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A G-MINIMAL MODEL FOR PRINCIPAL G-BUNDLES

by Shrawan KUMAR

Introduction.

Sullivan built a minimal model theory for simplicial complexes. He showed that given a simply connected simplicial complex X with all its Betti numbers being finite, there is associated to it a certain uniquely determined (up to DGA isomorphism) DGA over $\mathbb{Q}$ (called minimal model for the space X) which contains exactly the rational homotopy information of the space X . Actually large part of this theory goes through for nilpotent simplicial complexes as well. For a quick exposition of this theory, see [3; Sections 1 to 3], [4] or [7].

Suppose $E \xrightarrow{p} B$ is a principal G -bundle, then the C^∞ de-Rham complex $\Omega(E)$ of E acquires additional structures due to the action of G on E . $\Omega(E)$ becomes a $\mathfrak{G}$ ($=$ Lie-algebra of G) algebra [see section 1]. In this paper we formulate a certain « natural » model $\mu_G[E]$ (which we call the G -minimal model) for the space E which is a collection of mutually « $\mathfrak{G}$ -homotopic » $\mathfrak{G}$ -algebras $\{A_\theta\}$, such that the DGA of basic elements in A_θ is the minimal model for B and any A_θ has the complete rational homotopy information of the space E (and B) (see theorem (2.2)).

In general (probably) we don't get a $\mathfrak{G}$ -morphism from any A_θ to $\Omega(E)$ inducing isomorphism in cohomology. We analyze a more general question in theorem (2.3). It turns out that it is equivalent to the existence of a « special » connection in the bundle E . The nature of « special » connection seems interesting. For example, such a connection Φ_0 (if it exists) in a principal G -bundle E with highly connected base space B , would have the property that the corresponding (to Φ_0) lower characteristic forms themselves vanish. This actual vanishing of

characteristic forms figures in the definition of secondary characteristic classes by Chern-Simons [2].

Section 1 contains the various definitions and some examples. The main theorems of the paper (Theorems 2.2 and 2.3) are formulated in section 2. Section 3 contains the proofs and examples of some G -bundles which admit « special » connections. We add an appendix to give a spectral sequence which converges to the cohomology of B and which has $H(E) \otimes H(BG)$ as its E_1 term.

We intend to take up the question « which principal G -bundles admit a « special » connection » in a separate paper.

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Throughout G will denote a compact connected real Lie group and $\mathfrak{G}$ its real Lie-algebra. All the G -bundles will be principal and in the smooth ($=C^\infty$) category with simply-connected base space B . Further we assume that all the Betti numbers of B are finite. Vector spaces will be over reals and linear maps would mean $\mathbf{R}$ -linear maps. Isomorphism would always mean surjective isomorphism.

1. Definitions.

(1.1) DEFINITIONS. — (a) *A Differential Graded Algebra (abbreviated as DGA) is an associative graded algebra $A = \bigoplus_{k \geq 0} A^k$ with unity and a differential $d : A \rightarrow A$ of degree $+1$ satisfying*

1) *A is graded commutative i.e. $x \cdot y = (-1)^{k' \ell} y \cdot x$ for $x \in A^k$ and $y \in A^{\ell}$.*

2) d is derivation i.e.

$$d(x \cdot y) = (dx) \cdot y + (-1)^k x \cdot dy \quad \text{for } x \in A^k$$

and

3) $d^2 = 0$.

A is connected if $H^0(A)$ is the ground field. A is simply connected if in addition $H^1(A) = 0$.

(b). — Let A and B be two DGA with morphisms $f, g : A \rightarrow B$. f and g are said to be homotopic, if there exists a morphism $H : A \rightarrow B \otimes_{\mathbb{R}} \mathbb{R}(t, dt)$ such that $\varepsilon_0 \circ H = f$ and $\varepsilon_1 \circ H = g$, where $\varepsilon_0, \varepsilon_1 : B \otimes \mathbb{R}(t, dt) \rightarrow B$ are evaluations at 0 and 1 respectively.

(c) [1(a), section 4]. — By a $\mathfrak{G}$ -algebra we mean a DGA A with two linear maps $L : \mathfrak{G} \rightarrow \text{Der}_0 A$ and $i : \mathfrak{G} \rightarrow \text{Der}_{-1} A$ ($\text{Der}_{\ell} A$ denotes the set of all derivations of degree ℓ i.e. linear maps $\theta : A^k \rightarrow A^{k+\ell}$ satisfying $\theta(ab) = \theta(a)b + (-1)^k a\theta(b)$ for $a \in A^k$) satisfying

$$1) i(X) \circ i(X) = 0$$

$$2) L(X)i(Y) = i(Y)L(X) + i[X, Y]$$

$$3) L(X) = di(X) + i(X)d$$

for all $X, Y \in \mathfrak{G}$.

Remarks. — 1) i and L correspond to inner and Lie derivatives respectively.

2) As a consequence of (2) and (3) above, L is a Lie algebra homomorphism.

