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REGULAR HOLOMORPHIC IMAGES OF BALLS

by J. E. FORNAESS ⁽¹⁾ and E. L. STOUT ⁽²⁾

*Dedicated to Walter Rudin on the occasion
of his sixtieth birthday, May 2, 1981.*

A few years ago we proved [2] that every paracompact connected N -dimensional complex manifold is the image of an N -dimensional polydisc under a finite regular holomorphic map. When we wrote [2] it was not clear to us that an analogous result could be obtained for the ball. In the present paper we settle this question as follows :

THEOREM. — *For each $N = 1, 2, \dots$, there is a positive integer λ_N such that if $\mathfrak{M}$ is a connected, paracompact N -dimensional complex manifold, then there is a regular holomorphic map Φ from the unit ball B_N in $\mathbb{C}^N$ onto $\mathfrak{M}$ such that for each $p \in \mathfrak{M}$, the fiber $\Phi^{-1}(p)$ contains not more than λ_N points.*

As we shall see, for λ_N we can take the value

$$(1 + 4^{12N}(12N)^{2N})((2N+1)4^N + 2).$$

This is not the best possible value for λ_N ; at the cost of considerable complication in the proof below, we could improve this bound somewhat, but the gain would be more apparent than real. We have no inkling of the best possible value of λ_N nor of any way of finding it. (We point out however that D. H. Bushnell [1] has improved our estimate for the polydisc case from $(2N+1)4^N + 2$ to $(2N+1)^2 + 1$. This has the immediate effect of reducing our λ_N to $(1 + 4^{12N}(12N)^{2N})((2N+1)^2 + 1)$ as we shall see in the course of the proof.)

We treat the ball case by reducing it to the case of the polydisc. Two new ingredients in this work are these. First, we invoke a result from [3] to

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recognize that certain monotone unions of balls are again balls. Second, based on some estimates in the plane, we have a lemma to the effect that certain dumbbell-shaped sets in $\mathbb{C}^N$ admit rough approximations by domains biholomorphically equivalent to balls.

The paper is organized as follows : In Section 1 we state two lemmas. In Section 2 we give the proof of the theorem assuming the lemmas. Sections 3 and 4 are devoted to the proofs of the lemmas, and Section 5 contains some concluding remarks.

The case $N = 1$ of the theorem was proved in [2]. Hence we will assume in the rest of this paper that $N \geq 2$.

1. Two lemmas.

We shall use the notation that for $x \in \mathbb{C}$, $\mathbf{x}$ denotes the point $(x, 0, \dots, 0) \in \mathbb{C}^N$. Also, for $r \in (0, \infty)$, K_r denotes the compact set in $\mathbb{C}^N$ consisting of the union of $\bar{B}_N$, the interval $[0, 4]$ and the ball $4 + r\bar{B}_N$ of radius r , center 4 .

LEMMA 1. — *If U is a neighborhood of K_2 , then there is an open set W biholomorphically equivalent to B_N with $K_{1/16} \subset W \subset U$.*

It would be of interest to know whether the dumbbell K_1 has a fundamental neighborhood basis consisting of balls, i.e., of domains biholomorphically equivalent to balls.

Denote by E the union of the ray $\{x : x \in \mathbb{R}, x \geq 0\}$ and the balls $4j + \bar{B}_N$, $j = 0, 1, 2, \dots$

LEMMA 2. — *If D is a domain in $\mathbb{C}^N$, then there is a regular holomorphic map Φ from a neighborhood V of E onto D such that for each $p \in D$, the fiber $\Phi^{-1}(p)$ consists of no more than $c_N = 1 + 4^{12N}(12N)^{2N}$ points and such that $\bigcup_{j=0}^{\infty} \Phi(4j + \frac{1}{64} B_N) \supset D$.*

2. Proof of the theorem.

Let $\mathcal{M}$ be a connected paracompact complex manifold of dimension N . The Main Theorem of [2] provides a regular holomorphic map from

the unit polydisc in $\mathbb{C}^N$ onto $\mathfrak{M}$ the multiplicity of which does not exceed $(2N+1)4^N + 2$. Thus, to prove the theorem, it would suffice to construct a regular holomorphic map from B_N onto a polydisc with multiplicity no more than $1 + 4^{12N}(12N)^{2N}$. We shall do somewhat more : We shall treat not only the case of the polydisc but rather the case of an arbitrary domain in $\mathbb{C}^N$. Our reason for doing this is twofold. The case of a general domain in $\mathbb{C}^N$ is not especially harder than the case of the polydisc, and by treating the case of general domains in $\mathbb{C}^N$ directly, we obtain a relatively simple proof of our theorem for these special domains. Thus, we shall prove the following result.

