mumford curve $X = \Omega/\Gamma$. We suppose that the order of the group H (see (2.2)) is not divisible by $\text{char } \bar{K}$. Let $\mathcal{U}$ be a pure affinoid covering of X such that the reduction $(\bar{X}, \bar{\mathcal{U}})$ satisfies:

- (i) every component of $(\bar{X}, \bar{\mathcal{U}})$ is non-singular.
- (ii) every singular point of $(\bar{X}, \bar{\mathcal{U}})$ is an ordinary double point.

Then $\varphi^{-1}(\mathcal{U})$ is a pure affinoid covering of Y and the reduction $(\bar{Y}, \varphi^{-1}(\bar{\mathcal{U}}))$ of Y with respect to $\varphi^{-1}(\mathcal{U})$ also satisfies (i) and (ii). The canonical map of $(\bar{Y}, \varphi^{-1}(\bar{\mathcal{U}}))$ to $(\bar{X}, \bar{\mathcal{U}})$ is unramified outside the double points of $(\bar{Y}, \varphi^{-1}(\bar{\mathcal{U}}))$.

Proof. — Any small enough $U \in \mathcal{U}$ is isomorphic to an affinoid subset of $\mathbb{P}^1$. The proof of (2.5) follows from the next lemma.

LEMMA 2.6. — Let U be an affinoid subset of $\mathbb{P}^1$ given by the inequalities: $|\pi| \leq |z| \leq 1$; $|z - a_1| \geq 1, \dots, |z - a_s| \geq 1$; $|z - b_1| \geq |\pi|, \dots, |z - b_t| \geq |\pi|$ in which $0 < |\pi| < 1$; $|a_i| = 1$; $|a_i - a_j| = 1$ for $i \neq j$; $|b_i| = |\pi|$ and $|b_i - b_j| = |\pi|$ for $i \neq j$. Let $u_1, \dots, u_c \in \mathcal{O}(U)^*$ and let $n_1, \dots, n_c$ denote positive integers not divisible by $\text{char}(K)$. Let V denote the affinoid space be given by its affinoid algebra

$$\mathcal{O}(V) = \mathcal{O}(U) \langle X_1, \dots, X_c \rangle / (X_1^{n_1} - u_1, \dots, X_c^{n_c} - u_c).$$

Then the canonical reduction $\bar{V}$ of V has non-singular components. The only singularities of $\bar{V}$ are ordinary double points. The map $\bar{V} \longrightarrow \bar{U}$ is unramified outside the double points of $\bar{V}$.

Proof. — We may suppose that $\mathcal{O}(V)$ is an integral domain. Let M denote the subgroup of $\mathcal{O}(U)^*$ consisting of the elements m of the form

$$m = z^{k_0} (z - a_1)^{k_1} \dots (z - a_s)^{k_s} \left(\frac{\pi}{z} - \frac{\pi}{b_1} \right)^{q_1} \dots \left(\frac{\pi}{z} - \frac{\pi}{b_t} \right)^{q_t}.$$

The $k_0, k_1, \dots$ are integers and we write $k_0 = k_0(m)$. Then M is a free abelian group of rank $s + t + 1$. Every element of $\mathcal{O}(U)^*$ can uniquely be decomposed as $u \cdot m$ with $m \in M$ and $u = \lambda + h$, $\lambda \in K^*$ and $h \in \mathcal{O}(U)$ such that $\|h\| < |\lambda|$. Let $d = [\mathcal{O}(V) : \mathcal{O}(U)]$ and let N denote the group of elements of $\mathcal{O}(V)^*$ having their d -th power in M . Then $N = N_0 \oplus N_1$ where N_1 is the group of the d -th roots of unity and where N_0 is a free abelian group satisfying

$[N_0 : M] = d$. Take a basis $u_1, \dots, u_{s+t+1}$ of M such that N_0 is the free group generated by $\frac{1}{n_1} u_1, \dots, \frac{1}{n_{s+t+1}} u_{s+t+1}$ (in additive notation). With this choice one can write

$$\Theta(V) = \Theta(U) [X_1, \dots, X_{s+t+1}] / (X_i^{n_i} - u_i; i = 1, \dots, s+t+1).$$

It is possible to choose the $u_1, \dots, u_{s+t+1}$ such that $k_0(u_1) = 1$ and $k_0(u_i) = 0$ for $i = 2, \dots, t+s+1$.