Notation. — We denote by $A^{\mathfrak{G}} = \{a \in A : L(X)a = 0 = i(X)a \text{ for all } X \in \mathfrak{G}\}$ and call them *basic elements* and by

$$I(A) = \{a \in A : L(X)a = 0 \text{ for all } X \in \mathfrak{G}\}$$

and call them *invariant elements*. In the example (1) of (1.2) below, the basic elements correspond exactly to the forms on the base.

(d) Let A_1 and A_2 be two $\mathfrak{G}$ -algebras. A $\mathfrak{G}$ -morphism $\varphi : A_1 \rightarrow A_2$ is a DGA homomorphism commuting with L and i actions.

(1.2) *Examples of $\mathfrak{G}$ -algebras.* — (1) The main motivating example is the smooth de Rham complex $\Omega(E)$ of the total space E of a G -bundle.

(2) *Weil algebra of $\mathfrak{G}$* , which is defined to be the algebra $S(\mathfrak{G}^*) \otimes \Lambda(\mathfrak{G}^*)$ where $S(\mathfrak{G}^*)$ (respectively $\Lambda(\mathfrak{G}^*)$) denotes the total symmetric (respectively exterior) algebra of $\mathfrak{G}^*$ (=the dual of $\mathfrak{G}$).

For details of the operators d , L and i on $W(\mathfrak{G})$, see [1(a), section 6].

(3) $\Lambda(\mathfrak{G}^*)$ considered as a DGA with the operators

$$i(X)\omega = \omega(X) \quad \text{and} \quad [L(X)\omega]Y = -\omega[X, Y]$$

for $\omega \in \mathfrak{G}^*$ and $X, Y \in \mathfrak{G}$. Extend $i(X)$ and $L(X)$ as derivations on the whole of $\Lambda(\mathfrak{G}^*)$. We denote $I(\Lambda(\mathfrak{G}^*))$ by $I_A(\mathfrak{G})$.

(1.3) DEFINITIONS. — (1) *A connection in a $\mathfrak{G}$ -algebra A is, by definition, a $\mathfrak{G}$ -morphism from $W(\mathfrak{G})$ to A .*

It is not difficult to see that a connection in $\Omega(E)$ in this sense gives rise to a connection in the G -bundle E in the usual geometric sense and vice-versa. See [1(a); sections 5 and 6].

(2) *We call a $\mathfrak{G}$ -algebra A with connection to be irreducible if there does not exist a $\mathfrak{G}$ -subalgebra B (of A) admitting a connection such that $A \subsetneq B \supset A^{\mathfrak{G}}$.*

2. Formulations of the main results.

Let $E \xrightarrow{p} B$ be a principal G -bundle. We are tacitly assuming that the base space B is simply connected although this restriction is more of a convenience than necessity. One can have suitable formulations for non simply-connected B as well by taking ℓ -stage minimal model for the space B , which always exists for finite ℓ . See [3; theorem (1.1)]. We associate a « G -model » as below.

(2.1) *A « G -model » associated to E .* — Let us fix a minimal model $\rho: \mu \rightarrow \Omega(B)$ in the sense of Sullivan [3; section 1]. Let $S_E = \{\theta: \theta \text{ is a DGA morphism from } I = I_S(\mathfrak{G}) \text{ to } \mu \text{ such that the map induced in cohomology } I \rightarrow H^*(\mu) \simeq H^*(B) \text{ is the characteristic cohomology homomorphism induced from some (and hence any) connection in } E\}$.

$I_S(\mathfrak{G}) \subset W(\mathfrak{G})$ denotes the algebra of all the invariant polynomials on $\mathfrak{G}$. Since I is a polynomial algebra, any two morphisms in S_E are homotopic. (Actually, $\theta \in S_E$ is nothing but an induced map at the minimal model level corresponding to the unique homotopy class of maps: $B \rightarrow B(G)$ determined by E).

Given a $\theta \in S_E$, we associate a $\mathfrak{G}$ -algebra $A_\theta = W(\mathfrak{G}) \otimes_I \mu$, where μ is considered as an I -module via θ . The operators i_X and L_X , for all $X \in \mathfrak{G}$, are defined to be 0 on μ . i_X , L_X and d , being I -linear on both $W(\mathfrak{G})$ and μ , extend to operators on $W(\mathfrak{G}) \otimes_I \mu$. It is easy to see that A_θ becomes a $\mathfrak{G}$ -algebra.

Let $\Phi_{\text{res.}} : I \rightarrow \Omega(B)$ be the characteristic homomorphism (i.e. the evaluation of the invariant polynomial after substituting the curvature) corresponding to a smooth connection Φ on the bundle E .