LEMMA 3. — *If D is a domain in $\mathbb{C}^N$, then there is a regular holomorphic map Ψ from B_N onto D such that for each $p \in D$, the fiber $\Psi^{-1}(p)$ consist of no more than c_N points.*

The c_N of this lemma is the same as the c_N of lemma 2.

Proof. — Let E, V and $\Phi : V \rightarrow D$ be as in lemma 2.

We shall construct a domain Ω biholomorphically equivalent to B_N with

$$\bigcup_{j=0}^{\infty} \left(4j + \frac{1}{64} B_N \right) \subset \Omega \subset V.$$

We may assume without loss of generality that

$$V \subset \{z \in \mathbb{C}^N : z = (z_1, \dots, z_N) \text{ with } |\operatorname{Im} z_j| < 2 \text{ for } j = 1, 2, \dots, N\}.$$

The domain on the right is biholomorphically equivalent to a polydisc and so is *taut* in the sense of [4]. By lemma 1 of [3], a monotone union of balls in V is (biholomorphically equivalent to) a ball. Accordingly, we shall construct a sequence $\{\Omega_j\}_{j=0}^{\infty}$ of domains with the following properties :

- (1) $\Omega_j \subset V$ for all j .
- (2) $4j + \frac{1}{16} B_N \subset \subset \Omega_j$ for all j .
- (3) $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega_2 \subset \subset \dots$
- (4) $\bar{\Omega}_j \cap \bigcup_{k>j} (4k + \bar{B}_N) = \emptyset$ for all j .

- (5) Ω_j is biholomorphically equivalent to B_N under a biholomorphic map that extends to carry a neighborhood of B_N biholomorphically onto a domain in V .

Conditions (1), (2) and (3) are the ones of interest; (4) and (5) are used to smooth the induction. Granted such a sequence, $\Omega = \bigcup_{j=0}^{\infty} \Omega_j$ is a domain in V biholomorphically equivalent to a ball, and it is carried onto D by the map Φ with the desired bound on the multiplicity.

To construct the sequence $\{\Omega_j\}_{j=0}^{\infty}$, let $\Omega_0 = B_N$. Assume $\Omega_0, \Omega_1, \dots, \Omega_k$ have been constructed so as to satisfy conditions (1)-(5). Let $\psi_k: B_N \rightarrow \Omega_k$ be a biholomorphic map that extends, as in (5), to carry a neighborhood of $\bar{B}_N$ biholomorphically onto a neighborhood of $\bar{\Omega}_k$. Denote by χ a smooth map from a neighborhood of K_2 into V that agrees with ψ_k near $\bar{B}_N$ and that agrees on a neighborhood of $4 + 2\bar{B}_N$ with the affine map $z \rightarrow 4(k+1) + \frac{1}{2}(z-4)$ that carries the ball $4 + 2B_N$ onto the ball $4(k+1) + B_N$. We require, in addition, that χ carry the interval $(0,4)$ diffeomorphically into $V \setminus (\bar{\Omega}_k \cup \bigcup_{m \geq k+1} (4m + \bar{B}_N))$. We may approximate χ uniformly on a neighborhood of K_2 by a holomorphic map ψ_{k+1} from C^N to C^N -note that K_2 is polynomially convex- and we may insist, moreover, that the approximation be close in the $\mathcal{C}^1$ sense on $[0,4]$. By making the approximation close enough, we shall have that ψ_{k+1} is one-to-one on K_2 . By replacing ψ_{k+1} by ψ'_{k+1} defined by $\psi'_{k+1}(z) = \psi_{k+1}(z+z_0)$ for a suitably chosen, small z_0 if necessary, we can be sure that ψ_{k+1} is regular on a neighborhood of K_2 . As $\Omega_{k-1} \subset \subset \Omega_k$, there is a domain Ω'_k biholomorphically equivalent to a ball with $\Omega'_k \subset \subset \Omega_k$ and such that Ω'_k satisfies (1)-(5). Granted that our approximation of χ by ψ_{k+1} is close enough, we shall have $\psi_{k+1}(B_N) \supset \supset \Omega'_k$ and

