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On the space of maps inducing isomorphic connections


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ON THE SPACE
OF MAPS INDUCING ISOMORPHIC CONNECTIONS

by T.R. RAMADAS

1. Introduction.

In this paper we prove the following

**Theorem.** — Let $M$ be a smooth compact manifold, $P$ a principal bundle on $M$ with the unitary group $U(k)$ as structure group, $A$ a smooth connection on $P$, and Aut $A$ the group of gauge transformations (i.e., automorphisms of $P$ which act trivially on $M$) which leave $A$ invariant. Let $B$ be the Grassmanian of $k$-planes in a separable Hilbert space $\mathcal{H}$, $E$ the Stiefel bundle of orthonormal $k$ frames in $\mathcal{H}$, and $\omega$ the canonical universal connection on $E$. Denote by $\Sigma(A)$ the space of maps $p : M \to B$ such that the pull-back bundle $p^*(E)$, with the connection $p^*\omega$, is isomorphic to $(P, A)$.

Then the space $\Sigma(A)$, with the $C^\infty$ topology, has the homotopy type of $B_{(\text{Aut}A)}$ where $B_{(\text{Aut}A)}$ is the base-space of a universal bundle for Aut $A$.

The connectedness of $\Sigma(A)$ is shown in [6]. We use some ideas from this paper.

To motivate this result, consider the case when $P$ is a principal $G$-bundle with $G$ a compact Lie group. Let Aut $P$ denote the group of gauge transformations of $P$. Denote by $\mathcal{E}$ the space of $C^\infty$ connections on $P$. The group Aut $P$ acts on $\mathcal{E}$, though not freely in general. Denote by $\mathcal{E}$ the quotient.
By [4] there exists a finite dimensional principal $G$-bundle $E(G, M) \rightarrow B(G, M)$ with connection such that the following diagram commutes, and the map $\varphi$ is onto:

$$
\begin{array}{ccc}
\text{Mor}_G(P, E(G, M)) & \xrightarrow{\varphi} & \mathcal{C} \\
\downarrow \text{Aut } P & & \downarrow \\
\text{Mor}_P(M, B(G, M)) & \xrightarrow{\mathcal{L}} & \mathcal{C}
\end{array}
$$

Here $\text{Mor}_G(P, E(G, M))$ is the space of $G$-morphisms of $P$ into $E$ and $\text{Mor}_P(M, B(G, M))$ is the component of $C^\infty(M, B(G, M))$ which induces pull-back bundles isomorphic to $P$. $\mathcal{L}$ is the map given by pulling back the universal connection on $E(G, M)$.

We wish to investigate the fibres of the map $\varphi$. It is possible to do so when we consider instead of $E(G, M)$ a universal bundle $E_G$ with connection such that $E_G$ is contractible. Suppose then, that in the above diagram we replace $E(G, M)$ by $E_G$ and $B(G, M)$ by $B_G$. Let $A \in \mathcal{C}$ and $\mathcal{A}$ its class in $\mathcal{C}$. We argue heuristically:

The spaces $\mathcal{C}$ and $\text{Mor}_G(P, E_G)$ are both contractible. This would imply that $\varphi^{-1}(A)$ is contractible (all the mappings being assumed to be good fibrations). The group $\text{Aut } A$ acts on $\varphi^{-1}(A)$ to give $\varphi^{-1}(A)$. If all goes well this implies

a) $\varphi^{-1}(A) \rightarrow \mathcal{L}^{-1}(A)$ is a universal $\text{Aut } A$ bundle. The fibre over $A$ of the map $\mathcal{L}$ has the same homotopy type as $B_{(\text{Aut } A)}$.

b) If $G$ has trivial centre and all connections are generic (i.e. $\text{Aut } P$ acts freely on $\mathcal{C}$) $\mathcal{L}$ has a section.

The quotient space $\mathcal{L}$ is relevant in studies of Yang-Mills theories, at present very popular in Physics. It has been pointed out [1] that the Universal Connection Theorem could possibly provide connections between Yang-Mills theories and so-called $\sigma$-models which concern themselves with the space $\text{Mor}(M, B)$. Also in the cases when $\mathcal{L}$ has a section, it could give an alternative to "gauge-fixing" which has been shown to be impossible in general [3, 7, 5].