As the maps $\gamma = \rho \circ \theta$, $\Phi_{\text{res.}}$ are homotopic, there is a diagram of $\mathfrak{G}$ -algebras (and $\mathfrak{G}$ -morphisms)

$$\begin{array}{ccc}
 A_\theta = W(\mathfrak{G}) \otimes_I \mu & \xrightarrow{\text{Id.} \otimes \rho} & W(\mathfrak{G}) \otimes_I^\gamma \Omega(B) \\
 & & \searrow \varepsilon_0 \\
 \text{(D)} \quad & & W(\mathfrak{G}) \otimes_I [\Omega(B) \otimes_R \mathbf{R}(t, dt)] \\
 & & \searrow \varepsilon_1 \\
 \Omega(E) \xleftarrow{\tilde{\Phi}} & W(\mathfrak{G}) \otimes_I^{\Phi_{\text{res.}}} \Omega(B) &
 \end{array}$$

$W(\mathfrak{G}) \otimes_I^\gamma \Omega(B)$ denotes the tensor product, where $\Omega(B)$ is considered as an I -module via γ . The map $\tilde{\Phi}$ is extension of the connection $\Phi : W(\mathfrak{G}) \rightarrow \Omega(E)$ and the canonical inclusion $\Omega(B) \hookrightarrow \Omega(E)$.

In view of the lemma (3.3) of this paper, all the maps in diagram (D) induce isomorphism in cohomology. Since E is a nilpotent space (B being simply connected, by assumption), for any $\theta \in S_E$, the DGA A_θ contains all the rational homotopy information of the space E . (Of course, the minimal model μ , of the base space, sits inside A_θ as exactly the set of its basic elements and hence the $\mathfrak{G}$ -algebra A_θ contains the complete rational homotopy information of the base space as well).

If we choose another $\theta_1 \in S_E$ then, θ, θ_1 being homotopic, A_θ and A_{θ_1} are « $\mathbb{G}$ -homotopic » in the following sense.

$$\begin{array}{ccc} & & \begin{array}{c} \theta \\ W(\mathbb{G}) \otimes \mu = A_\theta \\ \downarrow \\ \mathbb{I} \end{array} \\ & \nearrow^{\varepsilon_0} & \\ W(\mathbb{G}) \otimes [\mu \otimes \mathbf{R}(t, dt)] & & \\ & \searrow_{\varepsilon_1} & \\ & & \begin{array}{c} \theta_1 \\ W(\mathbb{G}) \otimes \mu = A_{\theta_1} \\ \downarrow \\ \mathbb{I} \end{array} \end{array}$$

Now let $E \xrightarrow{p} B$ and $E' \xrightarrow{p'} B'$ be two bundles with a G -morphism $f: E \rightarrow E'$. This induces, of course, a morphism $\Omega(B') \rightarrow \Omega(B)$ and hence a map $\tilde{f}: \mu' \rightarrow \mu$ at the minimal model level. It is easy to see that, for any $\theta' \in S_{E'}$, $\tilde{f}\theta' \in S_E$. There exists a canonical $\mathbb{G}$ -morphism $\mu_G[f]:$

$$A_{\theta'} = W(\mathbb{G}) \otimes \mu' \xrightarrow{\text{Id.} \otimes \tilde{f}} A_{\tilde{f}\theta'} = W(\mathbb{G}) \otimes \mu.$$

We summarize all this in the following.

(2.2) THEOREM. — Let $E \xrightarrow{p} B$ be a G -bundle (B being simply connected and having all its betti nos. finite). There is associated a collection $\mu_G[E] = \{A_\theta\}_{\theta \in S_E}$ of mutually « $\mathbb{G}$ -homotopic » $\mathbb{G}$ -algebras admitting connections, as defined above. Moreover, for any $\theta \in S_E$, $H^*(A_\theta)$ is isomorphic with $H^*(E)$ and $A_\theta^\mathbb{G}$ is a minimal model for B . In fact, A_θ contains the complete rational homotopy information of the space E (and B).

Further, given two G -bundles E, E' and a G -morphism $f: E \rightarrow E'$, there exists a « natural » $\mathbb{G}$ -morphism $\mu_G[f]$ from $\mu_G[E']$ to $\mu_G[E]$ (that is, for any $\theta' \in S_{E'}$ there exists a $\theta \in S_E$ and a « natural » $\mathbb{G}$ -morphism $A_{\theta'} \rightarrow A_\theta$) as defined above.

We call $\mu_G[E]$ the G -minimal model associated to the bundle E .

Remark. — Similarly, we can associate a $\mathbb{G}$ -minimal model to any $\mathbb{G}$ -algebra A which is finite dimensional in each degree, admits a connection and such that $A^\mathbb{G}$ is simply-connected.

For this, we choose a connection $\Phi: W(\mathbb{G}) \rightarrow A$. This gives a $\mathbb{G}$ -morphism $W(\mathbb{G}) \otimes A^\mathbb{G} \rightarrow A$, which induces isomorphism in

cohomology by lemma (3.3). Now we choose a minimal model $\rho : \mu \rightarrow A^{\mathfrak{G}}$ and take various homotopy lifts θ to make the construction of $A_{\theta} = W(\mathfrak{G}) \underset{1}{\overset{\theta}{\otimes}} \mu$.

$$\begin{array}{ccc} & & \mu \\ & \nearrow \theta & \downarrow \mathfrak{G} \\ I & \xrightarrow{\Phi_{\text{res.}}} & A \end{array}$$

Observe that, homotopy class of the map $\Phi_{\text{res.}}$ does not depend upon the particular choice of connection in A . $\square$

Now we study the existence of a $\mathfrak{G}$ -morphism : $A_{\theta} \rightarrow \Omega(E)$.

