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IMPROVEMENT OF GRAUERT-RIEMENSCHNEIDER'S THEOREM FOR A NORMAL SURFACE

by Jean GIRAUD

1. Vanishing theorem.

1.1. A *surface* is a noetherian, excellent, normal scheme of dimension 2. A desingularization of X is a proper and birational map $f: \tilde{X} \to X$ such that $\tilde{X}$ is regular. The set

(1) $\quad \mathrm{Sing}\,(f) = \{x \in X,\ \dim(f^{-1}(x)) > 0\}$

is made up of finitely many closed points and f is an isomorphism above

(2) $X_f = X - \mathrm{Sing}\,(f) \subset X_{\mathrm{reg}} = \{x \in X,\ 0_{X,x} \text{ is regular}\}$.

We usually denote by E_i the irreducible components of

(3) $$E(f) = f^{-1}(\mathrm{Sing}\,(f))$$

and for $A = \mathbf{N}$, $\mathbf{Z}$ or $\mathbf{Q}$, we let

(4) $$NS(f,A) = \oplus\, A E_i .$$

We do not assume that $X_f = X_{\mathrm{reg}}$, hence X itself may be regular. For any $V = \Sigma V_i . E_i \in NS(f,\mathbf{Q})$, we write

(5) $V \geqslant 0$ when all V_i are $\geqslant 0$

(6) $V \gtrsim 0$ when all $-V.E_i$ are $\geqslant 0$.

Note that the minus sign is justified by

(7) $$V \gtrsim 0 \Rightarrow V \geqslant 0 .$$

To prove (7) we let $V = V_+ - V_-$; since $V \gtrsim 0$, we have

$0 \leqslant -V.V = -V_+.V_- + V_-^2 \leqslant V_-^2$, hence $V_- = 0$, since the intersection matrix is negative definitive. We introduce the dual basis of $NS(f,\mathbf{Q})$

$$\text{(8)} \qquad E_i^* \text{ defined by } E_i^*.E_j = -\delta_{ij}$$

and we observe that

$$\text{(9)} \qquad E_i^* \geqq 0, \quad dE_i^* \in NS(f,\mathbf{N})$$

where d is the absolute value of the determinant of the intersection matrix.

LEMMA 1.2. — *For any* $V \in NS(f,\mathbf{Q})$ *there exists a unique* $[V] \in NS(f,\mathbf{Z})$ *such that*

(i) $V \lesssim [V]$,

(ii) *if* $W \in NS(f,\mathbf{Z})$ *and if* $V \lesssim W$ *then* $[V] \leqslant W$.

We will prove that $[V]$ is the infimum for the usual order relation of $E(V) = \{W \in NS(f,\mathbf{Z}),\ V \lesssim W\}$. Let $N \in \mathbf{Z}$ be such that $dN \leqslant \inf(V.E_i)$; we have $-dN\Sigma E_i^* \in E(V)$, hence $E(V)$ is non empty. For $i = 1,2$, let $W_i = \Sigma W_{i,j}E_j \in E(V)$ and let $Z = \Sigma Z_jE_j$ with $Z_j = \inf(W_{1,j}, W_{2,j})$. By Artin's trick we prove that $Z \in E(V)$ as follows. For any j, we have $Z_j = W_{1,j}$ or $Z_j = W_{2,j}$. By symmetry we can assume that $Z_j = W_{1,j}$ and we get

$$Z.E_j = W_{1,j}E_j^2 + \sum_{k \neq j} Z_kE_k.E_j \leqslant W_1.E_j \leqslant V.E_j$$

hence $Z \gtrsim V$. To conclude, we note that the coordinates of any $W = \Sigma W_iE_i \in E(V)$ are bounded from below since $W_i = -W.E_i^* \geqslant -V.E_i^*$ since E_i^* is $\geqslant 0$. Observe the obvious

$$\text{(1)} \quad [V+W] \leqslant [V] + [W];\ [V+E] = [V] + E \text{ if } E \in NS(f,\mathbf{Z}).$$

We also let

$$\text{(2)} \qquad [\underline{V}] = -[-V] \text{ in such a way that } [\underline{V}] \lesssim V \lesssim [V].$$

1.3. Let L be an invertible sheaf on $\tilde{X}$. We define

(1) $e_f(L) \in NS(f,\mathbf{Q})$ by $e_f(L).E_i = \deg(L|E_i)$ for any i.