The paper is organized as follows. In § 2 we imbed $E$ and $B$ as closed submanifolds of Hilbert spaces. In § 3 we describe a one parameter family of isometries $A_t : \mathcal{H} \rightarrow \mathcal{H}$, and also give the
$\mathcal{C}^\infty$ topology to be used on the function spaces $\text{Mor}_{U(k)}(P, E)$ and $\text{Mor}_r(M, B)$. In § 4 we prove that $\varphi^{-1}(A)$ is contractible [Proposition 4.1] using the isometries $A_t$. Then we prove [Proposition 4.3] that $\varphi^{-1}(A) \rightarrow \mathcal{G}^{-1}(A)$ is a locally trivial principal fibre space with $\text{Aut} A$ as structure group. This involves, among other things, proving that the above projection is closed [Lemma 4.4], which is done by studying a certain differential equation. The completeness of the $\mathcal{C}^\infty$ topology is crucial, and the imbeddings obtained in § 2 simplify proofs throughout.

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2. The bundle of orthonormal $k$-frames in a Hilbert space.

Fix an integer $k > 0$. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space over the complex numbers. Denote by $E$ the space of orthonormal $k$-frames in $\mathcal{H}$. The group $U(k)$ acts on $E$ on the right and the quotient is the Grassmannian $B$ of $k$-dimensional subspaces of $\mathcal{H}$. In fact $E$ is a universal principal bundle for $U(k)$. It also carries a natural connection, which is a universal connection for $U(k)$.

It will be useful, in the following, to have characterizations of $E$ and $B$ as closed submanifolds of Hilbert spaces.

We shall identify a point $p$ in $B$ with the orthogonal projector onto the corresponding subspace, denoted by $H(p)$. Thus $H(p) = \{x \in \mathcal{H} \mid px = x\}$. For $p_0 \in B$, define

$$\mathcal{R}_0 = \{p \in B \mid H(p_0) \cap \ker p = \{0\}\}.$$ 

Then we have a bijection $L_0 : \mathcal{R}_0 \rightarrow \mathcal{L}(H(p_0), \ker p_0)$ such that for $p \in \mathcal{R}_0$ its image $L_0 = L_0(p)$ has $H(p)$ as graph.

**Lemma 2.1** [2]. *The charts $\{(\mathcal{R}_0, L_0)\}$ give $B$ the structure of a $\mathcal{C}^\infty$ Hilbert manifold.*

Let $\mathcal{H}_2$ denote the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$.
PROPOSITION 2.2. — Let \( \psi \) denote the injection \( B \rightarrow J_2 \) given by associating to each \( k \)-dimensional subspace its orthogonal projector. Then \( \psi \) is a \( C^\infty \) immersion, and a homeomorphism onto its image.

Proof. — Follows from Lemmas 2.3 and 2.4.

Remark. — This shows that \( B \), with the manifold structure given in Lemma 2.1 is a submanifold of \( J_2 \).

LEMMA 2.3. — On a chart \( (\mathcal{U}_0, L_0) \) \( \psi \) is given by \( (1 - 3) \). It is a \( C^\infty \) immersion.

Proof. — Let \( L \in \mathcal{L}(H(p_0), \ker p_0) \) and let \( p = \psi L_0^{-1}(L) \).

Write

\[
p = A + LA
\]

where \( A : \mathcal{H} \rightarrow H(p_0) \). Then we claim that \( A \) satisfies

\[
A = p_0 + L^+(1 - p_0) - L^+LA
\]

which can be solved to give

\[
A = \frac{1}{1 + L^+L} (p_0 + L^+(1 - p_0)).
\]

To see that \( p \) given by \( (2.1) - (2.3) \) is indeed equal to \( \psi L_0^{-1}(L) \), we verify:

a) Image of \( p = \{x + Lx | x \in H(p_0)\} \). The map is clearly into this set. In fact it is onto since \( A \) is invertible on \( H(p_0) \).

b) \( p^2 = p \). This follows since \( Ap = p \), which in turn is clear because \( Ap \) satisfies the same equation as \( p \).

\[
Ap = p_0p + L^+(1 - p_0) p - L^+LAp = A + L^+LA - L^+LAp
\]

\[
= p_0 + L^+(1 - p_0) - L^+LAp.
\]

c) \( p \) is an orthogonal projector, for

\[
\ker p = \{y - L^+y | y \in \ker p_0\}
\]

which is the orthogonal subspace to \( \text{Im} p \).

