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GEOMETRIC FOURIER ANALYSIS

by Antonio CORDOBA

In this paper we present several results related to maximal and square functions whose proofs have a similar flavour: after some algebraic manipulation and the use of the uncertainty principle they are reduced to certain properties of the geometry of "rectangles" in $\mathbb{R}^n$.

A. In $\mathbb{R}^2$ let us consider the angles

$$\omega_j = \frac{2\pi j}{N}, \quad j = 0, 1, \dots, N-1, \quad N \in \mathbb{Z}^+$$

and let us denote by H_j the Hilbert transform in the direction ω_j and by S_j the projection, at the Fourier transform side, over the angles

$$\Delta_j = \{\xi, 2\pi j/N \leq \arg(\xi) \leq 2\pi(j+1)/N\}$$

$$\text{i.e. } \widehat{S_j f}(\xi) = \chi_{\Delta_j}(\xi) \hat{f}(\xi).$$

THEOREM 1. — *There exist constants independent of N , $0 < a$, $c < \infty$, so that*

$$\text{i) } \left\| \left[\sum_{j=1}^N |H_j f_j|^2 \right]^{1/2} \right\|_4 \leq C(\log N)^a \left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_4$$

$$\text{ii) } \left\| \left(\sum_{j=1}^N |S_j f|^2 \right)^{1/2} \right\|_4 \leq C[\log N]^a \|f\|_4.$$

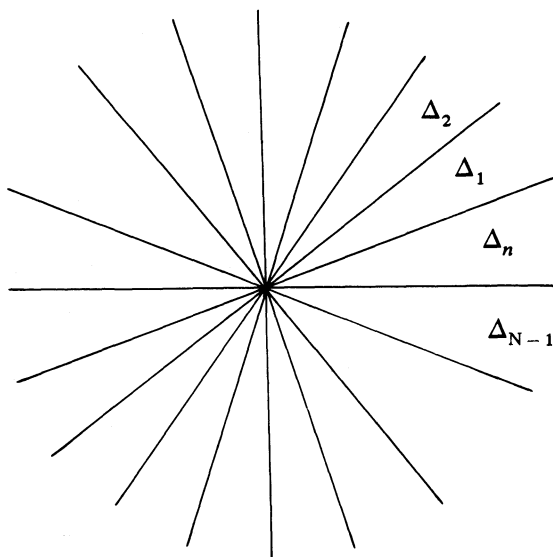
B. Let $\gamma : [0, 1] \rightarrow S^{n-1}$ be a smooth curve crossing a finite number of times each hyperplane of $\mathbf{R}^n$. Given a real number $N \gg 1$ let us consider the family $\mathcal{B}_N$ of cylinders of $\mathbf{R}^n$ having eccentricity = height/radius = N and direction in the curve γ . With a locally integrable function f we may consider its maximal function Mf given by the formula

$$Mf(x) = \sup_{x \in R \in \mathcal{B}_N} \frac{1}{\mu\{R\}} \int_R |f(y)| d\mu(y)$$

where μ denotes Lebesgue's measure in $\mathbf{R}^n$.

THEOREM 2. — *There exists a constant C_γ , independent of N , such that $\|Mf\|_2 \leq C_\gamma [\log N]^2 \|f\|_2$.*

A. The square function.



$$Sf(x) = \left(\sum |S_j f(x)|^2 \right)^{1/2},$$

$$\widehat{S_j f}(\xi) = \chi_{\Delta_j}(\xi) \hat{f}(\xi),$$

$$\Delta_j = \{ \xi : 2\pi j/N \leq \arg(\xi) \leq 2\pi(j+1)/N \}.$$

Part (i) of theorem 2 was proved in ref. [4] and, therefore, we shall concentrate in part (ii). Although we have not made a careful analysis of the nature of the best constant a , it has to be strictly positive, as an adequate Kakeya's set argument can show. On the other hand, interpolating with the L^2 -result, one may obtain $\|Sf\|_p \leq C[\log N]^{a(p)} \|f\|_p$, $2 \leq p \leq 4$, which it is the best range of p 's where such an inequality can hold. We shall proceed proving a previous lemma.