Consider the surjective map of $\Theta(U) \langle X_1, Y_1, X_2, X_3, \dots, X_{s+t+1} \rangle$ to $\Theta(V)$ given by $X_i \mapsto X_i$ and $Y_1 \mapsto \rho X_1^{-1}$ with $\rho \in K^*$ such that $\rho^{n_1} = \pi$. This map induces a norm on $\Theta(V)$ and the reduction R of $\Theta(V)$ with respect to this norm is

$$\overline{\Theta(U)} [X_1, Y_1, X_2, X_3, \dots, X_{s+t+1}]$$

divided by the ideal generated by the elements $X_1^{n_1} - \bar{u}_1$, $Y_1^{n_1} - \frac{\pi}{u_1}$, $X_1 Y_1$, $X_i^{n_i} - \bar{u}_i$ for $i \geq 2$. Further $\overline{\Theta(U)}$ is the localization of $\bar{K}[T, S]/TS$ at the element

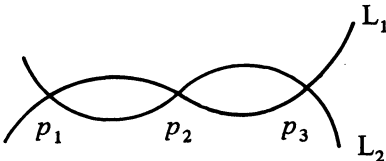
$$(T - \bar{a}_1) \dots (T - \bar{a}_s) \left(S - \frac{\pi}{b_1} \right) \dots \left(S - \frac{\pi}{b_t} \right).$$

A straightforward calculation shows that R has no nilpotents. Hence R is the reduction of $\Theta(V)$ with respect to the spectral norm. The only singular maximal ideals of R are

$$(X_1, Y_1, X_2 - c_2, \dots, X_{s+t+1} - c_{s+t+1})$$

in which $c_i \in \bar{K}$ satisfies $c_i^{n_i} = \bar{u}_i(\tau)$ with $|\pi| < |\tau| < 1$. The completion of the local ring of R at such a maximal ideal is $\cong \bar{K}[[X_1, Y_1]]/(X_1 Y_1)$. Further $\overline{\Theta(U)} \rightarrow R$ is unramified outside the ideal (S, T) of $\overline{\Theta(U)}$. This proves the lemma.

An example 2.7. — Let X be a Mumford curve of genus 2 with reduction $\bar{X}$



(Two rational curves L_1, L_2 intersecting in 3 points p_1, p_2, p_3 .)

We write $r: X \rightarrow \bar{X}$ for the reduction map. Let $\theta \in \Theta$ be a theta function for the curve X . On the affinoid part $r^{-1}(L_1 - \{p_1, p_2, p_3\})$ the function θ can be represented by a holomorphic invertible func-

tion u which is normalized by $\|u\| = 1$. The reduction $\bar{u}$ is a rational function on L_1 which is invertible and regular outside $\{p_1, p_2, p_3\}$. Let $\text{ord}(\theta)$ denote the triple $(a_1, a_2, a_3) \in \mathbb{Z}^3$ given by $a_i = \text{ord}_{p_i}(\bar{u})$. This induces a group homomorphism

$$\text{ord} : \Theta/K^* \longrightarrow \{(a_1, a_2, a_3) \in \mathbb{Z}^3 \mid a_1 + a_2 + a_3 = 0\}.$$

Using [5] one easily shows that it is an isomorphism. Let $\theta_1, \theta_2 \in \Theta$ be a basis for the theta functions. Put $\text{ord}(\theta_1) = (a_1, a_2, a_3)$ and $\text{ord} \theta_2 = (b_1, b_2, b_3)$. As in (2.2) the curve Y is given by $Y = \Omega_*/N$ in which