(i) \( \psi \) is \( C^\infty \): To see this split \( \psi \) into the steps:
$\mathcal{L}(\mathcal{H}, H(p_0)) \hookrightarrow \mathcal{L}(\mathcal{H}, \mathcal{H})$

$\{p_0 + L^+(1 - p_0)\} \{p_0 + L^+(1 - p_0)\}$

$\mathcal{L}(H(p_0), \ker p_0) \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{H})$

$\{L^+L\} \{p_0 \left[ \frac{1}{1 + L^+L} \right] p_0 \}$

$\psi$ is in fact real-analytic.

(ii) It is enough to check the differential at $L = 0$. Here $\delta p = \delta L^+(1 - p_0) + p_0 \delta L$ which is clearly injective. Also the image, being defined by $p_0 \delta pp_0 = (1 - p_0) \delta p(1 - p_0) = 0$ and $\delta p^+ = \delta p$, is closed, and hence admits a supplement.

**Lemma 2.4.** — The inverse $\psi^{-1}$ is given by (4) and is continuous.

**Proof.** — Consider a chart $(\mathcal{R}_0, L_0)$. Let $p \in \mathcal{R}_0$ and let $Q = (p_0 |_{H(p)})^{-1}$. Then for $x \in H(p)$, $Qx = x + (1 - p_0)pQx$. This gives, for $L = (1 - p_0)Q$, $L = (1 - p_0)p(1 + L)$.

This can be solved to give $p \xleftarrow{\psi^{-1}} L$ such that

$$Lx = (1 - p_0) \frac{1}{1 - (1 - p_0)p} x, x \in H(p_0). \quad (4)$$

The continuity of $\psi^{-1}$ follows easily.

We turn now to $E$. This can be identified with a closed subset of $\mathcal{L}(C^k, \mathcal{H})$: $E = \{U : C^k \longrightarrow \mathcal{H} \mid U^+U = 1\}$. Standard arguments show:

**Lemma 2.5.** — $E$ is a closed submanifold of $\mathcal{L}(C^k, \mathcal{H})$. It is a principal $U(k)$ bundle on $B$. The $u(k)$-valued one-form $U^+dU$ is a connection on $E$. 
Lemma 2.6. — $E$ is contractible and hence a universal $U(k)$ bundle. The connection is a universal $U(k)$ connection.

Proof. — Both statements are well-known. The first follows also from the remarks after Lemma 4.2. The second is a consequence of the Universal Connection Theorem.

3. Some preliminary remarks and definitions.

(i) A one-parameter-family of isometries on $\mathcal{H}$.

Following [6], we introduce, on $\mathcal{H}$, a one-parameter family of isometries which we will use later. Define, for $t \in [0, 1]$ an isometry $A_t : \mathcal{H} \to \mathcal{H}$ as follows. Fix an orthonormal basis, so that $\mathcal{H} \approx \{\text{square-summable sequences in } \mathbb{C}\}$. Then let $A_0 = \text{Identity}$

$$A_t(a_0, a_1, a_2, \ldots) = (a_0, a_1 \ldots a_{n-2}, a_{n-1} \cos \theta_n(t), a_{n-1} \sin \theta_n(t))$$

$$a_n \cos \theta_n(t), a_n \sin \theta_n(t) a_{n+1} \cos \theta_n(t), a_{n+1} \sin \theta_n(t) \ldots$$

for $\frac{1}{n+1} \leq t \leq \frac{1}{n}$ where $\theta_n(t) = \frac{\pi}{2} n[(n + 1) t - 1]$.

The $A_t$ are continuous in $t$ w.r. to the strong operator topology. Note that

$$A\left(\frac{1}{2}\right) (a_0, a_1, \ldots) = (a_0, 0, a_1 0, \ldots) \in \mathcal{H}_{\text{even}}$$

$$A(1) (a_0, a_1, \ldots) = (0, a_0, 0, a_1 \ldots) \in \mathcal{H}_{\text{odd}}$$

where $\mathcal{H}_{\text{even}}$ and $\mathcal{H}_{\text{odd}}$ denote obvious subspaces of $\mathcal{H}$.

(ii) The topology of the function spaces $\text{Mor}_{U(k)}(P, E)$ $\text{Mor}(M, B)$.