We also write $L \gtrsim 0$ instead of $e_f(L) \gtrsim 0$ and this means $L.E_i \leqslant 0$ for all i. We will often drop the subscript f. Sending V to $0_{\tilde{X}}(V)$ we identify $NS(f,\mathbf{Z})$ to a subgroup of $\mathrm{Pic}(\tilde{X})$ and since $V = e_f(0_{\tilde{X}}(V))$, $0_{\tilde{X}}(V) \gtrsim 0$ is equivalent to $V \gtrsim 0$. Hence, when we write $\mathrm{Pic}(\tilde{X})$ additively, we can safely write V in place of $0_{\tilde{X}}(V)$ and $L + V$ in place of $L(V) = L \otimes 0_{\tilde{X}}(V)$. We will sometimes write $[L]$ insteated of $[e_f(L)]$.

1.4. We can also give an algorithmic description of $[V]$ as follows. Start with $Z \in NS(f,\mathbf{Z})$ such that $Z \leqslant [V]$. For instance, if $V = \Sigma V_i E_i$ let $Z = \Sigma V_i' E_i$ where V_i' is the smallest integer $\geqslant V_i$. If $Z \neq [V]$ there must exist a i such that $Z.E_i > V.E_i$ and we still have $Z + E_i \leqslant [V]$. In fact, since $V \lesssim [V]$, we have $([V]-Z).\ E_i \leqslant (V-Z).E_i < 0$, hence $([V]-Z) \geqslant E_i$ since $[V] - Z$ is effective with integral coefficients. We now replace Z by $Z + E_i$ and reach $[V]$ in a finite number of steps.

VANISHING THEOREM 1.5. — *Let* $f : \tilde{X} \to X$ be a desingularization of a normal surface X, *let* $E = f^{-1}(\mathrm{Singl}\,(f))$ *and let* L *be an invertible sheaf on* $\tilde{X}$.

(i) *If* $[L] \gtrsim 0$ *then* $H^1_E(\tilde{X},L) = 0$.

(ii) *If* $[L] \geqslant 0$ *then* $f_*(L)$ *is reflexive.*

(iii) *Let* K *be the dualizing sheaf of* $\tilde{X}$. *If* $[K-L] \gtrsim 0$ *then* $\mathbf{R}^1 f_*(L) = 0$.

1.5.1. To prove (i) we let $M = [L]$ and $L' = L(-M)$ in such a way that $[L'] = 0$ and $M \gtrsim 0$, $M \in NS(f,\mathbf{N})$. For any $V \in NS(f,\mathbf{N})$, $V \neq 0$, there exists an E_i such that $(L'+V).\ E_i < 0$. Otherwise we would have $L' + V \lesssim 0$ hence $L' \lesssim -V$, hence $0 = [L'] \leqslant -V < 0$ which is impossible. We observe that E_i must be contained in the support of V, otherwise we would have $V.E_i \geqslant 0$, hence

$$(L'+V).E_i \geqslant ([L']+V).E_i = V.E_i \geqslant 0.$$

Furthermore, since $M \gtrsim 0$, we have

$$(L+V).E_i = (L'+M+V).E_i \leqslant (L'+V).E_i < 0.$$

As a consequence we get

$$(1) \qquad V - E_i \in NS(f,\mathbf{N}) \quad \text{and} \quad (L+V).E_i < 0.$$

As a consequence we get $H^0(E_i, L(V)|E_i) = 0$ hence the map

$$(2) \qquad H^0(V-E_i;\ L(V-E_i)|(V-E_i)) \to H^0(V; L(V)|V)$$

is surjective. By induction on V, we conclude that, if $[L] \gtrsim 0$, we have

(3) $H^0(V,L(V)|V) = 0$ for any $V \in NS(f,N)$

hence $H^1_E(\tilde{X};L) = \varinjlim H^0(V;L(V)|V) = 0$. This proves (i) and we get (iii) by duality.

1.5.2. To prove (ii), we can assume that $[L] = 0$ since $f_*(L)$ reflexive implies that, for any $V \in NS(f,N)$, the map $f_*(L) \to f_*(L(V))$ is an isomorphism. Let $u : f_*(L) \to f_*(L)^{vv}$ be the map from $f_*(L)$ to it's bidual. Since L is invertible, we know that u is an isomorphism over the open subset X_f of X. Since X is normal, we know that coker(u) is finite and since f is proper, this implies the existence of some $V \in NS(f;N)$ such that $f_*(L)^{vv} = f_*(L(V))$. Since $[L] = 0$, we know that $H^0(V,L(V)|V) = 0$ hence $f_*(L) \to f_*(L(V))$ is an isomorphism and this concludes the proof.