$\Omega_* = \{(\omega, \lambda_1, \lambda_2) \in \Omega \times (K^*)^2 \mid \lambda_1^{n_1} = \theta_1(\omega) \text{ and } \lambda_2^{n_2} = \theta_2(\omega)\}$ and where N maps bijectively to Γ . We assume further that $\text{char } \bar{K}$ does not divide $n_1 n_2$. The reduction of Y obtained in (2.5) is denoted by $\bar{Y}$. The étale map $\varphi : Y \longrightarrow X$ induces some $\tilde{\varphi} : \bar{Y} \longrightarrow \bar{X}$. We will use (2.5) and the proof of (2.6) in order to calculate the reduction $\bar{Y}$.

Let t be a parameter on $L_1 \cong \mathbb{P}^1$ such that $t = 0, 1, \infty$ corresponds to p_1, p_2, p_3 on L_1 . Then $\tilde{\varphi}^{-1}(L_1 - \{p_1, p_2, p_3\})$ is the affine variety over $\bar{K}$ with coordinate ring

$$\bar{K}[t]_{(t-1)}[X_1, X_2]/(X_1^{n_1} - t^{a_1}(t-1)^{a_2}, X_2^{n_2} - t^{b_1}(t-1)^{b_2}).$$

It is connected and non-singular. Its closure in $\bar{Y}$ is a curve M_1 . The curve M_1 is an abelian ramified covering of $L_1 = \mathbb{P}^1$. The genus g of M_1 is given by the Riemann-Hurwitz formula

$$2g - 2 = 2n_1 n_2 + \frac{n_1 n_2}{e_1} (e_1 - 1) + \frac{n_1 n_2}{e_2} (e_2 - 1) + \frac{n_1 n_2}{e_3} (e_3 - 1).$$

In this formula e_i denotes the ramification index of a point of M_1 above p_i in L . One easily verifies that $\frac{1}{e_i}Z = \frac{a_i}{n_1}Z + \frac{b_i}{n_2}Z$ for $i = 1, 2, 3$. One finds in the same manner that $M_2 = \tilde{\varphi}^{-1}(L_2)$ is a non-singular curve of the same genus. The two curves M_1 and M_2 meet in $\frac{n_1 n_2}{e_1} + \frac{n_1 n_2}{e_2} + \frac{n_1 n_2}{e_3}$ points (namely the $\tilde{\varphi}$ -pre-images of p_1, p_2, p_3). Hence the arithmetic genus of $\bar{Y}$ is equal to $2g - 1 + n_1 n_2 \left\{ \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} \right\}$.


 (Picture of $\bar{Y}$)

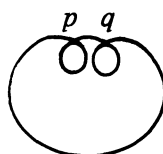
M_1 One easily computes that this number is equal to the genus of $M_2 Y$ (as it should be).

The universal analytic covering of Y (as constructed in (1.5)) has an automorphism group $\Gamma(Y)$ which is free on $n_1 n_2 \left\{ \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} \right\}$ generators. This number is equal to $\sum_{i=1}^3 \text{g.c.d.}(n_2 a_i, n_1 b_i)$ and so ≥ 3 .

This shows that Ω_* cannot be the universal analytic covering of Y .

2.8 The other examples of a Mumford curve of genus 2

a) X is a Mumford curve with stable reduction $\bar{X}$:



