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# ON THE SPACE OF MAPS INDUCING ISOMORPHIC CONNECTIONS

by T.R. RAMADAS

## 1. Introduction .

In this paper we prove the following

**THEOREM.** — *Let  $M$  be a smooth compact manifold,  $P$  a principal bundle on  $M$  with the unitary group  $U(k)$  as structure group,  $A$  a smooth connection on  $P$ , and  $\text{Aut } A$  the group of gauge transformations [i.e., automorphisms of  $P$  which act trivially on  $M$ ] which leave  $A$  invariant. Let  $B$  be the Grassmanian of  $k$ -planes in a separable Hilbert space  $\mathcal{H}$ ,  $E$  the Stiefel bundle of orthonormal  $k$  frames in  $\mathcal{H}$ , and  $\omega$  the canonical universal connection on  $E$ . Denote by  $\Sigma(A)$  the space of maps  $p : M \longrightarrow B$  such that the pull-back bundle  $p^*(E)$ , with the connection  $p^*\omega$ , is isomorphic to  $(P, A)$ .*

*Then the space  $\Sigma(A)$ , with the  $C^\infty$  topology, has the homotopy type of  $B_{(\text{Aut } A)}$  where  $B_{(\text{Aut } A)}$  is the base-space of a universal bundle for  $\text{Aut } A$ .*

The connectedness of  $\Sigma(A)$  is shown in [6]. We use some ideas from this paper.

To motivate this result, consider the case when  $P$  is a principal  $G$ -bundle with  $G$  a compact Lie group. Let  $\text{Aut } P$  denote the group of gauge transformations of  $P$ . Denote by  $\mathcal{C}$  the space of  $C^\infty$  connections on  $P$ . The group  $\text{Aut } P$  acts on  $\mathcal{C}$ , though not freely in general. Denote by  $\mathcal{C}/\sim$  the quotient.

By [4] there exists a finite dimensional principal  $G$ -bundle  $E(G, M) \rightarrow B(G, M)$  with connection such that the following diagram commutes, and the map  $\varphi$  is onto:

$$\begin{array}{ccc} \text{Mor}_G(P, E(G, M)) & \xrightarrow{\varphi} & \mathcal{C} \\ \text{Aut } P \downarrow & & \downarrow \\ \text{Mor}_P(M, B(G, M)) & \xrightarrow{\varrho} & \underline{\mathcal{C}} \end{array}$$

Here  $\text{Mor}_G(P, E(G, M))$  is the space of  $G$ -morphisms of  $P$  into  $E$  and  $\text{Mor}_P(M, B(G, M))$  is the component of  $C^\infty(M, B(G, M))$  which induces pull-back bundles isomorphic to  $P$ .  $\varrho$  is the map given by pulling back the universal connection on  $E(G, M)$ .

We wish to investigate the fibres of the map  $\varrho$ . It is possible to do so when we consider instead of  $E(G, M)$  a universal bundle  $E_G$  with connection such that  $E_G$  is contractible. Suppose then, that in the above diagram we replace  $E(G, M)$  by  $E_G$  and  $B(G, M)$  by  $B_G$ . Let  $A \in \mathcal{C}$  and  $\underline{A}$  its class in  $\underline{\mathcal{C}}$ . We argue heuristically:

The spaces  $\mathcal{C}$  and  $\text{Mor}_G(P, E_G)$  are both contractible. This would imply that  $\varphi^{-1}(A)$  is contractible (all the mappings being assumed to be good fibrations). The group  $\text{Aut } A$  acts on  $\varphi^{-1}(A)$  to give  $\varphi^{-1}(\underline{A})$ . If all goes well this implies

a)  $\varphi^{-1}(A) \rightarrow \varphi^{-1}(\underline{A})$  is a universal  $\text{Aut } A$  bundle. The fibre over  $A$  of the map  $\varrho$  has the same homotopy type as  $B_{(\text{Aut } A)}$ .

b) If  $G$  has trivial centre and all connections are generic (i.e.  $\text{Aut } P$  acts freely on  $\mathcal{C}$ )  $\varrho$  has a section.