In $\mathbf{R}^n$ let us consider a cubic lattice $\mathcal{Q} = \{Q_\nu\}_{\nu \in \mathbf{Z}^n}$ i.e. the Q_ν 's are congruent cubes with disjoint interiors and such that $\mathbf{R}^n = \cup Q_\nu$. Define, for each ν , the operators $\widehat{P_\nu f} = \chi_{Q_\nu} \cdot \hat{f}$ and the square function $Gf(x) = (\sum |P_\nu f(x)|^2)^{1/2}$.

LEMMA. — For each $s > 1$ there exists a finite constant C_s so that for every f , $\omega \in C_0(\mathbf{R}^n)$ we have:

$$\int_{\mathbf{R}^n} |Gf(x)|^2 \omega(x) dx \leq C_s \int_{\mathbf{R}^n} |f(x)|^2 A_s \omega(x) dx,$$

where $A_s \omega = [(\omega^s)^*(x)]^{1/s}$ and $*$ -denotes the Hardy-Littlewood maximal function.

Proof. — Without lack of generality we may assume that $\mathcal{Q}$ is the unit lattice i.e. Q_ν is centered at the point $\nu \in \mathbf{Z}^n$ and has volume equal to one. Let ψ be a smooth function with compact support and equal to 1 in Q_0 . For each $y \in Q_0$ let us consider the Fourier multiplier $m_y(z) = \sum_{\nu} e^{2\pi i \nu \cdot y} \psi(z - \nu)$, $z \in \mathbf{R}^n$. Then the kernel $\mu_y = \hat{m}_y$ is a measure of finite total variation uniformly in $y \in Q_0$. More concretely: $\mu_y = \sum_{\nu} \hat{\psi}(y + \nu) \delta_{y+\nu}$ where, as usual, δ_x denotes Dirac's function translated to the point x . Therefore, $|\mu_y * f(x)|^2 \leq C \sum_{\nu} (1 + |\nu|)^{2n} |\hat{\psi}(y + \nu)|^2 |f(x - y - \nu)|^2$ and, since $\hat{\psi}$ is rapidly decreasing, we have:

$$\begin{aligned} \int_{\mathbf{R}^n} |\mu_y * f(x)|^2 \omega(x) dx \\ \leq C_N \sum_{\nu} (1 + |\nu|)^{2n-N} \int_{\mathbf{R}^n} |f(x)|^2 \omega(x + y + \nu) dx \end{aligned}$$

(we may assume that $\omega \geq 0$).

Thus,

$$\begin{aligned}
& \int_{Q_0} \int_{\mathbb{R}^n} |\mu_y * f(x)|^2 \omega(x) dx \\
& \leq C_N \sum_{\nu} (1 + |\nu|)^{2n-N} \int_{\mathbb{R}^n} |f(x)|^2 \left| \int_{Q_0} \omega(x + y + \nu) dy \right| dx \\
& \leq C \int_{\mathbb{R}^n} |f(x)|^2 \omega^*(x) dx, \text{ taking } N \geq 4n + 1.
\end{aligned}$$

On the other hand, if $\widehat{T_\nu f(\xi)} = \psi(\xi - \nu) \hat{f}(\xi)$, we have:

$$\mu_y * f(x) = \sum_{\nu} e^{2\pi i \nu \cdot y} T_\nu f(x)$$

and therefore,

$$\int_{Q_0} \int_{\mathbb{R}^n} |\mu_y * f(x)|^2 \omega(x) dx dy = \sum_{\nu} \int_{\mathbb{R}^n} |T_\nu f(x)|^2 \omega(x) dx.$$

To finish we observe that $P_\nu f = P_\nu T_\nu f$ and we may apply the weighted inequality of reference [5].