1.5.3. We do not really need duality for surfaces to state and prove (iii). In fact, we can define

(1) $K_f \in NS(f,Q)$ by $(K_f + E_i).E_i = -2\chi(0_{E_i})$ for all i,

and write the hypothesis $[K_f - e_f(L)] \gtrsim 0$. As for the proof it runs parallel to the proof of (i) and uses the fact that $H^1(E_i,M) = 0$ if M is an invertible sheaf on the reduced and irreducible Gorenstein curve E_i with $\deg(M) > -2\chi(0_{E_i})$; details are left to the reader. We define $C(f)$ and C_+ in $NS(f,N)$ by

$$(2) \qquad [K_f] = C_+ - C(f).$$

Observe that if we denote by $K_{\tilde{X}}$ and K_X the dualizing sheaves of $\tilde{X}$ and X we have

$$(3) \qquad K_f = e_f(K_{\tilde{X}}) \quad \text{and} \quad K_X = f_*(K_{\tilde{X}}(C(f))).$$

The first formula comes from (1). For the second observe that $[K_{\tilde{X}}(C(f))] = K_{\tilde{X}}] + C(f) = C_+ \geqslant 0$ hence its direct image is reflexive by (1.5(ii)) and coïncide with $K_{\tilde{X}}$ over X_f, hence it must be K_X.

Corollary 1.6. — *Under the hypothesis of* (1.5), *let* L *be an invertible sheaf on* $\tilde{X}$ *such that* $[L] = 0$. *Then* $f_*(L)$ *is reflexive and the map* $u : \mathbf{R}^1 f_*(L) \to H^1(C(f),L|C(f))$ *is an isomorphism.*

We know that u is surjective. Let us introduce $V \in NS(f,\mathbf{Z})$ such that $[K_f+C(f)-e_f(L)-V] = 0$. We claim that $V \geqslant 0$. In fact $0 = [K_f+C(f)-e_f(L)-V] \geqq K_f + C(f) - e_f(L) - V$ hence $e_f(L) + V \geqq K_f + C(f)$ hence $V = [e_f(L) + V] \geqslant [K_f+C(f)] = C_+ \geqslant 0$. We have a diagram

$$\begin{array}{ccc} \mathbf{R}^1f_*(L) & \xrightarrow{u} & \mathbf{R}^1f_*(L|C(f)) \\ {\scriptstyle v}\downarrow & & \downarrow \\ \mathbf{R}^1f_*(L(V)) & \xrightarrow{w} & \mathbf{R}^1f_*(L(V)|C(f)). \end{array}$$

By (1.5.1(3)), the morphism v is injective hence it is enough to show that w is injective. This follows from $\mathbf{R}^1f_*(L(V-C(f))) = 0$ which comes from (1.5 (iii)) since $[K_f-e_f(L)-V+C(f)] = 0$.

Corollary 1.7. — *We have* $\mathbf{R}^1f_*(0_{\tilde{X}}) \simeq H^1(C(f);0_{C(f)})$ *and* $\mathbf{R}^1f_*(0_{\tilde{X}}) = 0$ *is equivalent to* $C(f) = 0$.

We get the isomorphism by (1.6) applied to $L = 0_{\tilde{X}}$. Hence $C(f) = 0$ implies $\mathbf{R}^1f_*(0_{\tilde{X}}) = 0$. Conversely, if $\mathbf{R}^1f_*(0_{\tilde{X}}) = 0$ and $C(f) \neq 0$, we have $\chi(0_{C(f)}) > 0$ which means

$$\begin{aligned} 0 > (K_{\tilde{X}}+C(f)).C(f) &= (K_f+C(f)).C(f) \\ &\geqslant ([K_f]+C(f)).C(f) = C_+.C(f) \geqslant 0 \end{aligned}$$

a contradiction.