We topologize $\text{Mor}_{U(k)}(P, E)$ as a (closed) subset of $\mathcal{C}^{\infty}(P, \mathcal{L}(\mathbb{C}^k, \mathcal{H}))$, and $\text{Mor}(M, B)$ as a (closed) subset of $\mathcal{C}^{\infty}(M, \mathcal{H})$. The $\mathcal{C}^{\infty}$ topology is described below:

Let $X$ be a compact manifold and $\mathcal{F}$ a Hilbert space. Let $X_1, \ldots, X_q$ be a set of vector fields on $X$ which together span the tangent space at each point of $X$. For a multi index $\alpha = (\alpha_1, \ldots, \alpha_2)$
set $D^\alpha = X_1^{\alpha_1}, \ldots, X_q^{\alpha_q}$. We make $C^\infty (X, \mathcal{F})$ a Frechet space w.r.
to the seminorms $\|f\|_\alpha = \sup_x \|D^\alpha f\|$ where the heavy bars $\|\|$ denote the Hilbert space norm. The topology is clearly independent of the choice of $X_1, \ldots, X_q$. If $N \subset \mathcal{F}$ is a closed submanifold then $C^\infty (X, N)$ is a closed subset of $C^\infty (X, \mathcal{F})$ and we give it the relative topology, which makes it a complete metric space.

We choose now, once and for all, a set of vector fields $X_1, \ldots, X_p$ spanning the tangent space of $M$ at each point. Let $\hat{X}_1, \ldots, \hat{X}_p$ be their lifts to $P$ w.r. to some connection, and let $\hat{Y}_1, \ldots, \hat{Y}_{k^2}$ be vertical vector fields on $P$, the images of a fixed basis $Y_1, \ldots, Y_{k^2}$ in $u(k)$ by the group action. We will use these to determine the seminorms. Note that $[\hat{X}_i, \hat{Y}_g] = 0$ $\forall X_i$ and $Y_g$. We will let $\alpha_L = (\alpha_1, \ldots, \alpha_{k^2})$ and $\alpha = (\alpha_1, \ldots, \alpha_p)$, and write the seminorms as $\|f\|_{\alpha_L, \alpha} = \sup_{x \in P} \|D^{\alpha_L} D^\alpha f\|$.

When there is no need to distinguish between the vertical and horizontal vectors we simply denote $(\alpha_L, \alpha)$ by $\gamma$.

**Lemma 3.1.** - $\text{Mor}_{U(k)}(P, E)$ and $\text{Mor}(M, B)$ are closed sub-
sets of $C^\infty (P, \mathcal{F}(C^k, \mathcal{E}))$ and $C^\infty (M, \mathcal{F})$ respectively. The map $\text{Mor}_{U(k)}(P, E) \rightarrow \text{Mor}(M, B)$ is continuous.

**Proof.** - For $g \in U(k)$ the map $C^\infty (P, E) \rightarrow C^\infty (P, E)$ given by $f \mapsto f^g$, $f^g(x) = f(xg) g^{-1}$ ($x \in P$), is continuous. This follows since

$$\|f_1^g - f_2^g\|_{\alpha_L, \alpha} = \sup_{x \in P} \|D^{\alpha_L} x D^\alpha (f_1 (xg) g^{-1} - f_2 (xg) g^{-1})\|$$

$$= \sup_{x \in P} \|D^{\alpha_L} x D^\alpha (f_1 (xg) - f_2 (xg))\|$$

$$= \sup_{x \in P} \|D^{[\alpha L, g]} x D^\alpha (f_1 (xg) - f_2 (xg))\|$$

$$= \|f_1 - f_2\|_{\alpha_L, \alpha}$$

where $D^{[\alpha L, g]}$ denotes the differential operator

$$D^{[\alpha L, g]} = (g^{-1} \hat{Y}_1 g)^{\alpha_1} \cdots (g^{-1} \hat{Y}_{k^2} g)^{\alpha_{k^2}}.$$

Here $g^{-1} \hat{Y}_i g$ is the image of the Lie algebra element $g^{-1} \hat{Y}_i g$. This proves the first statement. To prove the second statement, let $f_n \rightarrow f$ in $\text{Mor}_{U(k)}(P, E)$ and let $p_n = f_n f_n^+$. Then
\[ \| p_n - p \|_\alpha = \sup_{x \in B} \| D^\alpha (p_n - p) \| \quad \text{(where } D^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n}) \]

\[ = \sup_{x \in \mathcal{P}} \| D^\alpha (p_n - p) \| \quad \text{(where } D^\alpha = \hat{X}_1^{\alpha_1} \ldots \hat{X}_n^{\alpha_n}) \]

\[ = \sup_{x \in \mathcal{P}} \| \sum_{\beta < \alpha} \left( \alpha^\beta \right) (D^{\alpha - \beta} f_n D^\beta f_n + D^{\alpha - \beta} f D^\beta f + ) \| \]

\[ \leq \alpha \| f_n \|_{\beta} \| f_n - f \|_{\alpha - \beta} + \| f \|_{\alpha - \beta} \| f_n - f \|_{\beta}. \]

This proves \( p_n \rightarrow p \) in \( \text{Mor}(\mathcal{M}, \mathcal{B}) \).