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Let us consider for each $k \in \mathbb{Z}$ a decomposition of the strip $2^k \leq x_n \leq 2^{k+1}$ into congruent disjoint parallelepipeds $\{Q_\nu^k\}$ whose sides are parallel to the coordinate axis. Define:

$$Sf(x) = \left(\sum_{k, \nu} |P_\nu^k f(x)|^2 \right)^{1/2}$$

where $\widehat{P_\nu^k f(\xi)} = \chi_{Q_\nu^k}(\xi) \hat{f}(\xi)$. Combining the Littlewood-Paley theorem with the previous result we obtain:

COROLLARY 1. — For each p , $2 \leq p < \infty$, there exists a finite constant C_p so that $\|Sf\|_p \leq C_p \|f\|_p$, for every $f \in C_0(\mathbb{R}^n)$.

Proof of theorem 1. — We may assume, without lack of generality, that $0 \leq j \leq \frac{N}{8}$ so that $0 \leq \frac{2\pi j}{N} \leq \frac{\pi}{4}$.

We define

$$\Delta_j = \left\{ \xi = \xi_1 + i\xi_2, 1 \leq \xi_1 \leq 2, \frac{2\pi j}{N} \leq \arg(\xi) \leq \frac{2\pi(j+1)}{N} \right\}$$

$$j = 0, 1, \dots, \frac{N}{8},$$

$$\widehat{P_j f} = \chi_{\Delta_j} \cdot \hat{f}$$

and we want to compute:

$$\sum_{j,k} \int_{\mathbb{R}^2} |P_j f(x) P_k f(x)|^2 dx = \sum_{|j-k| < N^{1/2}} \int_{\mathbb{R}^2} |P_j f(x) P_k f(x)|^2 dx$$

$$+ \sum_{|j-k| \geq N^{1/2}} \int_{\mathbb{R}^2} |P_j f(x) P_k f(x)|^2 dx = I + II.$$

We decompose further each sector Δ_j into $N^{1/2}$ subsectors $\Delta_j^1, \dots, \Delta_j^{N^{1/2}}$, where

$$\Delta_j^\alpha = \{ \xi = \xi_1 + i\xi_2 \in \Delta_j \mid \alpha N^{-1/2} \leq \xi_1 - 1 \leq (\alpha + 1) N^{-1/2} \}.$$

It happens that if $|j - k| \geq N^{1/2}$ the overlapping of the sets $\Delta_j^\alpha + \Delta_k^\beta$, $\alpha, \beta = 1, \dots, N^{1/2}$, is finite (uniformly on N).

Therefore,

$$II \leq \sum_{|j-k| \geq N^{1/2}} \sum_{\alpha, \beta} \int_{\mathbb{R}^2} |P_j^\alpha f(x) P_k^\beta f(x)|^2 dx \leq \left\| \left(\sum_{j, \alpha} |P_j^\alpha f|^2 \right)^{1/2} \right\|_4^4$$

where the operators P_j^α have the obvious definition $\widehat{P_j^\alpha f} = \chi_{\Delta_j^\alpha} \cdot \hat{f}$. We claim that

$$\left\| \left(\sum_{j, \alpha} |P_j^\alpha f|^2 \right)^{1/2} \right\|_4 \leq C [\log N]^{1/4} \|f\|_4$$

for some universal constant C .

To see this we take $\omega \geq 0$ in $L^2(\mathbb{R}^n)$ and we consider:

$$\begin{aligned} \sum_{j,\alpha} \int_{\mathbb{R}^2} |P_j^\alpha f(x)|^2 \omega(x) dx &= \sum_{l=1}^{\frac{1}{8}N^{1/2}} \sum_{\alpha=1}^{N^{1/2}} \sum_{j=lN^{1/2}}^{(l+1)N^{1/2}} \int_{\mathbb{R}^2} |P_j^\alpha f(x)|^2 \omega(x) dx \\ &\leq C_s \sum_{l=1}^{\frac{1}{8}N^{1/2}} \sum_{\alpha=1}^{N^{1/2}} \int_{\mathbb{R}^2} |Q_l^\alpha f(x)|^2 M_s \omega(x) dx \end{aligned}$$

where $Q_l^\alpha f$ is given, at the Fourier transform side, as multiplication by the characteristic function of a rectangle, with sides parallel to the coordinates axis, and dimensions $N^{-1/2} \times 2N^{-1/2}$, $M_s \omega = (M\omega^s)^{1/s}$, $1 < s < \infty$, and M denotes the maximal function associated to the base of rectangles with directions in the set $2\pi j/N$, $j = 0, 1, \dots, N/8$ (see ref. [5]).