Proposition 1.8. — *Let* $f:\tilde{X} \to X$ *be a desingularization of a normal surface* X *and let* M *be a reflexive sheaf of rank one on* X. *There exists a pair* (L,u) *where* L *is an invertible sheaf on* $\tilde{X}$ *such that* $[e_f(L)] = 0$ *and* $u : f_*(L)|X_f \simeq M|X_f$ *is an isomorphism. The pair* (L,u) *is unique up to a unique isomorphism. Furthermore* $M = f_*(L)$.

1.8.1. It is clear that there exists a pair (L',u'), where L' is invertible on $\tilde{X}$ and $u' : f_*(L')|X_f \simeq M|X_f$ is an isomorphism. If (L'',u'') is another solution, we canonically have $L'' = L'(V)$, $V \in NS(f,\mathbf{Z})$, hence we get existence and uniqueness since $[e_f(L'(V))] = [e_f(L')] + V$. By (1.5(ii)), $f_*(L)$ is reflexive since $[e_f(L)] = 0$, hence $f_*(L) \simeq M$ since both are reflexive and coïncide over X_f.

1.8.2. We denote by $f^v(\mathrm{M})$ the invertible sheaf on $\tilde{\mathrm{X}}$ characterized by $[f^v(\mathrm{M})] = 0$ and $f_*(f^v(\mathrm{M})) = \mathrm{M}$. We observe that we have

$$(1) \qquad e_f(f^v(\mathrm{M})) \in \mathrm{NS}(f,\mathrm{Q}), \quad e_f(f^v(\mathrm{M})) \lesssim 0,$$

but this element is not necessarily zero. However, if M is *invertible*, we obviously have $f^v(\mathrm{M}) = f^*(\mathrm{M})$ since $e_f(f^*(\mathrm{M})) = 0$. More generally, it is useful to compare $f^v(\mathrm{M})$ with another lifting $\tilde{\mathrm{M}}$ defined as follows

$$(2) \qquad \mathrm{M}' = f^*(\mathrm{M})/\text{torsion} \quad \tilde{\mathrm{M}} = \mathrm{M}'^{vv} = \text{bidual of } \mathrm{M}'.$$

COROLLARY 1.8.3. — *Let* M *be a reflexive sheaf of rank one on* X. *Then* $\mathrm{M} \lesssim 0$ *and* $[\tilde{\mathrm{M}}] \leqslant 0$. *We have* $f^v(\mathrm{M}) = \tilde{\mathrm{M}}(-[\tilde{\mathrm{M}}])$.

Since M′ is torsion free of rank one it is invertible except at finitely many closed points; hence $\tilde{\mathrm{M}}$ is invertible. To prove that $\tilde{\mathrm{M}} \lesssim 0$, assume that there exists E_i such that $\tilde{\mathrm{M}}.\mathrm{E}_i < 0$. Then $f_*(\tilde{\mathrm{M}}(-\mathrm{E}_i)) = f_*(\tilde{\mathrm{M}}) = \mathrm{M}$. In a neighborhood U of the generic point of E_i, we have $\mathrm{M}' = \tilde{\mathrm{M}}$, hence $\tilde{\mathrm{M}}$ is generated on a possibly smaller neighborhood U′ by sections of M, hence we cannot have $f_*(\tilde{\mathrm{M}}(-\mathrm{E}_i)) = f_*(\tilde{\mathrm{M}})$. By definition of $[\tilde{\mathrm{M}}]$, we get $[\tilde{\mathrm{M}}] \leqslant 0$ out of $\tilde{\mathrm{M}} \lesssim 0$. We deduce $f^v(\mathrm{M}) = \tilde{\mathrm{M}}(-[\tilde{\mathrm{M}}])$ from $[\tilde{\mathrm{M}}(-[\tilde{\mathrm{M}}])] = 0$.

COROLLARY 1.8.4. *Assume that* $\tilde{\mathrm{X}}$ *dominates some desingularization* X′ *of* X. *We have* $f = gh$ *with* $\tilde{\mathrm{X}} \xrightarrow{h} \mathrm{X}' \xrightarrow{g} \mathrm{X}$. *For any reflective sheaf of rank one* M *on* X *we have* $f^v(\mathrm{M}) = h^*(g^v(\mathrm{M}))$.