The reduction is $\mathbf{P}^1$ parametrized by t where the two pairs of points $t = 0, t = \infty$ and $t = 1, t = d$ are identified. Again one has an isomorphism $\Theta/K^* \xrightarrow{\text{ord}} \mathbf{Z}^2$ given as follows: $\theta \in \Theta$ lift to a function u on $r^{-1}(\bar{X} - \{p, q\})$ with constant absolute value 1. The reduction $\bar{u}$ is a rational function on the normalization $\mathbf{P}^1$ of $\bar{X}$ and we put $\text{ord}(\theta) = (\text{ord}_0 \bar{u}, \text{ord}_1 \bar{u})$. Let θ_1, θ_2 be a basis of the theta functions and put $\text{ord}(\theta_1) = (a_1, a_2)$ and $\text{ord}(\theta_2) = (b_1, b_2)$. Let Y be the curve obtained from X by (2.2) with $\Omega_* = \{(\omega, \lambda_1, \lambda_2) | \lambda_1^{n_1} = \theta_1(\omega), \lambda_2^{n_2} = \theta_2(\omega)\}$ and N which maps bijectively to Γ . The reduction of Y is made by using (2.5). The canonical map $\varphi: Y \rightarrow X$ induces a $\tilde{\varphi}: \bar{Y} \rightarrow \bar{X}$. The pre-image $\tilde{\varphi}^{-1}(\bar{X} - \{p, q\})$ is affine with coordinate ring

$$\bar{K}[t]_{t(t-1)(t-d)}[X, Y] / \left(X^{n_1} - t^{a_1} \left(\frac{t-1}{t-d} \right)^{a_2}, Y^{n_2} - t^{b_1} \left(\frac{t-1}{t-d} \right)^{b_2} \right).$$

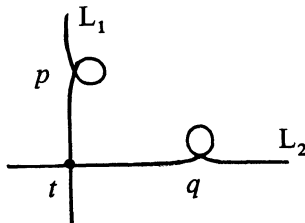
The corresponding non-singular projective curve (i.e. the normalization of $\bar{Y}$) has genus g given by the Riemann-Hurwitz formula

$$2g - 2 = -2n_1 n_2 + 2 \frac{n_1 n_2}{e_1} (e_1 - 1) + 2 \frac{n_1 n_2}{e_2} (e_2 - 1) \text{ and } \frac{1}{e_1} Z = \frac{a_1}{n_1} Z + \frac{b_1}{n_2} Z \text{ and } \frac{1}{e_2} Z = \frac{a_2}{n_1} Z + \frac{b_2}{n_2} Z.$$

The number of double points of $\bar{Y}$ is $\frac{n_1 n_2}{e_1} + \frac{n_1 n_2}{e_2}$. So $\bar{Y}$ is an irreducible curve with double points.

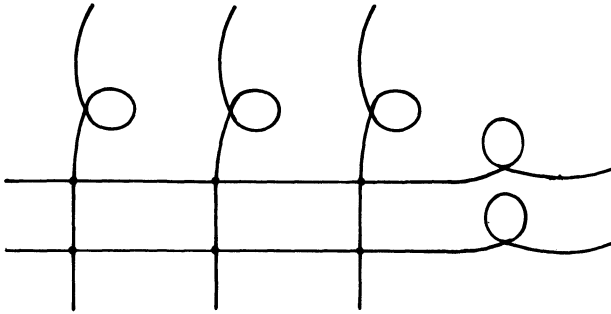
The group $\Gamma(Y)$ (see (1.5)) is free on $\frac{n_1 n_2}{e_1} + \frac{n_1 n_2}{e_2} - 1$ generates.

b) X is Mumford curve with stable reduction $\bar{X}$:



Let L_1 be described by a parameter t_1 where $t_1 = 1, -1$ corresponds to p and $t_1 = 0$ corresponds to r . A parameter t_2 describes L_2 in a similar way. A theta function θ for X is lifted to a function u on $r^{-1}(L_1 - \{p, r\})$. One can normalize u such that $\|u\| = 1$. Put $a_1 = \text{ord}_1 \bar{u}$. In a similar way a_2 is defined. One obtains again an isomorphism $\text{ord}: \Theta/K^* \rightarrow \mathbb{Z}^2$ with $\text{ord}(\theta) = (a_1, a_2)$ as given above.