In establishing the last estimate we have made a repeat use of the lemma. Using Holder's inequality together with the known estimates for M , we get: $II \leq C[\log N] \|f\|_4^4$.

We estimate I in the following manner:

$$I = \sum_{\nu=0}^{\frac{1}{2}\log N} \sum_{2^{-\nu}N^{1/2} \leq |j-k| < 2^{-\nu+1}N^{1/2}} \int_{\mathbb{R}^2} |P_j f(x) P_k f(x)|^2 dx + \sum_j \|P_j f\|_4^4.$$

Since we always have $\sum_{j=1}^N \|P_j f\|_4^4 \leq C \|f\|_4^4$ and we want an estimate

with a factor of $(\log N)^a$, we may estimate each block of the preceding sum independently:

For each ν we decompose the secteur Δ_j into subsectors

$$\Delta_j^\alpha = \{\xi = \xi_1 + i\xi_2 \in \Delta_j \mid \alpha 2^\nu N^{-1/2} \leq \xi_1 - 1 \leq (\alpha + 1) 2^\nu N^{-1/2}\}$$

and we repeat the same arguments used in the estimation of II .

To finish we observe that, by homogeneity, we have proved the following: $\left\| \left(\sum_j |P_{j,n} f|^2 \right)^{1/2} \right\|_4 \leq C(\log N)^{1/2} \|f\|_4$, uniformly on n , where, for each $n \in \mathbb{Z}$

$$\Delta_{j,n} = \{\xi = \xi_1 + i\xi_2 \in \Delta_j \mid 2^n \leq \xi_1 \leq 2^{n+1}\}$$

$$\widehat{P_{j,n} f} = \chi_{\Delta_{j,n}} \cdot \hat{f}.$$

We decompose $\Delta_j = \bigcup_{l=1}^{\log N} \bigcup_{n \equiv l \pmod{[\log N]}} \Delta_{j,n}$ which gives us the decomposition

$$\left(\sum_j |P_j f|^2 \right)^{1/2} \leq \sum_{l=1}^{\log N} \left(\sum_j |P_j^l f(x)|^2 \right)^{1/2};$$

here P_j^l is given by the multiplier $\bigcup_{n \equiv l \pmod{[\log N]}} \Delta_{j,n}$.

The point is that if $n_1 \equiv n_2 \pmod{[\log N]}$ and, says, $n_1 > n_2$, then $2^{n_1} \geq N 2^{n_2}$. That is: the smaller side $2^{n_1}/N$ of the rectangles corresponding to Δ_{j,n_1} , $j = 1, 2, \dots, N$ is bigger than the diameter of the set $\bigcup_{\substack{n \equiv n_1 \pmod{[\log N]} \\ n < n_1}} \Delta_{j,n}$.

Furthermore, we decompose each $\Delta_{j,n}$ into N "squares" $\{\Delta_{j,n}^\alpha\}$ of side $\simeq 2^n N^{-1}$ and following our convention we shall define the corresponding multiplier operators $P_{j,n}^\alpha$.

To simplify notation we shall keep l fixed in the following and we shall assume that the index n ranges in the set of integers congruent with $l \pmod{[\log N]}$. We have:

$$\begin{aligned} & \left\| \left(\sum_j |P_j^l f|^2 \right)^{1/2} \right\|_4^4 \sim \left\| \left(\sum_{j,n} |P_{j,n} f|^2 \right)^{1/2} \right\|_4^4 \\ &= \sum_n \left\| \left(\sum_j |P_{j,n} f|^2 \right)^{1/2} \right\|_4^4 + 2 \sum_{\substack{j,k \\ n_1 > n_2}} \int_{\mathbb{R}^2} |P_{j,n_1} f(x) P_{k,n_2} f(x)|^2 dx \\ &\leq C(\log N)^2 \|f\|_4^4 + 2 \sum_{\substack{j,k \\ n_1 > n_2}} \int_{\mathbb{R}^2} \left| \sum_\alpha P_{j,n_1}^\alpha f P_{k,n_2} f \right|^2 dx. \end{aligned}$$