The quotient space  $\underline{\mathcal{C}}$  is relevant in studies of Yang-Mills theories, at present very popular in Physics. It has been pointed out [1] that the Universal Connection Theorem could possibly provide connections between Yang-Mills theories and so-called  $\sigma$ -models which concern themselves with the space  $\text{Mor}(M, B)$ . Also in the cases when  $\varrho$  has a section, it could give an alternative to "gauge-fixing" which has been shown to be impossible in general [3, 7, 5].

The paper is organized as follows. In § 2 we imbed  $E$  and  $B$  as closed submanifolds of Hilbert spaces. In § 3 we describe a one parameter family of isometries  $A_t: \mathcal{H} \rightarrow \mathcal{H}$ , and also give the

$C^\infty$  topology to be used on the function spaces  $\text{Mor}_{U(k)}(P, E)$  and  $\text{Mor}_P(M, B)$ . In § 4 we prove that  $\varphi^{-1}(A)$  is contractible [Proposition 4.1] using the isometries  $A_t$ . Then we prove [Proposition 4.3] that  $\varphi^{-1}(A) \longrightarrow \varphi^{-1}(\underline{A})$  is a locally trivial principal fibre space with  $\text{Aut } A$  as structure group. This involves, among other things, proving that the above projection is closed [Lemma 4.4], which is done by studying a certain differential equation. The completeness of the  $C^\infty$  topology is crucial, and the imbeddings obtained in § 2 simplify proofs throughout.

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## 2. The bundle of orthonormal $k$ -frames in a Hilbert space.

Fix an integer  $k > 0$ . Let  $\mathcal{H}$  be an infinite dimensional separable Hilbert space over the complex numbers. Denote by  $E$  the space of orthonormal  $k$ -frames in  $\mathcal{H}$ . The group  $U(k)$  acts on  $E$  on the right and the quotient is the Grassmannian  $B$  of  $k$ -dimensional subspaces of  $\mathcal{H}$ . In fact  $E$  is a universal principal bundle for  $U(k)$ . It also carries a natural connection, which is a universal connection for  $U(k)$ .

It will be useful, in the following, to have characterizations of  $E$  and  $B$  as closed submanifolds of Hilbert spaces.

We shall identify a point  $p$  in  $B$  with the orthogonal projector onto the corresponding subspace, denoted by  $H(p)$ . Thus  $H(p) = \{x \in \mathcal{H} \mid px = x\}$ . For  $p_0 \in B$ , define

$$\mathcal{Q}_0 = \{p \in B \mid H(p_0) \cap \ker p = \{0\}\}.$$

Then we have a bijection  $L_0: \mathcal{Q}_0 \longrightarrow \mathcal{L}(H(p_0), \ker p_0)$  such that for  $p \in \mathcal{Q}_0$  its image  $L \equiv L_0(p)$  has  $H(p)$  as graph.

LEMMA 2.1 [2]. — *The charts  $\{(\mathcal{Q}_0, L_0)\}$  give  $B$  the structure of a  $C^\infty$  Hilbert manifold.*

Let  $\mathcal{J}_2$  denote the Hilbert space of Hilbert-Schmidt operators on  $\mathcal{H}$ .

PROPOSITION 2.2. — Let  $\psi$  denote the injection  $B \longrightarrow \mathcal{J}_2$  given by associating to each  $k$ -dimensional subspace its orthogonal projector. Then  $\psi$  is a  $C^\infty$  immersion, and a homeomorphism onto its image.

*Proof.* — Follows from Lemmas 2.3 and 2.4.

*Remark.* — This shows that  $B$ , with the manifold structure given in Lemma 2.1 is a submanifold of  $\mathcal{J}_2$ .

LEMMA 2.3. — On a chart  $(\mathcal{D}_0, L_0)$   $\psi$  is given by (1 - 3). It is a  $C^\infty$  immersion.