There exists a $\mathfrak{G}$ -morphism $\varphi : A_{\theta} \rightarrow \Omega(E)$ inducing the map ρ at the base if and only if there exists a connection Φ in the bundle E such that the following diagram is (actually) commutative

$$\begin{array}{ccc} I & \xrightarrow{\Phi_{\text{res.}}} & \Omega(B) \\ & \searrow \theta & \nearrow \rho \\ & \mu & \end{array}$$

More generally, we have the following result.

(2.3) THEOREM. — Let $E \xrightarrow{p} B$ be a G -bundle. There exists a $\mathfrak{G}$ -algebra $A = \bigoplus_{k \geq 0} A^k$ and a $\mathfrak{G}$ -morphism $\varphi : A \rightarrow \Omega(E)$ satisfying

- 1) A^k is finite dimensional for all $k \geq 0$ and A^0 is the ground field.
- 2) φ induces isomorphism in cohomology.

3) $\varphi|_{A^{\mathfrak{G}}} : A^{\mathfrak{G}} \rightarrow \Omega(B)$ is a minimal model in the sense of Sullivan if and only if there exist a connection $\Phi : W(\mathfrak{G}) \rightarrow \Omega(E)$, a minimal model $\rho : \mu \rightarrow \Omega(B)$ and a DGA morphism $\theta : I = I_s(\mathfrak{G}) \rightarrow \mu$ making the following diagram actually (not merely homotopically, which always exists) commutative.

$$\begin{array}{ccc} I & \xrightarrow{\Phi_{\text{res.}}} & \Omega(B) \\ & \searrow \theta & \nearrow \rho \\ & \mu & \end{array}$$

(D')...

Notes. — (1) We call any such connection a « special » connection.
 (2) Given such a diagram (D'), there is a « canonical » $\mathfrak{G}$ -morphism $\varphi_{\Phi, D'} : A_{\mathfrak{g}} \rightarrow \Omega(E)$, induced from the connection Φ and the map $\rho : \mu \rightarrow \Omega(B) \hookrightarrow \Omega(E)$, satisfying (1), (2) and (3) above.

(2.4) COROLLARY. — *If A is any $\mathfrak{G}$ -algebra with a $\mathfrak{G}$ -morphism $\varphi : A \rightarrow \Omega(E)$ satisfying (1), (2) and (3) above then there exist a « special » connection Φ in the algebra $\Omega(E)$ and a commutative diagram (D') with the property that there exists a $\mathfrak{G}$ -morphism $\alpha : A_{\mathfrak{g}} \rightarrow A$ satisfying $\varphi \circ \alpha = \varphi_{\Phi, D'}$.*

So, if A is irreducible, α is a surjective morphism. We prove theorem (2.3) and its corollary in the next section.

(2.5) Remark. — The following result due to Kostant [6; Theorem 0.2. and lemma 1] gives that, as a graded vector space over $\mathbf{R}$, $A_{\mathfrak{g}}$ can be identified with $\Lambda(\mathfrak{G}^*) \otimes_{\mathbf{R}} H \otimes_{\mathbf{R}} \mu$.

« Let H be any graded $\mathfrak{G}$ -submodule of $S(\mathfrak{G}^*)$ satisfying $I_S(\mathfrak{G})^+ S(\mathfrak{G}^*) \oplus H = S(\mathfrak{G}^*)$ ($I_S(\mathfrak{G})^+$ denotes the set of all the $\mathfrak{G}$ -invariant polynomials on $\mathfrak{G}$ with zero constant term). Then, the canonical map from $H \otimes I_S(\mathfrak{G})$ to $S(\mathfrak{G}^*)$, given by $f \otimes g \mapsto fg$, is a $\mathfrak{G}$ -module isomorphism.

H can be taken to be, for example, the set of all G -harmonic polynomials on $\mathfrak{G}$ where G is the adjoint group of $\mathfrak{G}$. »

3. Proofs and some examples.

First we prove the following lemmas.

(3.1) LEMMA. — *Any $\mathfrak{G}$ -algebra A , with a $\mathfrak{G}$ -morphism $\varphi : A \rightarrow \Omega(E)$ satisfying (1), (2) and (3) of theorem (2.3) admits a connection. In fact (3) can be replaced by a weaker assumption that $A^{\mathfrak{G}} \rightarrow \Omega(B)$ induces isomorphism in cohomology.*

Proof. — We show that there exists a linear map $\xi : \mathfrak{G}^* \rightarrow A^1$ commuting with the actions i and L .