we have,

$$\begin{aligned} & \sum_{\substack{j,k \\ n_1 > n_2}} \int \left| \sum_\alpha P_{j,n_1}^\alpha f P_{k,n_2} f \right|^2 dx = \sum_{\substack{j,k \\ n_1 > n_2}} \int \left| \sum_\alpha \widehat{P_{j,n_1}^\alpha f} * \widehat{P_{k,n_2} f} \right|^2 d\xi \\ &\leq C \sum_{\substack{j,k \\ n_1 > n_2}} \sum_\alpha \int |\widehat{P_{j,n_1}^\alpha f} * \widehat{P_{k,n_2} f}|^2 d\xi \\ &= C \sum_{\substack{j,k \\ n_1 > n_2}} \sum_\alpha \int |P_{j,n_1}^\alpha f(x)|^2 |P_{k,n_2} f(x)|^2 dx \end{aligned}$$

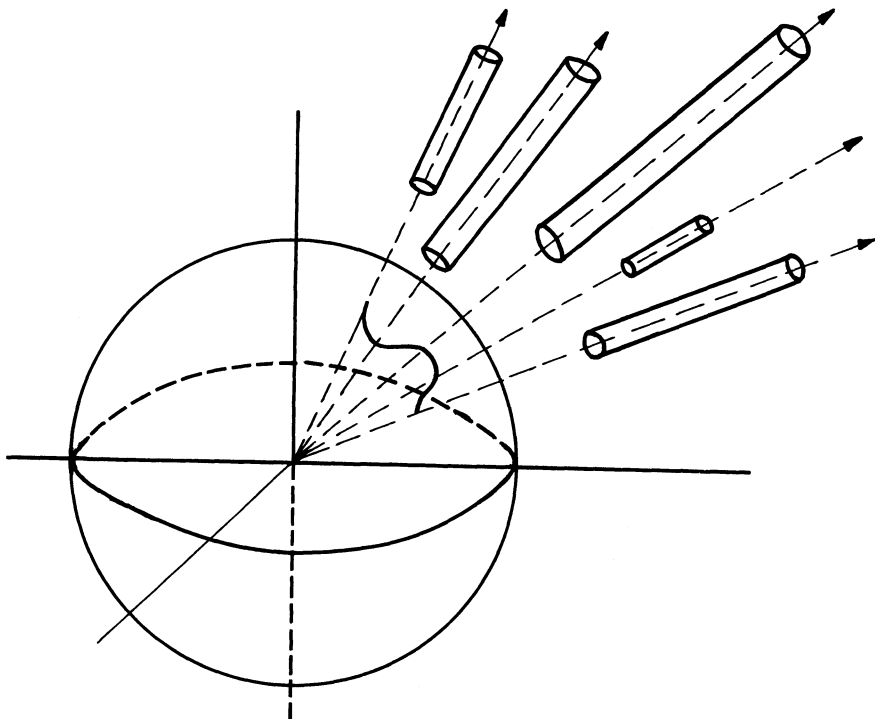
$$\begin{aligned}
&\leq C \left\| \left(\sum_{j,k,\alpha} |P_{j,n}^\alpha f|^2 \right)^{1/2} \right\|_4^2 \left\| \left(\sum_{k,n} |P_{k,n} f|^2 \right)^{1/2} \right\|_4^2 \\
&\leq C \|f\|_4^2 \left\| \left(\sum_j |P_j^l f|^2 \right)^{1/2} \right\|_4^2.
\end{aligned}$$

That is, we have obtained the inequality:

$$\begin{aligned}
\left\| \left(\sum_j |P_j^l f|^2 \right)^{1/2} \right\|_4^2 &\leq C(\log N)^2 \|f\|_4^4 \\
&\quad + C \|f\|_4^2 \left\| \left(\sum_j |P_j^l f|^2 \right)^{1/2} \right\|_4^2.
\end{aligned}$$

From which the desired result follows very easily.