Since $\tilde{\mathrm{X}}$ and X′ are regular and h proper and birational, we have $h_*h^*(g^v(\mathrm{M})) = g^v(\mathrm{M})$ hence $f_*h^*(g^v(\mathrm{M}))) = \mathrm{M}$, hence we only have to prove that $[e_f(h^*(g^v(\mathrm{M}))] = 0$. We use the map

$$(1) \qquad h^* : \mathrm{NS}(g,\mathrm{Q}) \to \mathrm{NS}(f,\mathrm{Q})$$

which preserves integrality, positivity and the intersection numbers. We still have to prove that we have, for any $\mathrm{V} \in \mathrm{NS}(g,\mathrm{Q})$

$$(2) \qquad h^*([\mathrm{V}]) = [h^*(\mathrm{V})].$$

For any $\mathrm{E} \in \mathrm{NS}(f,\mathrm{N})$, we have $h^*(\mathrm{V}).\mathrm{E} = \mathrm{V}.h_*(\mathrm{E}) \geqslant [\mathrm{V}].h_*(\mathrm{E}) = h^*([\mathrm{V}]).\mathrm{E}$, hence $h^*(\mathrm{V}) \lesssim h^*([\mathrm{V}])$, hence $[h^*(\mathrm{V})] \leqslant h^*([\mathrm{V}])$, in other words $h^*([\mathrm{V}]) = [h^*(\mathrm{V})] + \mathrm{A}$, $\mathrm{A} \in \mathrm{NS}(f,\mathrm{N})$.

From $h^*(V) \lesssim [h^*(V)]$, we deduce $V \lesssim h_*([h^*(V)]) = h_*h^*([V]) - h_*(A) = [V] - h_*(A)$. By definition of $[V]$, we deduce that $[V] \leqslant [V] - h_*(A)$, hence $h_*(A) = 0$, hence $A \in NS(h,N)$. We get $0 = h^*(V)$. $A \geqslant [h^*(V)]$. $A = h^*([V])$. $A - A^2 = -A^2$, hence $A = 0$.

PROPOSITION 1.9. — *Let* $f \colon \tilde{X} \to X$ *and assume that* $\mathbf{R}^1 f_*(O_{\tilde{X}}) = 0$.

(i) *Let* M *be a reflexive sheaf of rank one on* X. *We have* $f^v(M) = f^*(M)/$*torsion and* $\mathbf{R}^1 f_*(f^v(L)) = 0$.

(ii) *Let* L *be an invertible sheaf on* $\tilde{X}$ *such that* $L \lesssim 0$. *The map* $f^* f_*(L) \to L$ *is surjective and* $\mathbf{R}^1 f_*(L) = 0$.

We first prove (ii). We let $M = f_*(L)$, $L_0 = \mathrm{Im}(f^*(M) \to L)$, L_1 = bidual of L_0 and we get $L_0 \subset L_1 \subset L$ and $M \subset f_*(L_0) \subset f_*(L_1) \subset f_*(L) = M$. Since $\mathbf{R}^1 f_*(L_0) = 0$, we get $f_*(L_1/L_0) = 0$ and this implies $L_1/L_0 = 0$ since L_1/L_0 has finite support. Let us define $V \in NS(f,N)$ by $L = L_0(V)$. We have $f_*(L|V) = 0$, hence $\chi(K_V) = \chi(L|V) - L.V = -h^1(L|V) - L.V \leqslant -L.V$. Since $L \lesssim 0$, we get $-L.V \leqslant 0$ hence $\chi(O_V) \leqslant 0$, hence $V = 0$ since $h^1(O_V) = 0$. This means that $L_0 = L$, from which $\mathbf{R}^1 f_*(L) = 0$ follows.

To prove (i) we let $L = f^v(M)$ and apply (ii) to L (see (1.8.3)); recall that $M = f_* f^v(L)$ by (1.8).

As an exercise, we now deduce some well known facts about rational singularities.

PROPOSITION 1.10. *Let* $f \colon \tilde{X} \to X$ *be a desingularization and assume that* $\mathbf{R}^1 f_*(O_{\tilde{X}}) = 0$. *Let* I *be an ideal of* O_X. *The following conditions are equivalent*

(i) I *is integrally closed and* $IO_{\tilde{X}}$ *is invertible,*

(ii) $I = f_*((IO_{\tilde{X}})^{vv})$,

(iii) *There exists an effective divisor* D *on* $\tilde{X}$, *with* $O_{\tilde{X}}(-D) \gtrsim 0$ *such that* $I = f_*(O_{\tilde{X}}(-D))$.

Furthermore, if we have (iii), *we necessarily have* $IO_{\tilde{X}} = M_{\tilde{X}}(-D)$.