$$\psi_{k+1}(4 + \frac{1}{16} B_N) \supset \supset 4(k+1) + \frac{1}{64} B_N.$$

As ψ_{k+1} is one-to-one and regular on K_2 , there is a neighborhood U of K_2 carried biholomorphically into $V \setminus \bigcup_{m \geq k+1} (4m + \bar{B}_N)$ by ψ_{k+1} . Lemma 1 provides a domain W biholomorphically equivalent to B_N with $K_{1/16} \subset W \subset U$. For Ω_{k+1} we take $\psi_{k+1}(W)$. The sequence

$\Omega_0, \dots, \Omega_{k-1}, \Omega_k, \Omega_{k+1}$ enjoys the properties (1)-(5). Thus, our sequence $\{\Omega_j\}_{j=0}^\infty$ can be constructed inductively; at each step we alter the last Ω_k constructed, but having altered it once, it remains unaltered through the rest of the construction.

This proves lemma 3 and shows that to prove the theorem, it suffices to prove lemmas 1 and 2.

3. Proof of lemma 1.

At one point in the argument below we will need the following elementary geometric fact.

LEMMA 1.1. — For $d \in (0, \infty)$, let S_d be the union of all the discs in the open unit disc with centers on the real diameter and hyperbolic radius d . If $\alpha(d) = 2 \tanh d / (1 - \tanh^2 d)^2$, then $S_d \subset \{z = x + iy : |y| < \alpha(d)(1 - x)\}$.

Proof. — Denote by ρ the hyperbolic distance function on the disc so that if $[z, w] = \left| \frac{z - w}{1 - \bar{w}z} \right|$, then $\rho(z, w) = \frac{1}{2} \log \frac{1 + [z, w]}{1 - [z, w]}$. If $x \in (-1, 1)$ satisfies $\rho(0, x) = d$, then $x = \pm \tanh d$. Under the non-Euclidean translation $z \rightarrow \frac{z + t}{1 + tz} = \varphi_t(z)$, the circle $C_d(0) = \{z : \rho(z, 0) = d\}$ goes onto the circle $C_d(t) = \{z : \rho(z, t) = d\}$. If we denote by τ the number $\tanh d$, then the Euclidean radius δ of $C_d(t)$ is determined by $2\delta = \varphi_t(\tau) - \varphi_t(-\tau)$ so that

$$\delta = \frac{\tau(1 - t^2)}{1 - t^2\tau^2}.$$

This means that for $z = x + iy$ inside $C_d(t)$, we have

$$\begin{aligned} |y|/(1 - x) &< \tau(1 - t^2)(1 - t^2\tau^2)^{-1} \left(1 - \frac{t + \tau}{1 + t\tau} \right) \\ &= \tau(1 + t)/(1 - t\tau)(1 - \tau) \\ &\leq 2\tau/(1 - \tau)^2 \end{aligned}$$

which is the desired inequality.

The proof of lemma 1 depends on some estimates of conformal

mappings in the plane. Fix small δ_1 and δ_2 , and let Δ_n be the domain shown in figure 1. Thus, explicitly, Δ_n is the union of the open disc of radius

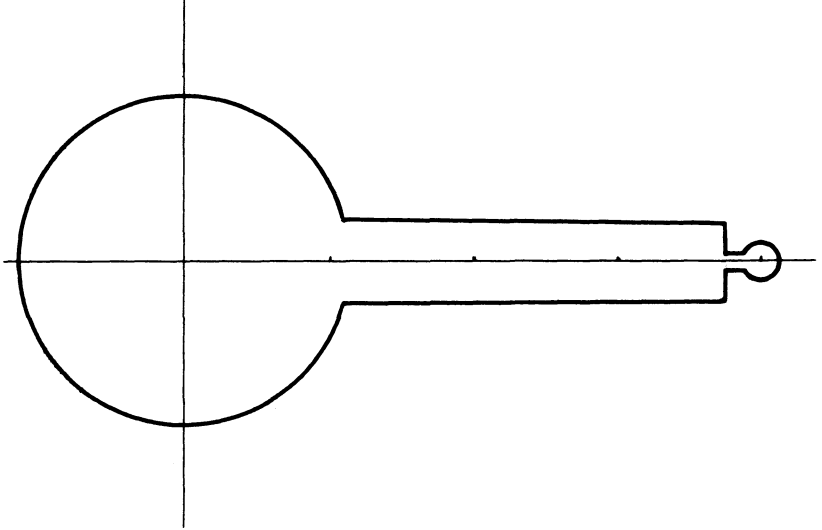


Fig. 1. — The domain Δ_n .