Let θ_1, θ_2 be a basis of the theta functions and let $Y \xrightarrow{\varphi} X$ be defined by " $\sqrt[n_1]{\theta_1}, \sqrt[n_2]{\theta_2}$ ". We study now the reduction $\bar{Y}$ and the map $\tilde{\varphi}: \bar{Y} \rightarrow \bar{X}$. The pre-image $\tilde{\varphi}^{-1}(L_1)$ is given by the equations $X^{n_1} - \left(\frac{t_1 - 1}{t_1 + 1}\right)^{a_1}, Y^{n_2} - \left(\frac{t_1 - 1}{t_1 + 1}\right)^{b_1}$. Here we have written $\text{ord}(\theta_1) = (a_1, a_2)$ and $\text{ord}(\theta_2) = (b_1, b_2)$. Let $e_1 \geq 1$ be defined by $\frac{1}{e_1} \mathbb{Z} = \frac{a_1}{n_1} \mathbb{Z} + \frac{b_1}{n_2} \mathbb{Z}$. Then $\tilde{\varphi}^{-1}(L_1)$ turns out to be the disjoint union of $\frac{n_1 n_2}{e_1}$ curves $M_1(1), \dots, M_1\left(\frac{n_1 n_2}{e_1}\right)$. Each $M_1(i)$ is a rational curve with one double point. The $M_1(i)$ are isomorphic to each other. The map $M_1(i) \rightarrow L_1$ has degree e_1 and is only ramified in the unique double point of $M_1(i)$. On each $M_1(i)$ lie e_1 pre-images of the point r . There is a similar description for $\tilde{\varphi}^{-1}(L_2) = M_2(1) \cup \dots \cup M_2\left(\frac{n_1 n_2}{e_2}\right)$ with $\frac{1}{e_2} \mathbb{Z} = \frac{a_2}{n_1} \mathbb{Z} + \frac{b_2}{n_2} \mathbb{Z}$. Every $M_1(i)$ meets e_1 of the curves $M_2(j)$ and every $M_2(j)$ meets e_2 of the curves $M_1(i)$. The reduction $\bar{Y}$ is totally split and stable. The curve Y is a Mumford curve. We have made a picture of $\bar{Y}$ for the values $a_1 = 1, a_2 = 0, b_1 = 0, b_2 = 1, n_1 = e_1 = 2$ and $n_2 = e_2 = 3$.



3. Mumford curves over a local field.

In this section k denotes a local field and K will be the completion of the algebraic closure of k . Let $\Gamma \subset \mathrm{PGL}(2, k)$ denote a Schottky group on g generators. Then $\mathcal{L}$ is a subset of $\mathbf{P}^1(k)$. Let Ω denote the analytic space over k , given by $\Omega = \mathbf{P}_k^1 - \mathcal{L}$. The action of Γ on Ω is k -rational and one can form the quotient $X = \Omega/\Gamma$. For every (finite) extension ℓ of k the set of ℓ -rational points of $X \times_k \ell$ is equal to $\mathbf{P}^1(\ell) - \mathcal{L}/\Gamma$. In particular the set of k -rational points of X is equal to $\mathbf{P}^1(k) - \mathcal{L}/\Gamma$. For our purposes we need that X has k -rational points. So we have to assume that $\mathcal{L}$ is a proper subset of $\mathbf{P}^1(k)$. The theta functions, corresponding to Γ , are elements of $\mathcal{O}(\Omega)$ since they can be written in the form

$$\theta_\delta = \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma\delta(a)}, \text{ where } a \in \mathbf{P}^1(k) - \mathcal{L} \text{ and } \delta \in \Gamma.$$

For every $\delta \in \Gamma$ the homomorphism $c_\delta: \Gamma \longrightarrow K^*$, given by $\theta_\delta(\gamma\omega) = c_\delta(\gamma) \theta_\delta(\omega)$, has also values in k^* . As in § 2 we want to calculate the abelian unramified field-extensions of $\mathfrak{N}(X) = H^0(\Gamma, \mathfrak{N}(\Omega))$. The field $\mathfrak{N}(X)$ is a function field of genus g with precise field of constants k .