*Proof.* — Let  $L \in \mathcal{E}(H(p_0), \ker p_0)$  and let  $p = \psi L_0^{-1}(L)$ . Write

$$p = A + LA \quad (1)$$

where  $A: \mathcal{H} \longrightarrow H(p_0)$ . Then we claim that  $A$  satisfies

$$A = p_0 + L^+(1 - p_0) - L^+LA \quad (2)$$

which can be solved to give

$$A = \frac{1}{1 + L^+L} (p_0 + L^+(1 - p_0)). \quad (3)$$

To see that  $p$  given by (2.1)-(2.3) is indeed equal to  $\psi L_0^{-1}(L)$ , we verify:

a) Image of  $p = \{x + Lx \mid x \in H(p_0)\}$ . The map is clearly into this set. In fact it is onto since  $A$  is invertible on  $H(p_0)$ .

b)  $p^2 = p$ . This follows since  $Ap = p$ , which in turn is clear because  $Ap$  satisfies the same equation as  $p$ .

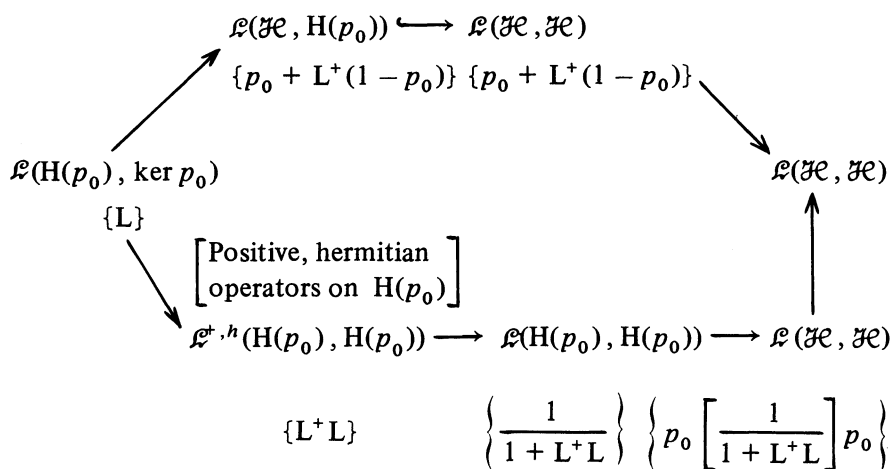
$$\begin{aligned} Ap &= p_0 p + L^+(1 - p_0) p - L^+LAp = A + L^+LA - L^+LAp \\ &= p_0 + L^+(1 - p_0) - L^+LAp. \end{aligned}$$

c)  $p$  is an orthogonal projector, for

$$\ker p = \{y - L^+y \mid y \in \ker p_0\}$$

which is the orthogonal subspace to  $\text{Im } p$ .

(i)  $\psi$  is  $C^\infty$ : To see this split  $\psi$  into the steps:



$\psi$  is in fact real-analytic.

(ii) It is enough to check the differential at  $L = 0$ . Here  $\delta p = \delta L^+(1 - p_0) + p_0 \delta L$  which is clearly injective. Also the image, being defined by  $p_0 \delta p p_0 = (1 - p_0) \delta p (1 - p_0) = 0$  and  $\delta p^+ = \delta p$ , is closed, and hence admits a supplement.

LEMMA 2.4. — *The inverse  $\psi^{-1}$  is given by (4) and is continuous.*

*Proof.* — Consider a chart  $(\mathcal{Q}_0, L_0)$ . Let  $p \in \mathcal{Q}_0$  and let  $Q = (p_0|_{H(p)})^{-1}$ . Then for  $x \in H(p)$ ,  $Qx = x + (1 - p_0)pQx$ . This gives, for  $L = (1 - p_0)Q$ ,  $L = (1 - p_0)p(1 + L)$ .

This can be solved to give  $p \xrightarrow[\psi^{-1}]{\quad} L$  such that

$$Lx = (1 - p_0) \frac{1}{1 - (1 - p_0)p} x, x \in H(p_0). \quad (4)$$

The continuity of  $\psi^{-1}$  follows easily.

We turn now to  $E$ . This can be identified with a closed subset of  $\mathcal{L}(\mathbf{C}^k, \mathcal{H})$ :  $E = \{U : \mathbf{C}^k \longrightarrow \mathcal{H} \mid U^+U = 1\}$ . Standard arguments show:

LEMMA 2.5. —  *$E$  is a closed submanifold of  $\mathcal{L}(\mathbf{C}^k, \mathcal{H})$ . It is a principal  $U(k)$  bundle on  $B$ . The  $u(k)$ -valued one-form  $U^+dU$  is a connection on  $E$ .*

LEMMA 2.6. —  $E$  is contractible and hence a universal  $U(k)$  bundle. The connection is a universal  $U(k)$  connection.