$1 + \delta_1$ centered at 0, the disc of radius $\frac{1}{8}$ centered at 4, the rectangle

$$\{x + iy : 0 < x < 3\frac{3}{4}, \quad -\delta_2 < y < \delta_2\}$$

and the rectangle

$$\left\{x + iy : 0 < x < 4, \quad -\frac{1}{n} < y < \frac{1}{n}\right\}.$$

Denote by φ_n the conformal map from the unit disc Δ to Δ_n , φ_n normalized so that $\varphi_n(0) = 0$ and $\varphi'_n(0) > 0$. Let $\psi_n : \Delta_n \rightarrow \Delta$ be the inverse of φ_n . By symmetry, φ_n and ψ_n both map the real line to itself, and we have that $\psi_n\left(4\frac{1}{8}\right) = 1$ for all n .

LEMMA 1.2. — *The sequence $\{\psi_n(4)\}_{n=1}^{\infty}$ converges to $1 \in b\Delta$.*

Proof. — By symmetry, we know that the points $\psi_n(4)$ are real.

A theorem of Carathéodory [5, p. 33] implies that the sequence $\{\varphi_n\}_{n=1}^{\infty}$ converges, uniformly on compacta in Δ , to the conformal map, φ , from Δ onto the *kernel* of the sequence $\{\Delta_n\}_{n=1}^{\infty}$ which is defined to be the largest domain D , $0 \in D$, with the property that if S is a closed subset of D with $S \subset \Delta_n$ for all large n , then $S \subset D$. The kernel of the sequence $\{\Delta_n\}_{n=1}^{\infty}$ is the union of the disc of radius $1 + \delta_1$ centered at 0 and the rectangle

$$\{x + iy : 0 < x < 3\frac{3}{4}, \quad -\delta_2 < y < \delta_2\}.$$

The map φ is normalized to satisfy $\varphi(0) = 0$, $\varphi'(0) > 0$, of course. Carathéodory's theorem asserts, moreover, that the sequence $\{\psi_n\}_{n=1}^{\infty}$ converges uniformly on compacta in D to φ^{-1} . This implies that if $K \subset \Delta$ is compact, then $K \subset \psi_n(D)$ for sufficiently large n . As ψ_n is one-to-one on Δ_n , it follows that $\psi_n(4)$ moves to $b\Delta$ as $n \rightarrow \infty$, and this implies that $\psi_n(4) \rightarrow 1$, as we wished to show.

A consequence of this lemma, by way of Schwarz's lemma, is that if V is the disc $\{z : |z - 4| < \frac{1}{16}\}$, then the sets $\psi_n(V)$ move to 1 as $n \rightarrow \infty$: If $\varepsilon > 0$, there is $n(\varepsilon)$ so that for $w \in V$, $|\psi_n(w) - 1| < \varepsilon$ provided $n > n(\varepsilon)$.

LEMMA 1.3. — *For large n , the set $\psi_n(V)$ is contained in the Stolz angle*

$$\{z : |z| < 1, \quad |\operatorname{Im} z| < 4(1 - \operatorname{Re} z)\}.$$

Proof. — Put $t_n = \psi_n(4)$ so that the sequence $\{t_n\}_{n=1}^{\infty}$ in $(0, 1)$ converges to 1. The points on the circle bV are all at the same distance, $d = \frac{1}{2} \log 3$, from the point 4, distance computed in the Carathéodory metric on the disc $V' = \{z : |z - 4| < \frac{1}{8}\}$. The holomorphic map $\psi_n : V' \rightarrow \Delta$ decreases the Carathéodory distance, so if ρ is again the Carathéodory (or hyperbolic) distance function on Δ , then

$$\psi_n(V) \subset \left\{z : \rho(t_n, z) < \frac{1}{2} \log 3\right\}.$$

The result now follows from lemma 1.1, for $\alpha\left(\frac{1}{2} \log 3\right) = 4$.