4. The topology of the fibres.

We will be interested in the fibres of the map \( \varphi \). Consider first a fibre of \( \varphi \).

**Proposition 4.1.** – Let \( A \in \mathfrak{A} \). Then \( \varphi^{-1}(A) \) is contractible. In other words the space of morphisms \( P \rightarrow E \) which induce a fixed connection on \( P \) is contractible.

**Proof.** – The proof proceeds in two steps.

(i) Define a map

\[ \xi: \varphi^{-1}(A) \times [0, 1/2] \rightarrow \varphi^{-1}(A) \]

by

\[ \xi_t(f)(x) = A_t \circ f(x) \quad x \in \mathcal{P}, \quad t \in [0, 1/2]. \]

The map is into \( \varphi^{-1}(A) \) since,

a) \( \xi_t(f)(xg) = A_t \circ f(xg) (g \in U(k)) = A_t \circ f(x) \circ g = \xi_t(f)(x) \circ gU \)

b) \( \xi_t(f)^+ d\xi_t(f) = f^+ df = A. \)

By lemma 4.2 below \( \xi \) is continuous.

(ii) There exists a \( f_0 \in \varphi^{-1}(A) \) s.t. \( \forall x \in \mathcal{P}, f_0(x) \) maps \( \mathfrak{C}^k \) into \( \mathfrak{H}_{odd} \) [Apply \( A_1 \) to any \( f \in \varphi^{-1}(A) \) to get such an \( f_0 \)]. Define for \( t \in [1/2, 1] \) a map \( \eta: \varphi^{-1}(A) \times [1/2, 1] \rightarrow \varphi^{-1}(A) \) by

\[ \eta_t(f)(x)v = (\sin t\pi) A_{1/2} f(x)v - \cos t\pi f_0(x)v. \]
Again the map is into $\varphi^{-1}(A)$. Note that $A_{1/2}f$ maps into $\mathcal{H}_{\text{even}}$. This means that $\forall (x, t), \eta_t f(x)$ defines an isometry of $C^k$ into $\mathcal{H}$, for, given $v, v' \in C^k$,

$$(\eta_t f(x) v, \eta_t f(x) v') = \sin^2 t\pi (A_{1/2} f(x) v, A_{1/2} f(x) v')$$

$$+ (\cos^2 t\pi) (f_0(x) v, f_0(x) v') = (v, v')$$

where $(,)$ denotes the inner product.

The points a), b) above can be checked easily. Lemma 4.2 gives continuity.

(iii) Compose $\xi$ and $\eta$ to get the contraction

$$\psi : \varphi^{-1}(A) \times [0, 1] \rightarrow \varphi^{-1}(A).$$

(See diagram)

**Lemma 4.2.** — The maps $\xi, \eta$ constructed in the proof of Proposition 4.1 are continuous (in the product topology).

**Proof.** — Consider the map $\xi$. Let $(f_n, t_n)$ be a sequence in $\varphi^{-1}(A) \times [0, 1/2]$. Then

$$\|\xi_{t_n}(f_n) - \xi_t(f)\|_\gamma = \sup_{x \in \mathcal{P}} \|A_{t_n} \circ D^\gamma f_n - A_t \circ D^\gamma f\|$$

$$= \sup_{x \in \mathcal{P}} \|A_{t_n} \circ D^\gamma (f_n - f) + (A_{t_n} - A_t) \circ D^\gamma f\|$$

$$\leq \|f_n - f\|_\gamma + \|(A_{t_n} - A_t)f\|_\gamma.$$

This shows continuity of $\xi$. The continuity of $\eta$ follows similarly.