Let us fix a point $e_0 \in E$. Consider the map $\varepsilon : G \rightarrow E$ defined by $\varepsilon(g) = e_0 g$. ε gives rise to a map $\varepsilon^* : \Omega(E) \rightarrow \Omega(G)$. We claim that $\varepsilon^* \varphi(A^k) \hookrightarrow \Lambda^k(\mathfrak{G}^*)$ (i.e. the left invariant k -forms on G). This is

because, for $X_1, \dots, X_k \in \mathfrak{G}$ and $a \in A^k$,

$$i(X_1) \circ \dots \circ i(X_k) \circ \varepsilon^* \varphi(a) = \varepsilon^* \varphi \circ i(X_1) \circ \dots \circ i(X_k)(a)$$

which is a constant function on G (since $A^0 \simeq \mathbf{R}$).

We further assert that $\varepsilon^* \varphi(A^1) = \mathfrak{G}^*$. Assuming this for a moment, let K be the kernel of the map $\varepsilon^* \varphi : A^1 \rightarrow \mathfrak{G}^*$ and $K^\perp$ be a $\mathfrak{G}$ -submodule (under the L action) of A^1 such that $K \oplus K^\perp = A^1$. $\varepsilon^* \varphi|_{K^\perp}$ is an isomorphism. Taking $(\varepsilon^* \varphi|_{K^\perp})^{-1} : \mathfrak{G}^* \rightarrow K^\perp \hookrightarrow A^1$ gives a desired map ξ .

Extend this map to an algebra morphism $\xi : \Lambda(\mathfrak{G}^*) \rightarrow A$. We define the curvature from $\mathfrak{G}^* \rightarrow A^2$ by $\omega \mapsto d(\xi(\omega)) - \xi(d_{\mathfrak{G}}(\omega))$, where $d_{\mathfrak{G}}$ denotes the differential in the complex $\Lambda(\mathfrak{G}^*)$, and extend this to $S(\mathfrak{G}^*)$. These two maps together give a unique algebra map (again denoted by) $\xi : W(\mathfrak{G}) \rightarrow A$. It is a routine checking that the map ξ is a connection in the $\mathfrak{G}$ -algebra A .

We return to prove that $\varepsilon^* \varphi(A^1) = \mathfrak{G}^*$. Let ω be a primitive element in $I_A^k(\mathfrak{G})$. As ω is universally transgressive, there exists a form $\tilde{\omega} \in \Omega^k(E)$ such that $\varepsilon^* \tilde{\omega} = \omega$ and $d\tilde{\omega} \in p^*(\Omega^{k+1}(B))$. We can further assume that $\tilde{\omega} \in I(\Omega^k(E))$, i.e. $L(X)\tilde{\omega} = 0$ for all $X \in \mathfrak{G}$. Since $H^{k+1}(A^{\mathfrak{G}}) \simeq H^{k+1}(B)$, there exists an element $y \in A^{\mathfrak{G}}$ such that $dy = 0$ and $\varphi(y) = d\tilde{\omega} + p^*(d\theta)$ for some $\theta \in \Omega^k(B)$. But then by taking $\tilde{\omega} + p^*(\theta)$ in place of $\tilde{\omega}$, we can assume that $\varphi(y) = d\tilde{\omega}$. By assumption $H(A) \simeq H(E)$, so that $y = dx$ for some $x \in A^k$. Since $H(I(A)) \simeq H(A)$ (as can be easily seen from the relation $L(X) = di(X) + i(X)d$), we can choose $x \in I(A^k)$.

Now $d(\varphi(x) - \tilde{\omega}) = 0$ and hence $\varphi(x) - \tilde{\omega} = \varphi(y') + d\theta'$ for some form $\theta' \in I(\Omega^{k-1}(E))$ and $y' \in I(A^k)$ (We are using $H(I(A)) \simeq H(I(\Omega(E)))$). This gives $\varepsilon^* \varphi(x) - \varepsilon^* \tilde{\omega} = \varepsilon^* \varphi(y') + d\varepsilon^*(\theta')$. Since $d\varepsilon^*(\theta')$ is a bi-invariant form on G which is a coboundary and hence is 0. So $\varepsilon^* \tilde{\omega} = \omega \in \varepsilon^* \varphi(A)$ and hence $\varepsilon^* \varphi(A)$ contains all the bi-invariant forms on G . But the image $\varepsilon^* \varphi(A)$ is closed under the actions of $i(X)$ and $L(X)$ which would imply that $\varepsilon^* \varphi(A^1) = \mathfrak{G}^*$, proving the lemma.

(3.2) LEMMA. — Let A be a $\mathfrak{G}$ -algebra admitting a connection Φ . Let Z denote the subalgebra of horizontal elements i.e.

$$Z = \{a \in A : i(X)a = 0 \text{ for all } X \in \mathfrak{G}\}.$$

Then the map $\beta : \Lambda(\mathfrak{G}^*) \otimes Z \rightarrow A$, defined by $\beta|_{\Lambda(\mathfrak{G}^*)} = \Phi|_{\Lambda(\mathfrak{G}^*)}$ and $\beta|_Z$ is the inclusion, is a graded algebra (but not DGA in general) isomorphism commuting with the natural i and L actions.