For $\delta \in (0, \pi)$, set $J_\delta = \{e^{i\theta} : |\theta| < \delta\}$, and define k_δ to be the harmonic extension of $\chi_{J_\delta} \log \left| \frac{1}{\sqrt{1-z_1}} \right|$ to the unit disc, Δ , where χ_{J_δ} denotes the characteristic function of J_δ .

LEMMA 1.4. — *There is $\delta_0 > 0$ small enough that for all $\delta \in (0, \delta_0)$,*

$$(1 - |z_1|^2) e^{2k_\delta(z_1)} \leq 2$$

for $z_1 \in \Delta$.

Proof. — The function $\log \left| \frac{1}{\sqrt{1-z_1}} \right|$ is harmonic on the unit disc and is positive at $z_1 \in \bar{\Delta}$ if and only if $|1 - z_1| < 1$. For $z_1 = e^{it}$, this happens if and only if $t \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \pmod{2\pi}$. Fix $\varepsilon > 0$ small enough that for $z_1 \in \Delta$ with $\operatorname{Re} z_1 \geq 1 - \varepsilon$, $\log \left| \frac{1}{\sqrt{1-z_1}} \right| > 0$. Since $k_\delta \rightarrow 0$ uniformly on $\{z_1 \in \bar{\Delta} : \operatorname{Re} z_1 \leq 1 - \varepsilon\}$ when $\delta \rightarrow 0$, it suffices to prove the inequality for $z_1 \in \bar{\Delta}$ with $\operatorname{Re} z_1 \geq 1 - \varepsilon$.

Fix $\delta_0 > 0$ so small that the harmonic extension h_δ of

$$(1 - \chi_{J_\delta}) \log \left| \frac{1}{\sqrt{1-z_1}} \right|$$

is positive on $\Delta \cap \{\operatorname{Re} z_1 = 1 - \varepsilon\}$ for all $\delta \in (0, \delta_0)$. The function h_δ is nonnegative on $\Delta \cap \{z_1 : \operatorname{Re} z_1 \geq 1 - \varepsilon\}$ because it has nonnegative values on the boundary of this set. For $\delta \in (0, \delta_0)$ and $\operatorname{Re} z_1 \geq 1 - \varepsilon$, $z_1 \in \Delta$, we have

$$\begin{aligned} (1 - z_1 \bar{z}_1) e^{2k_\delta(z_1)} &\leq (1 - z_1 \bar{z}_1) e^{2(k_\delta(z_1) + h_\delta(z_1))} \\ &= (1 - z_1 \bar{z}_1) \left| \frac{1}{1 - z_1} \right| \\ &= 2. \end{aligned}$$

The lemma is proved.

We can now prove lemma 1. Fix a neighborhood U of the set K_2 . Choose $\delta_0 > 0$ so small that $|z - w| > \delta_0$ if $z \in K_2$, $w \notin U$. Fix a small $\delta > 0$, and let J_δ be the arc of the unit circle as above. Define $\tilde{k}_\delta$ to be

the function holomorphic on the unit disc determined by $\text{Im } \tilde{k}_\delta(0) = 0$ and $\text{Re } \tilde{k}_\delta = k_\delta$, k_δ as in the preceding lemma. Thus,

$$(1) \quad \tilde{k}_\delta(z) = \frac{-1}{2\pi} \int_{-\delta}^{\delta} \frac{e^{it} + z}{e^{it} - z} \log |1 - e^{it}| dt.$$

Define $\tilde{\varphi}_n : B_N \rightarrow C^N$ by

$$\tilde{\varphi}_n(z_1, \dots, z_N) = (\varphi_n(z_1), (1 + \delta_2)z_2 e^{k_\delta(z_1)}, \dots, (1 + \delta_2)z_N e^{k_\delta(z_1)})$$

with φ_n the conformal map considered above and δ_2 as in the definition of Δ_n . The map $\tilde{\varphi}_n$ is regular and one-to-one from B_N onto a domain W in C^N . We shall show that for a suitable choice of the parameters on which φ_n depends and a suitable choice of the interval J_δ , the domain W has the properties we seek.

We shall need the fact that as $\delta_2 \rightarrow 0$, $\varphi_n(z_1) \rightarrow (1 + \delta_1)z_1$ uniformly on compacta in Δ as follows from [5, p. 33]. It is clear that the convergence is, in fact, uniform on compacta in $\bar{\Delta} \setminus \{1\}$.

Fix $\delta_2 < \delta_1 < \delta_0$ and $n > \frac{1}{\delta_2}$.