**Remark.** — The proof of Proposition 4.1 can be extended to prove contractivility of $\text{Mor}_{U(k)}(P, E)$. In particular, taking $P = U(k)$, we see that $E$ itself is contractible.
We turn now to the fibres of the map \( \varphi \). Note that if \( A \in \mathcal{C} \) and \( A \in \mathcal{A} \) is its class, then \( \varphi^{-1}(A) \) projects onto \( \varphi^{-1}(A) \). Also if \( \text{Aut} A \) is the subgroup of \( \text{Aut} \) that leaves \( A \) fixed \( \text{Aut}(A) \) acts freely on \( \varphi^{-1}(A) \), the quotient being in bijection with \( \varphi^{-1}(A) \).

\( \text{Aut} A \) is the space of maps \( \hat{g} : P \rightarrow U(k) \) such that

(i) \( \hat{g}(xh) = h^{-1}g(x)h \quad x \in P, \quad h \in U(k) \)

(ii) \( A = \hat{g}^{-1}A\hat{g} + \hat{g}^{-1}d\hat{g} \)

Since \( \hat{g} \in \text{Aut} A \) is determined by its value at a fixed point in \( P \), we shall, fixing \( y_0 \in P \) (projecting onto \( x_0 \in M \) identify \( \text{Aut} A \equiv \hat{g}(y_0) \in U(k) \).

Thus \( \text{Aut} A \) is a closed subgroup of \( U(k) \) [This is seen either from the equation (ii) above, or noting the fact that under the above identification \( \text{Aut} A \) is the centralizer of the holonomy group at \( y_0 \)] and hence a Lie subgroup.

From now on we assume that the vector fields \( \hat{X}_1 \ldots \hat{X}_p \) have been lifted to \( P \) w.r. to \( A \). Note that then \( \hat{X}_i(\hat{g}) = 0 \) for \( \hat{g} \in \text{Aut} A \).

**Proposition 4.3.** \( \varphi^{-1}(A) \rightarrow \varphi^{-1}(A) \) is a locally trivial principal fibre space with \( \text{Aut} A \) as structure group.

**Proof.** The proof proceeds in four steps.

a) \( \text{Aut}(A) \) acts continuously on \( \varphi^{-1}(A) \). For suppose \( (f_n, \hat{g}_n) \in \varphi^{-1}(A) \times \text{Aut} A \) and \( (f, \hat{g}) \). Then for any \( \alpha_L, \alpha \)

\[
\|f_n \circ \hat{g}_n - f \circ \hat{g}\|_{\alpha_L, \alpha} \leq \|(f_n - f) \circ \hat{g}_n\|_{\alpha_L, \alpha} + \|f \circ (\hat{g}_n - \hat{g})\|_{\alpha_L, \alpha} \\
= \sup_x \|D^\alpha_L(\alpha f (f_n - f)) \hat{g}_n\| + \sup_x \|D^\alpha_L(\alpha f) (\hat{g}_n - \hat{g})\|
\]

(since \( D^\alpha \hat{g} = 0 \))

\[
= \sup_x \left| \sum_{\beta_L < \alpha_L} (\alpha_L) D^\alpha_L D^\beta_L (f_n - f) D^\beta_L \hat{g}_n \right| \\
+ \sup_x \left| \sum_{\beta_L < \alpha_L} (\alpha_L) D^\alpha_L D^\beta_L D^\alpha f D^\beta_L (\hat{g}_n - \hat{g}) \right| \\
\leq \alpha_L \left| \sum_{\beta_L < \alpha_L} \|f_n - f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n\|_{\beta_L} + \|f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n - \hat{g}\|_{\beta_L} \right|.
\]
Now, for any $\hat{Y}_i, \hat{g} \in \text{Aut } A$

$$\hat{Y}_i(\hat{g}) = \lim_{t \to 0} \frac{\hat{g}(x \exp t Y_i) - \hat{g}(x)}{t} = [\hat{g}(x), Y_i].$$

Also, if $\hat{g}_1, \hat{g}_2$ are in $\text{Aut } A$, $d(\text{Tr}(\hat{g}_1 - \hat{g}_2)) = 0$, so that $\|\hat{g}_1(x) - \hat{g}_2(x)\| = \|\hat{g}_1(\gamma_0) - \hat{g}_2(\gamma_0)\|.$

So, we have

$$\|f_n \circ \hat{g}_n - f \circ \hat{g}\|_{\alpha_L, \alpha} \leq \alpha_L! \sum_{\beta_L \leq \alpha_L} \|f_n - f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n\|_{\beta_L}$$

$$+ \|f\|_{\alpha_L - \beta_L, \alpha} C_{\beta_L} \|\hat{g}_n(p_0) - \hat{g}(p_0)\|$$

where $C_{\beta_L}$ is a constant depending on the multiindex $\beta_L$.