B. The maximal function.



Our hypothesis over γ means that for each $\omega \in S^{n-1}$ and $b \in \mathbf{R}$ the function: $t \rightarrow \gamma(t) \cdot \omega - b$ has a finite number of changes of sign, uniformly in ω and b . It should be noted that,

in general, C_γ grows to infinity with this number. However, an estimate of the form $\|Mf\|_n \leq C(\log N)^a \|f\|_n$, should be true with C independent of γ . This is an interesting open problem.

For every positive integer m let us consider the points of γ given by $\omega_m^j = \gamma(j/2^m)$, $j = 1, 2, \dots, 2^m$. Let $\Psi \geq 0$ be a smooth function on the real line, supported on $|t| \leq 2$ and equal to 1 on $|t| \leq 1$.

We define $A_m^j f(x) = \int_{-\infty}^{+\infty} f(x - t\omega_m^j) \Psi(t) dt$

and $A_m f(x) = \sup_{j=1,2,\dots,2^m} |A_m^j f(x)|$.

Claim. — $\|A_m f\|_2 \leq C m \|f\|_2$, for every $f \in L^2(\mathbb{R}^n)$, where C is independent of m .

We shall prove the claim by induction. The case $m = 1$ is a consequence of the Hardy-Littlewood maximal theorem. Let us assume that the result is true for $k \leq m - 1$. It is very easy to check that $A_m f(x) \leq A_{m-1} f(x) + B_m f(x)$ where

$$B_m f(x) = \left(\sum_{j=1}^{2^m} |A_m^j f(x) - A_m^{j-1} f(x)|^2 \right)^{1/2}.$$

Therefore our claim is a consequence of the estimate: $\|B_m f\|_2 \leq C \|f\|_2$, uniformly on m . To see this we use Plancherel's theorem:

$$\begin{aligned} \int_{\mathbb{R}^n} |B_m f(x)|^2 dx &= \sum_{j=1}^{2^m} \int_{\mathbb{R}^n} |A_m^j f(x) - A_m^{j-1} f(x)|^2 dx \\ &= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{j=1}^{2^m} |\hat{\Psi}(\xi \cdot \omega_m^j) - \hat{\Psi}(\xi \cdot \omega_m^{j-1})|^2 d\xi; \end{aligned}$$

and we observe that, because of our hypotheses on γ , we have:

$$\sum_{j=1}^{2^m} |\hat{\Psi}(\xi \cdot \omega_m^j) - \hat{\Psi}(\xi \cdot \omega_m^{j-1})|^2 \leq C_\gamma < \infty \text{ uniformly on } m.$$

To continue let us observe that, in order to prove theorem 2, we can, without lack of generality, restrict to the case $r = 2^n$, $n \in \mathbb{Z}$ and, because of the fixed eccentricity, we may also consider cylinders with direction in the set $\gamma\left(\frac{j}{N}\right)$, $j = 1, \dots, N$. Finally we may take N of the form $N = 2^m$, $m \in \mathbb{Z}^+$.

Let us define:

i) $T_{2^\nu}^j f(x) = \sup_{x \in R} \frac{1}{\mu\{R\}} \int_R |f(y)| d\mu(y)$, where the supremum is taken over all cylinders with dimensions $(2^\nu)^{n-1} \times N 2^\nu$ and direction $\gamma\left(\frac{j}{N}\right)$.

ii) $T_{2^\nu} f(x) = \sup_j T_{2^\nu}^j f(x)$

$T^j f(x) = \sup_\nu T_{2^\nu}^j f(x)$

$Mf(x) = \sup_j T^j f(x) = \sup_\nu T_{2^\nu} f(x)$.