We define $\tilde{\varphi}'_n : B_N \rightarrow C^N$ by

$$\tilde{\varphi}'_n(z) = (\varphi_n(z_1), (1 + \delta_1)z_1, \dots, (1 + \delta_1)z_N).$$

Provided δ_1 and δ_2 are small enough, $\tilde{\varphi}'_n$ carries B_N into U : Since $|z_1|^2 + \dots + |z_N|^2 < 1$ when $z \in B_N$, it follows that for $z \in B_N$, if z_1 is near 1, then $|z_2|^2 + \dots + |z_N|^2$ is small. Also, when z_1 is away from 1, $\varphi_n(z_1)$ is near $(1 + \delta_1)z_1$, provided n is large enough. It follows that, as soon as δ_1 is small enough, we have $B_N \subset \subset \tilde{\varphi}'_n(B_N) \subset U$.

From (1) we see that $\tilde{k}_\delta(z) \rightarrow 0$ uniformly on compacta in $\bar{\Delta} \setminus \{1\}$ as $\delta \rightarrow 0^+$. Therefore, for small δ , $\tilde{\varphi}_n$ is a small perturbation of $\tilde{\varphi}'_n$ on a given compact subset of $\bar{B}_N \setminus \{1\}$, and so, provided δ is small enough, $\tilde{\varphi}_n$ carries the part of B_N on which $\text{Re } \varphi_n(z_1) < 3\frac{1}{2}$ onto a domain in U that contains $\bar{B}_N$.

For $z \in B_N$, the distance, $d(z)$, from $\tilde{\varphi}_n(z)$ to 4 is given by

$$d^2(z) = |\varphi_n(z_1) - 4|^2 + (1 + \delta_1)^2(|z_2|^2 + \dots + |z_N|^2)e^{2\text{Re } k_\delta(z_1)}.$$

When $\operatorname{Re} \varphi_n(z_1) \geq 3\frac{1}{2}$, $d(z)$ is less than 2, for $|z_1 - 4|$ is bounded by $\frac{3}{4}$ and

$$(1 + \delta_1)^2(|z_2|^2 + \cdots + |z_N|^2)e^{2\operatorname{Re} k_\delta(z_1)}$$

is no more than $2(1 + \delta_1)^2$ as follows from lemma 1.4. Thus, for $z \in B_N$ with $\operatorname{Re} \varphi_n(z_1) \geq 3\frac{1}{2}$, $\tilde{\varphi}_n(z) \in U$. The conclusion is that $\tilde{\varphi}_n(B_N) \subset U$ as soon as δ_1 , δ_2 and δ are sufficiently small.

Finally, by taking n large enough, we find that $\tilde{\varphi}_n(B_N)$ contains a ball of radius $\frac{1}{16}$ centered at 4. To see this, denote by $\tilde{V}_n$ the set

$$\left\{ z \in B_N : |\varphi_n(z_1) - 4| < \frac{1}{16} \right\},$$

and write $b\tilde{V}_n = \Sigma_1 \cup \Sigma_2$ where $\Sigma_1 = b\tilde{V}_n \cap bB_N$ and

$$\Sigma_2 = b\tilde{V}_n \cap \{z \in B_N : |\varphi_n(z_1) - 4| = \frac{1}{16}\}.$$

For $z \in \Sigma_2$, $d(z) \geq \frac{1}{16}$, and for $z \in \Sigma_1$, we have

$$d^2(z) = |\varphi_n(z_1) - 4|^2 + (1 + \delta_1)^2(1 - |z_1|^2)e^{2\operatorname{Re} k_\delta(z_1)}.$$

The function $\log \left| \frac{1}{\sqrt{1 - z_1}} \right|$ is harmonic on the unit disc so

$$\begin{aligned} 2\operatorname{Re} \tilde{k}_\delta(z_1) &= \log \left| \frac{1}{\sqrt{1 - z_1}} \right| - \frac{1}{\pi} \int_{e^{it} \notin J_\delta} \frac{1 - |z_1|^2}{|e^{it} - z_1|^2} \log |1 - e^{it}| dt \\ &= \log \left| \frac{1}{\sqrt{1 - z_1}} \right| + v(z_1) \end{aligned}$$