*Proof.* — Both statements are well-known. The first follows also from the remarks after Lemma 4.2. The second is a consequence of the Universal Connection Theorem.

### 3. Some preliminary remarks and definitions.

(i) A one-parameter-family of isometries on  $\mathcal{H}$ .

Following [6], we introduce, on  $\mathcal{H}$ , a one-parameter family of isometries which we will use later. Define, for  $t \in [0, 1]$  an isometry  $A_t: \mathcal{H} \rightarrow \mathcal{H}$  as follows. Fix an orthonormal basis, so that  $\mathcal{H} \approx \{\text{square-summable sequences in } \mathbb{C}\}$ . Then let  $A_0 = \text{Identity}$

$$A_t(a_0, a_1, a_2, \dots) = (a_0, a_1, \dots, a_{n-2}, a_{n-1} \cos \theta_n(t), a_{n-1} \sin \theta_n(t), \\ a_n \cos \theta_n(t), a_n \sin \theta_n(t), a_{n+1} \cos \theta_n(t), a_{n+1} \sin \theta_n(t), \dots)$$

$$\text{for } \frac{1}{n+1} \leq t \leq \frac{1}{n} \text{ where } \theta_n(t) = \frac{\pi}{2} n[(n+1)t - 1].$$

The  $A_t$  are continuous in  $t$  w.r. to the strong operator topology. Note that

$$A\left(\frac{1}{2}\right)(a_0, a_1, \dots) = (a_0, 0, a_1, 0, \dots) \in \mathcal{H}_{\text{even}}$$

$$A(1)(a_0, a_1, \dots) = (0, a_0, 0, a_1, \dots) \in \mathcal{H}_{\text{odd}}$$

where  $\mathcal{H}_{\text{even}}$  and  $\mathcal{H}_{\text{odd}}$  denote obvious subspaces of  $\mathcal{H}$ .

(ii) The topology of the function spaces  $\text{Mor}_{U(k)}(P, E)$   $\text{Mor}(M, B)$ .

We topologize  $\text{Mor}_{U(k)}(P, E)$  as a (closed) subset of

$$C^\infty(P, \mathcal{L}(\mathbb{C}^k, \mathcal{H})),$$

and  $\text{Mor}(M, B)$  as a (closed) subset of  $C^\infty(M, \mathcal{H}_2)$ . The  $C^\infty$  topology is described below:

Let  $X$  be a compact manifold and  $\mathcal{H}$  a Hilbert space. Let  $X_1, \dots, X_q$  be a set of vector fields on  $X$  which together span the tangent space at each point of  $X$ . For a multi index  $\alpha = (\alpha_1, \dots, \alpha_q)$

set  $D^\alpha = X_1^{\alpha_1}, \dots, X_q^{\alpha_q}$ . We make  $C^\infty(X, \mathcal{F})$  a Frechet space w.r. to the seminorms  $\|f\|_\alpha = \sup_x \|D^\alpha f\|$  where the heavy bars  $\|\cdot\|$  denote the Hilbert space norm. The topology is clearly independent of the choice of  $X_1, \dots, X_q$ . If  $N \subset \mathcal{F}$  is a closed submanifold then  $C^\infty(X, N)$  is a closed subset of  $C^\infty(X, \mathcal{F})$  and we give it the relative topology, which makes it a complete metric space.

We choose now, once and for all, a set of vector fields  $X_1, \dots, X_p$  spanning the tangent space of  $M$  at each point. Let  $\hat{X}_1, \dots, \hat{X}_p$  be their lifts to  $P$  w.r. to some connection, and let  $\hat{Y}_1, \dots, \hat{Y}_{k^2}$  be vertical vector fields on  $P$ , the images of a fixed basis  $Y_1, \dots, Y_{k^2}$  in  $u(k)$  by the group action. We will use these to determine the seminorms. Note that  $[\hat{X}_i, \hat{Y}_\ell] = 0 \quad \forall X_i \text{ and } Y_\ell$ . We will let  $\alpha_L = (\alpha_1, \dots, \alpha_{k^2})$  and  $\alpha = (\alpha_1, \dots, \alpha_p)$ , and write the seminorms as  $\|f\|_{\alpha_L, \alpha} = \sup_{x \in P} \|D^{\alpha_L} D^{\alpha} f\|$ .