A contribution to those extensions are the abelian extensions of the field of constants k . Restrictions with respect to the extensions in § 2 are:

- (i) k contains only finitely many roots of unity; let n denote their number.

- (ii) For a theta function θ with $\theta(\gamma\omega) = a(\gamma) \theta(\omega)$, there exists in general no homomorphism $b: \Gamma \longrightarrow k^*$ with $b^n = a$.

For any lattice T (again T is a lattice in $\Theta/k^* \times \otimes_{\mathbb{Z}} \mathbf{Q}$ containing Θ/k^*) there is an analytic space $\Omega(T)$ over k defined by the more or less symbolic formula

$$\Omega(T) = \{(\omega, \lambda_1, \dots, \lambda_g) \in \Omega \times (k^*)^g \mid \lambda_i^{n_i} = \theta_i(\omega), i = 1, \dots, g\}.$$

The function field $\mathfrak{N}(\Omega(T))$ of $\Omega(T)$ is equal to $\mathfrak{N}(\Omega)[x_1, \dots, x_g]$ where $x_i^{n_i} = \theta_i$. Let us write $a_i \in \text{Hom}(\Gamma, k^*)$ for the homomorphism $\gamma \longmapsto \theta_i(\gamma\omega) \theta_i(\omega)^{-1}$. Let $b_i \in \text{Hom}(\Gamma, K^*)$ denote a homomorphism satisfying $b_i^{n_i} = a_i$. Let ℓ be a finite Galois extension of k containing all the values $b_i(\gamma)$. The analytic space (over k) $\Omega(T) \times_k \ell$ has a group of automorphism G given by: an automorphism δ belongs to G if δ extends some automorphism $\gamma \in \Gamma$ of Ω .

From our choice of the field ℓ it follows that we have an exact sequence:

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\pi} \Gamma \longrightarrow 1 \text{ with } H = \text{Aut}(\Omega(T) \times_k \ell \longrightarrow \Omega).$$

Let M denote the subgroup $\text{Aut}(\Omega(T) \times_k \ell \longrightarrow \Omega \times_k \ell)$ of H and let N denote the subgroup $\text{Aut}(\Omega(T) \times_k \ell \longrightarrow \Omega(T)) \cong \text{Gal}(\ell \mid k)$ of H . Then M is a normal subgroup and we have an exact sequence $1 \longrightarrow M \longrightarrow H \longrightarrow \text{Gal}(\ell \mid k) \longrightarrow 1$ and H is the semi-direct product of M and N .

According to § 2 every finite abelian unramified covering of X has the form $\Omega(T) \times \ell/N$ for suitable, T , ℓ and N and in which N is a normal subgroup of G and G/N is a finite abelian group.

One clearly has $[G, G] \cap H$ is contained in N . In particular $[H, H]$ is contained in N . We will need the following lemma.

LEMMA 3.1. — *Let H denote the automorphism group of $\Omega(T) \times_k \ell \mid \Omega$ and let $[H, H]$ denote the commutator subgroup of H . Then $\Omega(T) \times_k \ell / [H, H] \cong \Omega(T') \times_k \ell'$ where*

- (i) ℓ' is the maximal abelian subextension of ℓ .
- (ii) T' is a sublattice of T , and T' satisfies $n T' \subset \Theta/k^*$.

Proof. — We choose a basis $\theta_1, \dots, \theta_g$ of Θ such that T is the $\mathbb{Z}$ -module generated by $\frac{1}{n_1} (\theta_1 \bmod k^*), \dots, \frac{1}{n_g} (\theta_g \bmod k^*)$.