where $v(z_1) \rightarrow 0$ as $z \rightarrow e^{it} \in \operatorname{int} J_\delta$ and uniformly when e^{it} is constrained to lie in a compact subset of $\operatorname{int} J_\delta$. Thus, provided only that n is large, we have that for $z \in \Sigma_1$,

$$2\operatorname{Re} \tilde{k}_\delta(z_1) > \log \left| \frac{1}{\sqrt{1 - z_1}} \right| - 1$$

which implies

$$d^2(z) > (1 + \delta_1)^2 \frac{1 - |z_1|^2}{|1 - z_1|} e^{-1}.$$

By lemma 1.3, z_1 lies in the Stolz angle $|\operatorname{Im} z_1| < 4(1 - \operatorname{Re} z_1)$, so we have the estimate that for large n ,

$$d^2(z) > (1 + \delta_1)^2 \frac{1}{3e}$$

whence $d(z) > \frac{1}{\sqrt{3e}} > \frac{1}{16}.$

This completes the proof of lemma 1.

4. Proof of lemma 2.

We begin with a covering lemma.

LEMMA 2.1. — *Given a domain $D \subset \mathbb{C}^N$, there is a family $\{V_j''\}_{j=0}^\infty$ of open balls in $\mathbb{C}^N$ with these properties :*

- (1) $V_j'' \subset \subset D$.
- (2) Every point in D has a neighborhood intersecting at most $c(N) = 4^{12N}(12N)^{2N}$ of the V_j'' .
- (3) If V_j' denotes the ball concentric with V_j'' and of radius $\frac{1}{128}$ times that of V_j'' , then $\bigcup_{j=0}^\infty V_j' = D$.

Proof. — According to [6, pp. 167 ff.], there is a locally finite sequence $Q_0, Q_1, \dots$ of closed cubes with mutually disjoint interiors such that

$$(4) \quad Q_j \subset D.$$

$$(5) \quad \bigcup_{j=0}^\infty Q_j = D.$$

(6) The sides of the Q_j are parallel to the coordinate axes.

(7) If Q_j^* is the cube centered at the center of Q_j and of edge $\frac{9}{8}$ the edge of Q_j , then $Q_j^* \subset \subset D$, and no point of D lies in more than $(12)^{2N}$ of the Q_j^* .

Fix attention on Q_0 . By a change of coordinates, we may suppose it to be the unit cube in the underlying $\mathbf{R}^{2N}$ so that

$$Q_0^* = \left\{ x \in \mathbf{R}^{2N} : -\frac{1}{16} \leq x_1, \dots, x_{2N} \leq 1 + \frac{1}{16} \right\}.$$

Put $v = 4^6 N$ and let $K_1, \dots, K_{v^{2N}}$ be the subcubes of Q_0 of edge $\frac{1}{v}$ with vertices $\left(\frac{\mu_1}{v}, \dots, \frac{\mu_{2N}}{v}\right)$ for integers $\mu_1, \dots, \mu_{2N}$. Let $V'_{0,r}$ be the open ball of radius $\frac{1}{v}\sqrt{2N}$ centered at the center p_r of K_r , and let $V''_{0,r}$ be the open ball centered at p_r and of radius $\frac{128}{v}\sqrt{2N}$. The balls $V'_{0,r}$ cover Q_0 and the balls $V''_{0,r}$ are contained in Q_0^* . To see the latter point, notice that the distance from p_r to bQ_0^* is at least $\frac{1}{2v} + \frac{1}{16}$, so for $x \in V''_{0,r}$ to lie outside Q_0^* , we must have $\frac{1}{2v} + \frac{1}{16} < \frac{128}{v}\sqrt{2N}$. As $v = 4^6 N, N \geq 2$, this inequality is not satisfied.

We perform the analogous construction, appropriately scaled, on each of the other cubes $Q_1, Q_2, \dots$. In this way, we obtain two families of concentric open balls

$$\{V'_{j,r}\}_{\substack{j=0,1,2,\dots \\ r=1,2,\dots,v^{2N}}} \quad \text{and} \quad \{V''_{j,r}\}_{\substack{j=1,2,\dots,v^{2N} \\ r=1,2,\dots,v^{2N}}}$$

such that $\bigcup_{j,r} V'_{j,r} \subset \subset D$ for all j, r , and the diameter of $V''_{j,r}$ is 128 times the diameter of $V'_{j,r}$.