When there is no need to distinguish between the vertical and horizontal vectors we simply denote  $(\alpha_L, \alpha)$  by  $\gamma$ .

LEMMA 3.1. —  $\text{Mor}_{U(k)}(P, E)$  and  $\text{Mor}(M, B)$  are closed subsets of  $C^\infty(P, \mathcal{L}(\mathbb{C}^k, \mathcal{H}))$  and  $C^\infty(M, \mathcal{B})$  respectively. The map  $\text{Mor}_{U(k)}(P, E) \longrightarrow \text{Mor}(M, B)$  is continuous.

*Proof.* — For  $g \in U(k)$  the map  $C^\infty(P, E) \longrightarrow C^\infty(P, E)$  given by  $f \longmapsto f^g$ ,  $f^g(x) \equiv f(xg)g^{-1}$  ( $x \in P$ ), is continuous. This follows since

$$\begin{aligned} \|f_1^g - f_2^g\|_{\alpha_L, \alpha} &= \sup_{x \in P} \|D_x^{\alpha_L} D_x^{\alpha} (f_1(xg)g^{-1} - f_2(xg)g^{-1})\| \\ &= \sup_{x \in P} \|D_x^{\alpha_L} D_x^{\alpha} (f_1(xg) - f_2(xg))\| \\ &= \sup_{xg \in P} \|D_{xg}^{[\alpha_L, g]} D_{xg}^{\alpha} (f_1(xg) - f_2(xg))\| \\ &= \|f_1 - f_2\|_{[\alpha_L, g], \alpha} \end{aligned}$$

where  $D^{[\alpha_L, g]}$  denotes the differential operator

$$D^{[\alpha_L, g]} = (g^{-1} \hat{Y}_1 g)^{\alpha_1} \dots (g^{-1} \hat{Y}_{k^2} g)^{\alpha_{k^2}}.$$

Here  $g^{-1} \hat{Y}_i g$  is the image of the Lie algebra element  $g^{-1} \hat{Y}_i g$ . This proves the first statement. To prove the second statement, let  $f_n \longrightarrow f$  in  $\text{Mor}_{U(k)}(P, E)$  and let  $p_n = f_n f_n^+$ . Then



$$\begin{aligned}
\|p_n - p\|_\alpha &= \sup_{x \in B} \|D^\alpha(p_n - p)\| \quad (\text{where } D^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}) \\
&= \sup_{x \in P} \|D^\alpha(p_n - p)\| \quad (\text{where } D^\alpha = \hat{X}_1^{\alpha_1} \dots \hat{X}_n^{\alpha_n}) \\
&= \sup_{x \in P} \left\| \sum_{\beta < \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} f_n D^\beta f_n^+ - D^{\alpha-\beta} f D^\beta f^+) \right\| \\
&\leq \alpha! \sum_{\beta < \alpha} \|f_n\|_\beta \|f_n - f\|_{\alpha-\beta} + \|f\|_{\alpha-\beta} \|f_n - f\|_\beta.
\end{aligned}$$

This proves  $p_n \longrightarrow p$  in  $\text{Mor}(M, B)$ .

#### 4. The topology of the fibres.

We will be interested in the fibres of the map  $\varphi$ . Consider first a fibre of  $\varphi$ .

**PROPOSITION 4.1.** — *Let  $A \in \mathcal{C}$ . Then  $\varphi^{-1}(A)$  is contractible. In other words the space of morphisms  $P \longrightarrow E$  which induce a fixed connection on  $P$  is contractible.*

*Proof.* — The proof proceeds in two steps.

(i) Define a map

$$\xi: \varphi^{-1}(A) \times [0, 1/2] \longrightarrow \varphi^{-1}(A)$$

by

$$\xi_t(f)(x) = A_t \circ f(x) \quad \begin{cases} f \in \varphi^{-1}(A) \\ x \in P \\ t \in [0, 1/2]. \end{cases}$$

The map is into  $\varphi^{-1}(A)$  since,

$$\begin{aligned}
\text{a) } \xi_t(f)(xg) &= A_t \circ f(xg) \quad (g \in U(k)) = A_t \circ f(x) \circ g \\
&= \xi_t(f)(x) \circ gU
\end{aligned}$$

$$\text{b) } \xi_t(f)^+ d\xi_t(f) = f^+ df = A.$$

By lemma 4.2 below  $\xi$  is continuous.