If $IO_{\tilde{X}}$ is invertible, then $\tilde{X}$ dominates the normalized blowing up of I, hence $f_*(IO_{\tilde{X}})$ is the integral closure of I. Hence (i) $\Rightarrow$ (ii), since in that case $IO_{\tilde{X}} = IO_{\tilde{X}}{}^{vv}$. Since $(IO_{\tilde{X}})^{vv} \lesssim 0$, we have $IO_{\tilde{X}}{}^{vv} = O_{\tilde{X}}(-D)$, with D effective (not necessarily vertical) and $D \gtrsim 0$; hence (ii) $\Rightarrow$ (iii). If we assume (iii), then I is integrally closed and (1.9 (ii)) implies that

$IO_{\tilde{X}} = O_{\tilde{X}}(-D)$, hence (iii) $\Rightarrow$ (i) and we have also proven the last assertion.

It follows that we have a 1-1-correspondance between ideals I of O_X which satisfy the above conditions and effective divisors D on X with $D \gtrsim 0$. We have that I is primary if and only if D is vertical $(\dim f(D) = 0)$ and I is reflexive (i.e. the ideal of a Weil divisor) if and only if $[D] = 0$. Observe that (1.9(i)) tells us that a reflexive I satisfy (i). Observe that if I is the maximal ideal of some closed point x, then we must have (ii), hence the corresponding D must be the connected component of the fundamental cycle corresponding to x. To complete the picture, recall Lipman's result saying that the set of ideals satisfying (i) is stable by multiplication, which means that $f_*(O_{\tilde{X}}(-D-E)) = f_*(O_{\tilde{X}}(-D))f_*(O_{\tilde{X}}(-E))$ if D and E are effective and $D \gtrsim 0$, $E \gtrsim 0$.

Example 1.11. — We now assume that $f: \tilde{X} \to X$ is the *minimal* desingularization and that X is the spectrum of a local ring R with algebraically closed residue field, in such a way that $K_{\tilde{X}} \lesssim 0$; this implies $[K_f] = -C(f)$. Assume that K_X *is invertible* which means that R is a *Gorenstein ring.* Since $f^*(K_X) = K_{\tilde{X}}(V)$ for some vertical V and $e_f(f^*(K_X)) = 0$, we conclude that $V = K_f$, hence K_f has integral coefficients, hence $K_f = -C(f)$ and $K_{\tilde{X}}(C(f)) = f^*(K_X) \approx O_{\tilde{X}}$.

If we have rational singularity, we know that $C(f) = 0$, hence $K_f = 0$, hence we get the well known result that $E_i^2 = -2$ for all i. If $C(f) \neq 0$, we still have that the dualizing sheaf $K_{C(f)} = K_{\tilde{X}}(C(f)) \otimes O_{C(f)}$ is isomorphic to $O_{C(f)}$. The converse is also true, see for instance [2].

2. Genus formula.

2.1. Let k be a field and X be a proper k-scheme of dimension 2 which is normal. We want to study Weil divisors of X, or equivalently reflexive sheaves of rank one on X. Such a sheaf M is determined by the invertible sheaf $i^*(M)$ since $M \to i_*i^*(M)$ is an isomorphism where $i: X_{reg} \to X$ is the inclusion of the open set X_{reg} made up of regular points of X. In other words, we study $\mathrm{Pic}(X_{reg})$. Let $f: \tilde{X} \to X$ be a desingularization of X, we have an exact sequence

$$(1) \qquad 0 \to NS(f,\mathbf{Z}) \xrightarrow{a} \mathrm{Pic}(\tilde{X}) \xrightarrow{b} \mathrm{Pic}(X_{reg}) \to 0$$

where $a(\mathrm{D})$ is the class of $\mathrm{O}_{\tilde{\mathrm{X}}}(\mathrm{D})$ and b is induced by the inclusion $j : \mathrm{X}_{\mathrm{reg}} \to \mathrm{X}$. The canonical lifting $f^v(\mathrm{M})$ of a reflexive sheaf of rank one M on X defined in (1.8.2) gives us a *non-linear* section of b. By composition with the usual map

$$\text{(2)}\ e_f : \mathrm{Pic}(\tilde{\mathrm{X}}) \to \mathrm{NS}(f,\mathbf{Z})^* \subset \frac{1}{d}\mathrm{NS}(f,\mathbf{Z}) \subset \mathrm{NS}(f,\mathbf{Q}),\ (1.3)$$