As no point of D is in more than $(12)^{2N}$ of the Q_j^* , a given point can belong to $V''_{j,r}$ for at most $(12)^{2N}$ distinct indices j . For a given j , there are $v^{2N} = 4^{12N} N^{2N}$ of the $V''_{j,r}$, so we see that no point of D lies in more than $c(N)$ of the $V''_{j,r}$. The lemma is proved.

Finally, we turn to the proof of lemma 2.

Let the sequence $\{V'_j\}_{j=0}^\infty$ and $\{V''_j\}_{j=0}^\infty$ be as in lemma 2.1. For each j , let $V_j \subset \subset V''_j$ be a concentric ball whose radius is very nearly the same as that of V''_j .

Specify a pair of diametrically opposite points on bV_j , say s_j and t_j . We may choose these points so that all of the s_j 's are distinct, all of the t_j 's are distinct, and so that no s_j is a t_j . For $j = 0, 1, 2, \dots$, let α_j be an invertible affine map of $\mathbb{C}^N$ to itself that carries $4j + B_N$ to V_j with $4j - 1$ and $4j + 1$ going to s_j and t_j respectively. Let $\varphi : \mathbb{R} \rightarrow D$ be a smooth, regular one-to-one map such that for all x in a neighborhood (in $\mathbb{R}$) of the interval $[4j-1, 4j+1]$, $\varphi(x) = \alpha_j(x)$ and such that the curves

$$L_j = ([4j-1, 4j+1]), \quad j = 0, 1, 2, \dots$$

have mutually disjoint neighborhoods $U_j \subset\subset D$. (NB. We do not require that the family $\{L_j\}_{j=0}^\infty$ be locally finite in D ; this additional condition cannot be met unless bD is connected.)

If $\Phi : \mathbb{C}^N \rightarrow \mathbb{C}^N$ is an entire map that approximates φ sufficiently well on $\mathbb{R}$ and α_j sufficiently well on a neighborhood of $4j + \bar{B}_N$, then Φ will be regular and one-to-one on a neighborhood of $4j + \bar{B}_N$ and will be one-to-one from a neighborhood of $[4j-1, 4j+1]$ into U_j . Moreover, we may take Φ to be regular on a neighborhood in $\mathbb{C}^N$ of the ray $\{x : x > 0\}$. (For the details of this kind of approximation, see [2, lemma 2.3] where the context is only formally different from the present one.) Notice finally that if the approximation is close enough on $4j + B_N$, we shall have that

$$\Phi(4j + \frac{1}{64} B_N) \supset V'_j \quad \text{and} \quad \Phi(4j + \bar{B}_N) \subset V''_j.$$

It follows that if V is a suitably small neighborhood of the set E , then Φ is regular on V , that $\Phi(V) = D$, and that no point is the image of more than $1 + 4^{12N}(12N)^{2N}$ points in V .

This completes the proof of lemma 2 and hence the proof of the theorem.

5. Remarks.

As we noted in the introduction, our estimate for the bound λ_N is not sharp, but we do not know what the sharp bound is.

In the other direction, it seems likely that on *purely topological* grounds, there should be an interesting *a priori* lower bound for λ_N . The fact that e.g., the ball and the polydisc are holomorphically distinct but topologically

equivalent shows that topological considerations are probably insufficient to determine the true lower bound for λ_N .

Granted the result of this paper and that of [2], it seems not unreasonable to ask for a pair of domains Ω', Ω'' in $\mathbb{C}^N$, Ω' a bounded domain of holomorphy, such that Ω' admits no regular holomorphic map onto Ω'' . Of course, if such Ω' and Ω'' exist, then for Ω'' we may take either a polydisc or a ball.

BIBLIOGRAPHY

- [1] D. H. BUSHNELL, Mapping a polydisc onto a complex manifold, Senior Thesis, Princeton University, 1976 (Princeton University Library).
- [2] J. E. FORNAESS and E. L. STOUT, Spreading polydiscs on complex manifolds, *Amer. J. Math.*, 99 (1977), 933-960.
- [3] J. E. FORNAESS and E. L. STOUT, Polydiscs in complex manifolds, *Math. Ann.*, 227 (1977), 145-153.
- [4] S. KOBAYASHI, *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, New York, 1970.
- [5] A. I. MARKUSHEVICH, *Theory of Functions of a Complex Variable*, vol. III, Prentice-Hall, Englewood Cliffs, 1967.
- [6] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

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