(ii) There exists a  $f_0 \in \varphi^{-1}(A)$  s.t.  $\forall x \in P, f_0(x)$  maps  $\mathbf{C}^k$  into  $\mathcal{H}_{\text{odd}}$  [Apply  $A_1$  to any  $f \in \varphi^{-1}(A)$  to get such an  $f_0$ ]. Define for  $t \in [1/2, 1]$  a map  $\eta: \varphi^{-1}(A) \times [1/2, 1] \longrightarrow \varphi^{-1}(A)$  by

$$\eta_t(f)(x)v = (\sin t\pi) A_{1/2} f(x)v - \cos t\pi f_0(x)v.$$

Again the map is into  $\varphi^{-1}(A)$ . Note that  $A_{1/2}f$  maps into  $\mathcal{H}_{\text{even}}$ . This means that  $\forall (x, t)$ ,  $\eta_t f(x)$  defines an isometry of  $\mathbb{C}^k$  into  $\mathcal{H}$ , for, given  $v, v' \in \mathbb{C}^k$ ,

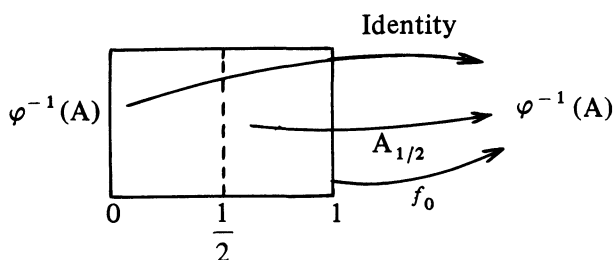
$$\begin{aligned} (\eta_t f(x) v, \eta_t f(x) v') &= \sin^2 t\pi (A_{1/2} f(x) v, A_{1/2} f(x) v') \\ &\quad + (\cos^2 t\pi) (f_0(x) v, f_0(x) v') = (v, v') \end{aligned}$$

where  $(,)$  denotes the inner product.

The points a), b) above can be checked easily. Lemma 4.2 gives continuity.

(iii) Compose  $\xi$  and  $\eta$  to get the contraction

$$\psi : \varphi^{-1}(A) \times [0, 1] \longrightarrow \varphi^{-1}(A). \quad (\text{See diagram})$$



LEMMA 4.2. — *The maps  $\xi, \eta$  constructed in the proof of Proposition 4.1 are continuous (in the product topology).*

*Proof.* — Consider the map  $\xi$ . Let  $(f_n, t_n)$  be a sequence in  $\varphi^{-1}(A) \times [0, 1/2]$ . Then

$$\begin{aligned} \|\xi_{t_n}(f_n) - \xi_t(f)\|_\gamma &= \sup_{x \in P} \|A_{t_n} \circ D^\gamma f_n - A_t \circ D^\gamma f\| \\ &= \sup_{x \in P} \|A_{t_n} \circ D^\gamma (f_n - f) + (A_{t_n} - A_t) \circ D^\gamma f\| \\ &\leq \|f_n - f\|_\gamma + \|(A_{t_n} - A_t)f\|_\gamma. \end{aligned}$$

This shows continuity of  $\xi$ . The continuity of  $\eta$  follows similarly.

*Remark.* — The proof of Proposition 4.1 can be extended to prove contractivity of  $\text{Mor}_{U(k)}(P, E)$ . In particular, taking  $P = U(k)$ , we see that  $E$  itself is contractible.

We turn now to the fibres of the map  $\varphi$ . Note that if  $A \in \mathcal{C}$  and  $\underline{A} \in \underline{\mathcal{C}}$  is its class, then  $\varphi^{-1}(A)$  projects onto  $\varphi^{-1}(\underline{A})$ . Also if  $\text{Aut } A$  is the subgroup of  $\text{Aut}$  that leaves  $A$  fixed  $\text{Aut}(A)$  acts freely on  $\varphi^{-1}(A)$ , the quotient being in bijection with  $\varphi^{-1}(\underline{A})$ .