As before the function field of $\Omega(T) \times_k \ell$ has the form

$$\mathfrak{N}(\Omega) \otimes_k \ell[x_1, \dots, x_g] \text{ with } \theta_i = X_i^{n_i}.$$

The commutator subgroup $[H, H]$ is generated by the elements $\{\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \mid \sigma_1, \sigma_2 \in N\}$ and $\{\sigma h \sigma^{-1} h^{-1} \mid \sigma \in N \text{ and } h \in M\}$. Let h_i denote the element of M given by the action $h_i(X_j) = X_j$ if $j \neq i$ and $h_i(X_i) = \xi_i X_i$ where ξ_i is a primitive n_i -th-root of unity. An easy calculation shows that $\sigma h_i \sigma^{-1} h_i^{-1} = h_i^{a_i(\sigma)}$ where $a_i(\sigma)$ is an integer depending on i and σ . Let $e_i = \text{g.c.d.}(n_i, \text{all } a_i(\sigma))$. One easily shows that $[H, H]$ is equal to the semi-direct product $\langle h_1^{e_1}, \dots, h_g^{e_g} \rangle \cdot [N, N]$. Let T' denote the sublattice of T generated by $\frac{1}{e_1}(\theta_1 \bmod k^*), \dots, \frac{1}{e_g}(\theta_g \bmod k^*)$ and let ℓ' denote the maximal abelian extension of k contained in ℓ . The function field of $\Omega(T') \times \ell'$ is $\mathfrak{N}(\Omega) \otimes_k \ell'[X_1^{d_1}, \dots, X_g^{d_g}]$ with $d_i e_i = n_i$. The automorphism group of $\Omega(T) \times \ell$ over $\Omega(T') \times \ell'$ turns out to be $[H, H]$. Hence $\Omega(T) \times \ell / [H, H] = \Omega(T') \times \ell'$. Let us write $y_i = x_i^{d_i}$. The automorphism group of $\Omega(T') \times \ell' \mid \Omega$ is commutative. In particular, any

$$\sigma \in \text{Gal}(\ell' \mid k) = \text{Aut}(\Omega(T') \times \ell' \mid \Omega(T'))$$

must commute with any $h \in \text{Aut}(\Omega(T') \times \ell' \mid \Omega \times \ell')$. Take h given by the formula $h(Y_i) = \tau_i Y_i$ ($i = 1, \dots, g$) where τ_i is a primitive e_i -th root of unity. Then $\sigma h(Y_i) = \sigma(\tau_i) Y_i$ and $h \sigma(Y_i) = \tau_i Y_i$. So $\tau_i \in k$ and each e_i divides $n =$ the number of roots of unity of k . This finally shows that $n T' \subset \Theta/k^*$.

LEMMA 3.2. — *Let H denote the automorphism group of $\Omega(T) \times_k \ell \mid \Omega$. Let H_1 be a subgroup of H , containing $[H, H]$ and such that the image of H_1 in $\text{Gal}(\ell \mid k)$ is contained in $[\text{Gal}(\ell \mid k), \text{Gal}(\ell \mid k)]$. Then $\Omega(T) \times \ell / H_1 \cong \Omega(T'') \times \ell'$ with*

- a) ℓ' is the maximal abelian extension of k , contained in ℓ .
- b) T'' is a sublattice of T such that $n T'' \subset \Theta/k^*$.

Proof. — One divides first by $[H, H]$. The result $\Omega(T') \times \ell'$ is further divided by the group $H_1/[H, H]$ which lies by assumption in $\text{Aut}(\Omega(T') \times \ell' \mid \Omega \times \ell')$. The result is $\Omega(T'') \times \ell'$ where T'' is a sublattice of T' .

(3.3) We apply (3.2) to the group $H_1 = [G, G] \cap H$. Let $\varphi: \Gamma \rightarrow G$ be a left-inverse of the canonical surjection $G \rightarrow \Gamma$. One can define the action of $\varphi(\gamma)$ on the function field of $\Omega(T) \times \ell$ by: $\varphi(\gamma)(f) = f \circ \gamma$ for any $f \in \mathcal{N}(\Omega)$; $\varphi(\gamma)\lambda = \lambda$ for any $\lambda \in \ell$ and $\varphi(\gamma)X_i = b_i(\gamma)X_i$.