b) Denote by $G$ the graph of the equivalence relation defined by $\text{Aut } A$ on $\varphi^{-1}(A)$. Then the map $G \longrightarrow \text{Aut } A$ is continuous. This follows since the map is given by $(f_1, f_2) \longmapsto f_1^*(\gamma_0) f_2(\gamma_0)$ which is clearly continuous.

c) The projection $\varphi^{-1}(A) \longrightarrow \mathcal{L}^{-1}(\hat{A})$ is continuous and closed. Continuity follows from lemma 3.1 and lemma 4.4 shows that it is closed. Thus $\mathcal{L}^{-1}(\hat{A})$ has the quotient topology w.r. to the projection.

d) Thus we have shown that $\varphi^{-1}(A) \longrightarrow \mathcal{L}^{-1}(\hat{A})$ is a principal fibre space. Now note that there is a $\text{Aut } A$-morphism

$$\varphi^{-1}(A) \longrightarrow E$$

$$\varphi^{-1}(A) \longrightarrow E/\text{Aut } A$$

given by $f \longmapsto f(\gamma_0)$. Since $E \longrightarrow E/\text{Aut } A$ is locally trivial, the proof is complete.

**Lemma 4.4.** — The map $\varphi^{-1}(A) \longrightarrow \mathcal{L}^{-1}(\hat{A})$ is closed.

**Proof.** — Let $f_n \in \varphi^{-1}(A)$ s.t. $p_n = f_n f_n^+ \longrightarrow p$ in $\mathcal{L}^{-1}(\hat{A})$.

It is enough to prove that $\{f_n\}$ contains a convergent subsequence. Since $p_n(x_0) \longrightarrow p(x_0)$ and $E$ has compact fibres one
can assume \( f_n(y_0) \rightarrow g_0 \in E \) without loss of generality. Note that the \( f_n \) satisfy
\[
df_n = f_n A + dp_n f_n.
\] (5)
We now prove that the \( f_n \) are Cauchy in the \( C^0 \) norm so that \( \exists \) a \( C^0 \) function \( f \) such that \( f_n \rightarrow f \). Put \( D = f_n - f_m \). Then from (5) we have
\[
d(DD^+) = DD^+ dp_n + dp_n DD^+ + d(p_n - p_m) f_m D^+ + Df_m^+ d(p_n - p_m).
\]
Evaluating on a vector field \( X_t \), taking the trace and then absolute value of both sides we get
\[
|X_t \text{Tr}(DD^+)| \leq |\text{Tr}(DD^+ X_t p_n)| + |\text{Tr}(X_t(p_n) DD^+)|
+ |\text{Tr}(X_t(p_n - p_m) f_m D^+)| + |\text{Tr}(Df_m^+ X_t(p_n - p_m))|
\leq 2 \{ ||D||^2 \ ||X_t p_n|| + ||D|| \ ||X_t(p_n - p_m)||\}
\]
or,
\[
|X_t \ ||D||^2 | \leq 2 \{ ||D||^2 \ ||X_t p_n|| + ||X_t(p_n - p_m)||\}.
\] (6)
Consider now the set \( \{X_t, Y_q\} \) which we collectively denote by \( \{Z_i\} \). They give a map from \( P \times \mathbb{R}^N \) (where \( N = k^2 + p \)) to the tangent bundle \( TP \) which is onto:
\[
(x, (t_1 \ldots t_N)) \mapsto (x, \sum_i t_i Z_i(x)).
\]
Take the obvious metric on the vector bundle \( P \times \mathbb{R}^n \). This induces a splitting of the above map as well as a Riemannian metric on \( P \). Then we have the following obvious result: if \( X \) is a vector field on \( P \) of norm \( \leq 1 \) and we express \( X = \Sigma a_i Z_i \) with respect to the above splitting then \( |a_i| \leq 1 \) \( \forall i \).