Proof. — Let us choose a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{G}$ and let $\{X_1^*, \dots, X_n^*\}$ be the dual basis (of $\mathfrak{G}^*$).

(a) β is injective. — For let

$$\beta \left(\sum_{\substack{0 \leq k \leq \ell \\ i_1 < \dots < i_k}} X_{i_1}^* \wedge \dots \wedge X_{i_k}^* \otimes h_{i_1, \dots, i_k} \right) = 0.$$

By operating $i(X_{j_\ell}) \circ \dots \circ i(X_{j_1})$ on both the sides, we get $h_{j_1, \dots, j_\ell} = 0$ and hence β is injective.

(b) β is surjective. — Let A_ℓ denote the set

$$\{a \in A : i(Y_1) \circ \dots \circ i(Y_\ell)a = 0 \quad \text{for all} \quad Y_1, \dots, Y_\ell \in \mathfrak{G}\}.$$

Clearly $A = A_{n+1} \supset A_n \supset \dots \supset A_1 = Z$. Assume, by induction, that A_ℓ is in the image of β (of course A_1 is in the image of β) and let $a \in A_{\ell+1}$. Consider the element

$$b = \sum_{i_1 < \dots < i_\ell} \beta(X_{i_1}^* \wedge \dots \wedge X_{i_\ell}^*) \cdot i(X_{i_\ell}) \circ \dots \circ i(X_{i_1})a.$$

By operating $i(X_{j_\ell}) \circ \dots \circ i(X_{j_1})$ on both the sides, we get

$$i(X_{j_\ell}) \circ \dots \circ i(X_{j_1})b = i(X_{j_\ell}) \circ \dots \circ i(X_{j_1})a.$$

This implies that $b - a \in A_\ell$ and hence, by induction hypothesis, $b - a \in \text{Image } \beta$, but $b \in \text{Image } \beta$ and hence a also is in the image.

We prove the following lemma which is analogue of Leray-Serre spectral sequence for fibrations.

(3.3) LEMMA. — *Let A be a $\mathfrak{G}$ -algebra admitting a connection which is finite dimensional in each degree. Then there exists a convergent spectral sequence with $E_2^{p,q} \simeq H^q(\mathfrak{G}) \otimes H^p(A^\mathfrak{G})$ and converging to the cohomology of A .*

Remarks. — (1) Observe that a principal G -bundle (for G a connected group, which we are always assuming) is always orientable.

(2) The hypothesis that A admits a connection is necessary. For, take a $\mathfrak{G}$ -algebra A with connection and then define

$$B = \sum_{\ell \geq 1} \Lambda(\mathfrak{G}^*) \otimes Z^\ell \oplus A^0.$$

For « appropriate » A , B will provide a counter example.

Proof (of the lemma). — Let Φ be a connection in A . By the previous lemma (3.2), this induces an isomorphism $\Lambda(\mathfrak{G}^*) \otimes Z \xrightarrow{\sim} A$. Consider the filtration $A = A_0 \supset A_1 \supset \cdots \supset A_p \supset \cdots$ where $A_p = \sum_{\ell \geq p} \Lambda(\mathfrak{G}^*) \otimes Z^\ell$. This is of course a convergent filtration bounded above. We compute $E_r^{p,q}$ for $r = 0, 1, 2$.

Clearly $E_0^{p,q} \simeq \Lambda^q(\mathfrak{G}^*) \otimes Z^p$. Further $E_1^{p,q} \simeq H^q(\mathfrak{G}, Z^p) \simeq H^q(\mathfrak{G}, (A^\mathfrak{G})^p)$. We are using the fact that the Lie-algebra cohomology of a reductive Lie-algebra $\mathfrak{G}$, with coefficients in a nontrivial finite dimensional irreducible $\mathfrak{G}$ -module V_p , vanishes i.e. $H(\mathfrak{G}, V_p) = 0$. See [5; Section 5-theorem 10]. Lastly $E_2^{p,q} \simeq H^q(\mathfrak{G}) \otimes H^p(A^\mathfrak{G})$.

Note. — The above given filtration does not depend upon the choice of the connection in A .

Now the proofs of the theorem (2.3) and its corollary are immediate.

(3.4) *Proof* (of theorem (2.3)). — The existence of a « special » connection is necessary, for take any connection Φ' in A (which exists by the Lemma 3.1) and compose this with the $\mathfrak{G}$ -morphism $\varphi : A \rightarrow \Omega(E)$ to get a connection $\Phi = \varphi \circ \Phi'$ in the bundle E . It is easy to see that Φ is a « special » connection.