we get a class

$$\text{(3)} \qquad e_f(f^v(\mathrm{M})) \in \frac{1}{d}\,\mathrm{NS}(f,\mathbf{Z})$$

which can only take *a finite number of values* since $[e_f(f^v(\mathrm{M}))] = 0$. Of course, this is still non linear. To recover the classical linear theory of [6], we recall that, for $\mathrm{A} = \mathbf{Z}$ or $\mathbf{Q}$, the quadratic module $\mathrm{NS}(f, \mathrm{A})$ lies inside the Néron-Severi group $\mathrm{NS}(\tilde{\mathrm{X}},\mathrm{A})$ and we define

$$\text{(4)} \qquad \mathrm{NS}(\mathrm{X},\mathrm{A}) = \text{orthogonal of } \mathrm{NS}(f,\mathrm{A}) \text{ inside } \mathrm{NS}(\tilde{\mathrm{X}},\mathrm{A})$$

which gives an orthogonal decomposition

$$\text{(5)} \qquad \mathrm{cl}(f^v(\mathrm{M})) = \mathrm{cl}(\mathrm{M}) + e_f(f^v(\mathrm{M}))$$

inside $\mathrm{NS}(\tilde{\mathrm{X}},\mathbf{Q}) = \mathrm{NS}(\mathrm{X},\mathbf{Q}) \oplus \mathrm{NS}(f,\mathbf{Q})$. We also have another linear invariant

$$\text{(6)} \qquad d_f(\mathrm{M}) = \text{class of } e_f(f^v(\mathrm{M})) \text{ in } \mathrm{NS}(f,\mathbf{Z})^*/\mathrm{NS}(f,\mathbf{Z}).$$

It is clear that the two linear invariants $\mathrm{cl}(\mathrm{M})$ and $d_f(\mathrm{M})$ can be computed with any lifting L of M, namely $\mathrm{cl}(\mathrm{M})$ is the orthogonal projection on $\mathrm{NS}(\mathrm{X},\mathbf{Q})$ of $\mathrm{cl}(\mathrm{L})$ and $d_f(\mathrm{M})$ is the image of $e_f(\mathrm{L})$; proof: $\mathrm{L} = f^v(\mathrm{M})(\mathrm{D})$ for some $\mathrm{D} \in \mathrm{NS}(f,\mathbf{Z})$. For instance, if K_{X} and $\mathrm{K}_{\tilde{\mathrm{X}}}$ are the dualizing sheaves of X and $\tilde{\mathrm{X}}$ we have an orthogonal decomposition

$$\text{(7)} \qquad \mathrm{cl}(\mathrm{K}_{\tilde{\mathrm{X}}}) = \mathrm{cl}(\mathrm{K}_{\mathrm{X}}) + \mathrm{K}_f \quad (1.5.3)$$

and

$$\text{(8)} \qquad e_f(\mathrm{K}_{\mathrm{X}}) = \mathrm{K}_f - [\mathrm{K}_f]\,.$$

If we introduce the effective divisor $\mathrm{C}(f) = [\mathrm{K}_f]_-$ as in (1.5.3) we know that the multi-degree of $f^v(\mathrm{M})|\mathrm{C}(f)$ can only take a finite number of values, hence the same holds for the length of

$$(9) \qquad \mathbf{R}^1 f_*(f^v(\mathrm{M})) = \mathrm{H}^1(\mathrm{C}(f);\ f^v(\mathrm{M})|\mathrm{C}(f)), \quad (1.6).$$

THEOREM 2.2. — *Let* M *be a reflexive sheaf of rank one on* X. *We have*

$$(1)\ \chi(\mathrm{M}) = \frac{1}{2}(\mathrm{cl}(\mathrm{M}), \mathrm{cl}(\mathrm{M}) - \mathrm{cl}(\mathrm{K_X})) + \chi(\mathrm{O_X}) + \frac{1}{2}\, e(\mathrm{M})\, d(\mathrm{M})$$

where the scalar product is computed in $\mathrm{NS}(\mathrm{X},\mathbf{Q})$ *and for any desingularization* $f : \tilde{\mathrm{X}} \to \mathrm{X}$ of X *we have*

$$(2) \qquad e(\mathrm{M}) = (e_f(f^v(\mathrm{M})),\ e_f(f^v(\mathrm{M})) - \mathrm{K}_f)$$

$$\begin{aligned}(3) \qquad d(\mathrm{M}) &= \lg \mathbf{R}^1 f_*(f^v(\mathrm{M})) - \lg \mathbf{R}^1 f_*(\mathrm{O}_{\tilde{\mathrm{X}}}) \\ &= h^1(\mathrm{C}(f);\ f^v(\mathrm{M})|\mathrm{C}(f)) - h^1(\mathrm{C}(f);\ \mathrm{O}_{\mathrm{C}(f)})\ (1.5.3).\end{aligned}$$