Now let \( y \in P \) and let \( \Gamma(y) \) be a minimal geodesic joining \( y_0 \) to \( y \) [such a geodesic exists for \( P \) compact] parametrized with respect to arc-length. Then the length of \( \Gamma(y) \) \( < T \) for some constant \( T \) independent of \( y \). Now let \( X_t \) be the tangent vector field to \( \Gamma \) (which is necessarily of norm one). This gives
\[
||X_t(p_n - p_m)|| = \sum_i ||p_n - p_m||_i \quad \text{where} \quad \||p||_i = \sup_x ||Z_i p||
= \sum_{|\alpha|=1} ||p_n - p_m||_\alpha.
\]
Thus we have, from (6)
\[
|X_t \ ||D||^2 | = 2 \{ a \ ||D||^2 + b \ ||D||\}
\]
with
\[ a = \sum_{|\alpha|=1} \| p \|_\alpha + c , \ c > 0 \]
and
\[ b = \sum_{a} \| p_n - p_m \|_\alpha . \]

Consider the ordinary differential equation
\[ \frac{du^2}{dt^2} = 2(au^2 + bu) \]
\[ u(0) = D(y_0) . \]
The solution is clearly:
\[ u(t) = D(y_0) e^{at} + \frac{(e^{at} - 1)}{a} b . \]

Consider the set \( K = \{ t \geq 0 \mid \| D(t) \| > u(t) \} \). \( K \) is open, and hence a union of disjoint open intervals. Let \( t_0 \) be its least boundary point. Clearly \( D(t_0) = u(t_0) \). From the polygonal approximations to \( \| D(t_0) \|^2 \) and \( u^2(t) \) it is clear that in an interval \( (t_0, t_0 + \varepsilon) \) we have \( \| D(t) \| < u(t) \). Thus \( K = \emptyset \). We have finally,
\[ \| D(y) \| < D(y_0) e^{aT} + \frac{(e^{aT} - 1)}{a} b \]
which clearly shows that \( \{ f_n \} \) are Cauchy in the \( C^0 \) norm.

Let \( f \) be the \( C^0 \) limit. We now turn back to (5) and ‘bootstrap’ the above result to show that \( f \) is \( C^w \) and \( f_n \to f \) in the \( C^w \) topology. Assume, therefore, that \( f \) is \( C^k \) and \( f_n \to f \) in \( C^k \).

For any multi-index \( \gamma (|\gamma| \geq 1) \) define \( \gamma' \) and \( X(\gamma) \) [here \( X(\gamma) \) is one of the vector fields \( Z_i \)] by \( D^\gamma = D^\gamma X(\gamma) \) so that \( D^\gamma \) is of order \( |\gamma| - 1 \). Let \( |\gamma| = k + 1 \). Then
\[ D^\gamma f_n = D^\gamma X(\gamma)(f_n) = D^\gamma (f_n, A(X(\gamma)) + X(\gamma)(p_n)) f_n \]
\[ = \sum_{\delta < \gamma'} \binom{\gamma'}{\delta} [D^{\gamma' - \delta} f_n D^\delta A(X(\gamma)) + D^{\gamma' - \delta} X(\gamma)(p_n) D^\delta f_n] . \]

Then
\[ \| D f_n - \sum_{\delta < \gamma'} \binom{\gamma'}{\delta} [D^{\gamma' - \delta} f D^\delta A(X(\gamma)) + D^{\gamma - 6} X(\gamma)(p) D^\delta f] \| \]
\[ \leq \gamma ! \sum_{\delta < \gamma'} \| f_n - f \|_{\gamma' - \delta} \| A(X(\gamma)) \|_\delta + p_n \|_{\gamma' - \delta, X(\gamma)} \| f_n - f \|_\delta \]
\[ + \| p_n - p \|_{\gamma' - \delta, X(\gamma)} \| f \|_\delta \]
where \( \| f \|_{\gamma' - \delta, X(\gamma)} \equiv \sup_x \| D^{\gamma' - \delta} X(\gamma)f \| . \)
This shows $D^y f_n$ tends uniformly to a $C^0$ function, and hence $f$ is $C^{k+1}$. By induction $f$ is $C^\infty$ and $f_n \to f$ in $C^\infty(P, E)$. The proof also shows $df = fA + pf$.

Since $\text{Mor}_{U(k)}(P, E)$ is closed, $f \in \text{Mor}_{U(k)}(P, E)$ and $p = ff^+$ by continuity of the projection $\text{Mor}_{U(k)}(P, E) \to \text{Mor}_p(M, B)$. (One can now easily show that $f^+ df = A$, thus showing that the fibre $\mathcal{L}^{-1}(A)$ is closed. This is because we have nowhere in the proof used the fact that $p \in \mathcal{L}^{-1}(A)$).

The Theorem stated in the Introduction now follows.

**BIBLIOGRAPHY**


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