Conversely, we fix a « special » connection Φ in E and a commutative diagram (D') as stated in the theorem. We have a $\mathfrak{G}$ -morphism $\varphi_{\Phi, D'} : A_\theta \rightarrow \Omega(E)$ as defined in Note (2) of the theorem. Since the map $\varphi_{\Phi, D'} : A_\theta \rightarrow \Omega(E)$ preserves the filtrations (given in the proof of lemma 3.3) of A_θ and $\Omega(E)$, it induces maps

$$\varphi_{\Phi, D'}^* : E_r^{p,q}(A_\theta) \rightarrow E_r^{p,q}(\Omega(E)).$$

Moreover $\varphi_{\Phi, D'}^* : E_2^{p,q}(A_\theta) \rightarrow E_2^{p,q}(\Omega(E))$ is an isomorphism for all p and q (lemma 3.3) and hence $\varphi_{\Phi, D'}$ induces isomorphism in cohomology. This proves the theorem.

(3.5) *Proof of the corollary (2.4).* — Let us fix a connection Φ' in A (exists by lemma 3.1). Then $\Phi = \varphi \circ \Phi'$ is a special connection in the bundle E . Consider the commutative diagram

$$\begin{array}{ccc} & I & \\ \Phi'_{\text{res.}} = \theta \swarrow & & \searrow \Phi_{\text{res.}} \\ \mu = A^{\mathfrak{G}} & \xrightarrow[\varphi_{\text{res.}} = \rho]{} & \Omega(B) \end{array}$$

It is easily seen that the map $\alpha : A_{\theta} \rightarrow A$, defined by $\alpha|_{W(\mathfrak{G})} = \Phi'$ and $\alpha|_{\mu}$ is the inclusion, is a $\mathfrak{G}$ -morphism satisfying $\varphi \circ \alpha = \varphi_{\Phi, D'}$. $\square$

Let $\mathcal{A}(E)$ denote the set of $\mathfrak{G}$ -isomorphism classes of all the irreducible $\mathfrak{G}$ -algebras A with a $\mathfrak{G}$ -morphism $A \rightarrow \Omega(E)$ satisfying (1), (2) and (3) of theorem (2.3). The following remark describes $\mathcal{A}(E)$, in fact it gives slightly sharper result.

(3.6) *Remark.* — Let J and J' be graded ideals in A_{θ} and $A_{\theta'}$ respectively which are closed under d , i and L , so that A_{θ}/J (respectively $A_{\theta'}/J'$) itself is a $\mathfrak{G}$ -algebra. Assume further that $J \cap A_{\theta}^{\mathfrak{G}} = 0 = J' \cap A_{\theta'}^{\mathfrak{G}}$ (and hence $(A_{\theta}/J)^{\mathfrak{G}} \simeq A_{\theta}^{\mathfrak{G}}$). If there exists a $\mathfrak{G}$ -morphism $f : A_{\theta}/J \rightarrow A_{\theta'}/J'$ inducing isomorphism in cohomology, then there exists a DGA isomorphism $\tilde{f} : \mu \rightarrow \mu$ making the following diagram commutative.

$$\begin{array}{ccc} & I_S(\mathfrak{G}) & \\ \theta \swarrow & & \searrow \theta' \\ \mu & \xrightarrow{\tilde{f}} & \mu \end{array}$$

and hence A_{θ} is $\mathfrak{G}$ -isomorphic with $A_{\theta'}$. To prove this, observe the following

(1) A_{θ} admits a unique connection.

(2) Let A, A' be two $\mathfrak{G}$ -algebras with connection which are finite dimensional in each degree and f a $\mathfrak{G}$ -morphism from A to A' which induces isomorphism in cohomology, then the map $f_{\text{res.}} : A^{\mathfrak{G}} \rightarrow A'^{\mathfrak{G}}$ also induces isomorphism in cohomology. This follows from the spectral sequence given in the appendix.

(3) A morphism of minimal differential algebras inducing an isomorphism in cohomology is itself an isomorphism, see [4; lecture 12].

(3.7) *Examples.* — We give below some examples of G -bundles which admit special connections.

(1) If G is abelian (i.e. G is a torus) then any G -bundle admits a special connection.

Since $\mathfrak{G}$ acts trivially on $S(\mathfrak{G}^*)$, the characteristic ring is the total algebra $S(\mathfrak{G}^*)$. Choose a basis $C = \{C_1, \dots, C_n\}$ of $\mathfrak{G}^*$. Let Φ_0 be a connection in E and let $\{\beta_1, \dots, \beta_n\}$ be the corresponding characteristic forms with respect to the basis C (i.e. $\beta_i = \Phi_0(C_i)$). Let $\{\alpha_1, \dots, \alpha_n\}$ be arbitrary elements in $\Omega^1(B)$. It can be easily seen that there exists a connection Φ in the bundle E such that the characteristic forms, with respect to the connection Φ , are $\{\beta_i + d\alpha_i\}_{1 \leq i \leq n}$. This ensures that E admits special connections. Moreover, it can be seen that the $\mathfrak{G}$ -algebra A_θ does not depend (upto $\mathfrak{G}$ -isomorphism) on θ .