Then $H_1 = H \cap [G, G]$ is generated by $[H, H]$ and the commutators $\varphi(\gamma)h\varphi(\gamma)^{-1}h^{-1}$ with $\gamma \in \Gamma$ and $h \in H$. This expression is 1 for any $h \in M$. For $h = \sigma \in \text{Gal}(\ell|k) = \text{Aut}(\Omega(T) \times \ell|\Omega(T))$ one easily sees that the commutator lies in M . This means that H_1 satisfies the condition of (3.2). Let $\Omega(T'') \times \ell'$ denote the quotient of $\Omega(T) \times \ell$ by H_1 . This quotient is invariant under any $\varphi(\gamma)$. In other words, the action of Γ on Ω can be extended to action of Γ on $\Omega(T'') \times \ell'$.

Let us describe the function field of $\Omega(T'') \times \ell'$ by

$$F = \mathcal{N}(\Omega) \otimes_k \ell'[Y_1, \dots, Y_g] \quad \text{with} \quad Y_i^{n_i} = \theta_i.$$

Then each n_i divides n .

The automorphism $\tilde{\gamma}$ on F which lifts the automorphism γ on $\mathcal{N}(\Omega)$ must satisfy $\tilde{\gamma}(Y_i) = b_i(\gamma)Y_i$ for certain elements $b_i(\gamma) \in \ell'$. Moreover $\tilde{\gamma}$ must commute with the action of $\text{Gal}(\ell'|k)$ on F . This implies that $b_i(\gamma) \in k$. We draw the conclusion that T'' is a sublattice of $\frac{1}{n}(\Theta/k^*)$ such that the canonical homomorphism $c: \Theta/k^* \rightarrow \text{Hom}(\Gamma, k^*)$ which is given by

$$c(\theta \bmod k^*)(\gamma) = \theta(\gamma\omega)\theta(\omega)^{-1},$$

extends to a group homomorphism $T'' \rightarrow \text{Hom}(\Gamma, k^*)$. This proves the main result.

THEOREM 3.3. — *Every finite abelian, unramified extension of X has uniquely the form $\Omega(T) \times \ell/N$ where*

- (i) ℓ is a finite abelian extension of k
- (ii) T is a sublattice of $\frac{1}{n}(\Theta/k^*)$ such that the canonical homomorphism $c: \Theta/k^* \rightarrow \text{Hom}(\Gamma, k^*)$ extends to T .
- (iii) N is a normal subgroup of G with $N \cap H = \{1\}$. The image πN of N in Γ is a normal subgroup with abelian factor group.

COROLLARY 3.4. (G. Frey). — *The profinite Galois group D of the maximal abelian unramified extension of the function field $\mathfrak{K}(X)$ of X is isomorphic to the direct product*

$$\text{Gal}(k^{ab}/k) \oplus \hat{\mathbb{Z}}^g \oplus \mathbb{Z}/n_1 \oplus \dots \oplus \mathbb{Z}/n_g.$$

The numbers $n_1, \dots, n_g$ satisfy $n_1 | n_2 | \dots | n_g | n$ where $n =$ the number of roots of unity in k and they are determined by the curve X .

Proof of (3.4). — One easily sees that there exists a largest lattice T , with $\Theta/k^* \subset T \subset \frac{1}{n} \Theta/k^*$ such that the map

$$c: \Theta/k^* \longrightarrow \text{Hom}(\Gamma, k^*)$$

extends to T . The finite group in (3.4) is the cokernel of the injection $\Theta/k^* \subset T$.

Remark 3.5. — The corollary (3.4) has been proved by G. Frey [2]. His proof is quite different from the one presented here. It is based upon a detailed study of the action of the Galois group $\text{Gal}(k^{ab}/k)$ on the points of finite order (or the Tate-modules) of the Jacobian variety (or a generalized Jacobian variety) of the Mumford curve $X = \Omega/\Gamma$.

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