Given $\alpha > 0$ we obtain, for each j , a sequence of disjoint cylinders $\{R_\lambda^j\}_{\lambda=1,2,\dots}$ with direction $\gamma(j/N)$ and such that:

$E_\alpha^j = \{x: T^j f(x) > \alpha\} \subset \bigcup_\lambda \widetilde{R}_\lambda^j$ where $\widetilde{R}$ denotes the result of expanding R by the factor two. We have,

$$E_\alpha = \{x: Mf(x) > \alpha\} = \bigcup_{j=1}^N E_\alpha^j.$$

The heights of the N collections of cylinders, $\{R_\lambda^j\}$, $j = 1, \dots, N$, are bounded from above. By induction we may obtain, for each k , a family of cylinders B_k with dimensions $(2^{\nu_k})^{n-1} \times 2^{\nu_k} N$, $\nu_0 > \nu_1 > \dots$ in such a way that:

1) No cylinder of B_k is contained in the double of another cylinder of B_j , $j \leq k$.

2) If $R \in \bigcup_{j=1}^N \{R_\lambda^j\}$ and if

$$\dim(R) = (2^\nu)^{n-1} \times 2^\nu \cdot N, \quad \nu_{k-1} > \nu \geq \nu_k,$$

then either $R \in B_k$ or R is contained in the double of a cylinder in $\bigcup_{j \leq k} B_j$. Obviously: $E_\alpha \subset \bigcup_{R \in \bigcup_k B_k} \widetilde{R}$.

Let us denote by Δ_k the union of the families B_j , where $\nu_0 - k \log N \geq \nu_j \geq \nu_0 - (k+1) \log N$ and let $E_i = \bigcup_{R \in \Delta_i} R$, $\widetilde{E}_i = \bigcup_{R \in \Delta_i} \widetilde{R}$. We know that $E_\alpha \subset \bigcup \widetilde{E}_i$.

We can now observe that the family $\{E_i\}$ is almost disjoint; more concretely, if $|i-j| \geq 2$ then $E_i \cap E_j = \emptyset$. This is true

because if $R_i \in \Delta_i$, $R_j \in \Delta_j$, $i - j \geq 2$, then the radius of R_i is greater than the height of R_j and, therefore, if $R_i \cap R_j \neq \emptyset$ then $R_j \subset \tilde{R}_i$ which is impossible.

Let $f_i = f|_{E_i}$, $i = 0, 1, \dots$ and let S_i be the maximal function given in the following way: $S_i g(x) = \sup_{x \in R} \frac{1}{\mu\{R\}} \int_R |g(y)| d\mu(y)$, where the sup is taken over the set of cylinders of dimensions $(2^\nu)^{n-1} \times 2^\nu N$, where $\nu_0 + 2 - i \log N \geq \nu \geq \nu_0 + 2 - (i+1) \log N$.

The previously obtained estimate $\|A_m f\|_2 \leq C_m \|f\|_2$ implies that S_i is bounded on $L^2(\mathbf{R})$ with bound less than $C_\gamma (\log N)^{3/2}$.

If $x \in \tilde{E}_i$ there exists a cylinder $R \in \Delta_i$ so that $x \in \tilde{R}$ and, therefore:

$$\begin{aligned} S_i f_i(x) &\geq \frac{1}{\mu\{\tilde{R}\}} \int_R |f_i(y)| d\mu(y) \\ &\geq \left(\frac{1}{4}\right)^n \frac{1}{\mu\{R\}} \int_R |f_i(y)| d\mu(y) \geq \left(\frac{1}{4}\right)^n \alpha. \end{aligned}$$

That is, $\tilde{E}_i \subset \{x : S_i f_i(x) \geq 4^{-n} \alpha\}$, which implies

$$\begin{aligned} \mu\{E_\alpha\} &\leq \sum_i \mu\{\tilde{E}_i\} \leq C_\gamma (\log N)^3 \alpha^{-2} \sum_j \|f_i\|_2^2 \\ &\leq C_\gamma (\log N)^3 \alpha^{-2} \|f\|_2^2. \end{aligned}$$

A standard use of the Marcinkiewicz interpolation theorem would yield the strong type inequality of Theorem 1. q.e.d.

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