(2) Let $E(G) \xrightarrow{p} B(G)$ be a universal G -bundle. Let Φ be a connection in $E(G)$. As is well known, the homomorphism $\Phi_{\text{res.}} : I_S(\mathfrak{G}) \rightarrow \Omega(B(G))$ induces isomorphism in cohomology (this follows easily from the spectral sequence given in the appendix) and $I_S(\mathfrak{G})$ is a polynomial algebra. Hence $\Phi_{\text{res.}}$ is a minimal model for the base space $B(G)$. This implies that the bundle $E(G)$ admits special connections. Moreover, it can be easily seen that any A_θ is $\mathfrak{G}$ -isomorphic with $W(\mathfrak{G})$.

Note. — This bundle is not in the finite dimensional smooth category, but the underlying difficulty is not serious and we omit the precise formulation.

(3) Let $E \xrightarrow{p} B$ be a G -bundle which admits a special connection and let $f : B' \rightarrow B$ be a map inducing isomorphism at de-Rham cohomology level, then $f^*(E)$ (the pull-back bundle) also admits a special connection.

(4) Let $E_i \xrightarrow{p_i} B_i$ be G_i bundles which admit special connections for $i = 1, 2$. Then the $G_1 \times G_2$ bundle $E_1 \times E_2 \xrightarrow{p_1 \times p_2} B_1 \times B_2$ also admits a special connection.

(5) Let $E \xrightarrow{p} B$ be a G -bundle admitting a special connection and let $\rho : G \rightarrow H$ be a Lie-group homomorphism. Let E_ρ denote the associated principal H -bundle, then E_ρ also admits a special connection. In particular

a G -bundle, which admits a reduction of its structural group to a maximal torus of G , has a special connection.

(6) Let $E \xrightarrow{p} B$ be a G -bundle. Suppose that a compact connected Lie-group H operates on E by bundle morphisms and hence H acts on the base B . Let $I_H(\Omega(B))$ denote the set of H -invariant forms on B . Then, of course, $I_H(\Omega(B)) \hookrightarrow \Omega(B)$ induces isomorphism in cohomology. If we can choose a minimal model $\rho: \mu_B \rightarrow I_H(\Omega(B))$ for the algebra $I_H(\Omega(B))$ so that ρ is surjective (e.g. if B is a symmetric space under the action of H) then E admits a special connection, because an H invariant connection in E can be checked to be « special ».

Appendix.

THEOREM. — *Let A be a $\mathfrak{G}$ -algebra, which is finite dimensional in each degree and which admits a connection. Then, there is a « natural » spectral sequence with $E_1^{p,q} \simeq H^{q-p}(A) \otimes I_S^p(\mathfrak{G})$ and converging to the cohomology of $A^{\mathfrak{G}}$.*

$I_S^p(\mathfrak{G})$ denotes the set of all the invariant homogeneous polynomials on $\mathfrak{G}$ of degree p (and hence grade degree $2p$).

Proof. — We sketch the derivation of this spectral sequence. Consider the tensor product of two $\mathfrak{G}$ -algebras $A \otimes W(\mathfrak{G})$. There is a canonical inclusion $A \rightarrow A \otimes W(\mathfrak{G})$. Restriction of this map from $A^{\mathfrak{G}} \rightarrow [A \otimes W(\mathfrak{G})]^{\mathfrak{G}}$ induces isomorphism in cohomology, see [1(b); Theorem 3]. The projection

$$A \otimes W(\mathfrak{G}) = A \otimes \Lambda(\mathfrak{G}^*) \otimes S(\mathfrak{G}^*) \rightarrow A \otimes S(\mathfrak{G}^*)$$

induces bijection of $[A \otimes W(\mathfrak{G})]^{\mathfrak{G}}$ onto $I(A \otimes S(\mathfrak{G}^*))$ (i.e. the set of invariants). So, by transporting, we get a differential D in the algebra $I(A \otimes S(\mathfrak{G}^*))$ to make it a DGA. Explicitly, this differential D is given by

$$D(a \otimes b) = (da) \otimes b - \sum_{j=1}^n i(X_j)a \otimes X_j^*b$$

for $a \in A$ and $b \in S(\mathfrak{G}^*)$, where $\{X_j\}_{1 \leq j \leq n}$ is a basis of $\mathfrak{G}$ and $\{X_j^*\}$ is the dual basis. (Although D is defined

on $A \otimes S(\mathfrak{G}^*)$, D^2 may not be 0 on the whole of $A \otimes S(\mathfrak{G}^*)$. Consider the filtration $F_0 \supset F_1 \supset \cdots \supset F_p \supset \cdots$.

$$F_p = \sum_{i \geq p} I(A \otimes S'(\mathfrak{G}^*)).$$

Now it is not difficult to see that

$$E_1^{p,q} \simeq H^{q-p}(A) \otimes I_s^p(\mathfrak{G}).$$

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