Proof. — Apply the usual Riemann-Roch formula to $f^v(\mathrm{M}) = \mathrm{L}$. Since $\mathrm{M} = f_*(f^v(\mathrm{M}))$, we get

$$\begin{aligned}\chi(\mathrm{M}) &= \chi(\mathrm{L}) + \lg \mathbf{R}^1 f_*(\mathrm{L}) = (\mathrm{L},\mathrm{L}-\mathrm{K}_{\tilde{\mathrm{X}}})/2 + \chi(\mathrm{O}_{\tilde{\mathrm{X}}}) + \lg \mathbf{R}^1 f_*(\mathrm{L}) \\ &= \chi(\mathrm{L},\mathrm{L}-\mathrm{K}_{\tilde{\mathrm{X}}})/2 + \chi(\mathrm{O_X}) + \lg \mathbf{R}^1 f_*(\mathrm{L}) - \lg \mathbf{R}^1 f_*(\mathrm{O}_{\tilde{\mathrm{X}}})\end{aligned}$$

and split the scalar product $(\mathrm{L},\mathrm{L}-\mathrm{K}_{\tilde{\mathrm{X}}})$ according to the orthogonal decomposition $\mathrm{NS}(\tilde{\mathrm{X}},\mathbf{Q}) = \mathrm{NS}(\mathrm{X},\mathbf{Q}) + \mathrm{NS}(f,\mathbf{Q})$.

According to (1.8.4), the terms $e(\mathrm{M})$ and $d(\mathrm{M})$ do not depend on the choice of the desingularization. Furthermore we have

$$(4) \qquad e(\mathrm{M}) = \sum_{x \in \mathrm{Sing}(\mathrm{X})} e(\mathrm{M},x), \quad d(\mathrm{M}) = \sum_{x \in \mathrm{Sing}(\mathrm{X})} d(\mathrm{M},x)$$

where $e(\mathrm{M},x)$ and $d(\mathrm{M},x)$ are defined by replacing X by $\mathrm{Spec}(\mathrm{O}_{\mathrm{X},x})$, or even by $\mathrm{Spec}(\hat{\mathrm{O}}_{\mathrm{X},x})$ as is easily seen. Furthermore $e(\mathrm{M},x) = d(\mathrm{M},x) = 0$ if M *is invertible in a neighborhood of* x. Furthermore $d(\mathrm{M},x) = 0$ if $\mathrm{O}_{\mathrm{X},x}$ is a *rational singularity* (1.7). We also know that $e(\mathrm{M})$ and $d(\mathrm{M})$ can only take a finite number of values.

For $n \in \mathbf{Z}$, we let $\mathrm{M}^n = i_*(i^*(\mathrm{M})^n) =$ bidual of $\mathrm{M}^{\otimes n}$ and we have

$$\begin{aligned}(5)\ \ \chi(\mathrm{M}^n) = \frac{n^2}{2}(\mathrm{cl}(\mathrm{M}), \mathrm{cl}(\mathrm{M})) - \frac{n}{2}(\mathrm{cl}(\mathrm{M}), \mathrm{cl}(\mathrm{K_X})) \\ + \chi(\mathrm{O_X}) + e(\mathrm{M}^n)/2 + d(\mathrm{M}^n).\end{aligned}$$

Observe that $e(M^n) = 0$ if the determinant of the intersection matrix divides n. In fact, in that case, we have $d_f(M^n) = 0$ hence $e_f(f^v(M)) = [e_f(f^v(M))] = 0$. For instance, if X is the Satake compactification of some Hilbert-Blümenthal surface and $M = K_X$, we can get an a priori proof of the formula for the rank of the vector spaces $H^0(X, K_X^n)$ of automorphic forms [3].

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