$\text{Aut } A$  is the space of maps  $\hat{g} : P \longrightarrow U(k)$  such that

$$(i) \hat{g}(xh) = h^{-1} \hat{g}(x) h \quad x \in P, h \in U(k)$$

$$(ii) A = \hat{g}^{-1} A \hat{g} + \hat{g}^{-1} d\hat{g}.$$

Since  $\hat{g} \in \text{Aut } A$  is determined by its value at a fixed point in  $P$ , we shall, fixing  $y_0 \in P$  (projecting onto  $x_0 \in M$ ) identify  $\text{Aut } A \ni \hat{g} \sim \hat{g}(y_0) \in U(k)$ .

Thus  $\text{Aut } A$  is a closed subgroup of  $U(k)$  [This is seen either from the equation (ii) above, or noting the fact that under the above identification  $\text{Aut } A$  is the centralizer of the holonomy group at  $y_0$ ] and hence a Lie subgroup.

From now on we assume that the vector fields  $\hat{X}_1 \dots \hat{X}_p$  have been lifted to  $P$  w.r. to  $A$ . Note that then  $\hat{X}_i(\hat{g}) = 0$  for  $\hat{g} \in \text{Aut } A$ .

**PROPOSITION 4.3.** —  $\varphi^{-1}(A) \longrightarrow \varphi^{-1}(\underline{A})$  is a locally trivial principal fibre space with  $\text{Aut } A$  as structure group.

*Proof.* — The proof proceeds in four steps.

a)  $\text{Aut}(A)$  acts continuously on  $\varphi^{-1}(A)$ . For suppose  $(f_n, \hat{g}_n) \in \varphi^{-1}(A) \times \text{Aut } A$  and  $(f_n, \hat{g}_n) \longrightarrow (f, \hat{g})$ . Then for any  $\alpha_L, \alpha$

$$\begin{aligned} \|f_n \circ \hat{g}_n - f \circ \hat{g}\|_{\alpha_L, \alpha} &\leq \|(f_n - f) \circ \hat{g}_n\|_{\alpha_L, \alpha} + \|f \circ (\hat{g}_n - \hat{g})\|_{\alpha_L, \alpha} \\ &= \sup_x \|D^{\alpha_L}([D^\alpha(f_n - f)] \hat{g}_n)\| + \sup_x \|D^{\alpha_L}([D^\alpha f](\hat{g}_n - \hat{g}))\| \\ &\quad \text{(since } D^\alpha \hat{g} = 0) \\ &= \sup_x \left\| \sum_{\beta_L \leq \alpha_L} \binom{\alpha_L}{\beta_L} D^{\alpha_L - \beta_L} D^\alpha(f_n - f) D^{\beta_L} \hat{g}_n \right\| \\ &\quad + \sup_x \left\| \sum_{\beta_L \leq \alpha_L} \binom{\alpha_L}{\beta_L} D^{\alpha_L - \beta_L} D^\alpha f D^{\beta_L} (\hat{g}_n - \hat{g}) \right\| \\ &\leq \alpha_L! \left\| \sum_{\beta_L \leq \alpha_L} \|f_n - f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n\|_{\beta_L} + \|f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n - \hat{g}\|_{\beta_L} \right\|. \end{aligned}$$

Now, for any  $\hat{Y}_i, \hat{g} \in \text{Aut } A$

$$\hat{Y}_i(\hat{g}) = \lim_{t \rightarrow 0} \frac{\hat{g}(x \exp t Y_i) - \hat{g}(x)}{t} = [\hat{g}(x), Y_i].$$

Also, if  $\hat{g}_1, \hat{g}_2$  are in  $\text{Aut } A$ ,  $d(\text{Tr}(\hat{g}_1 - \hat{g}_2)^+ (\hat{g}_1 - \hat{g}_2)) = 0$ , so that  $\|\hat{g}_1(x) - \hat{g}_2(x)\| = \|\hat{g}_1(y_0) - \hat{g}_2(y_0)\|$ .

So, we have

$$\begin{aligned} \|f_n \circ \hat{g}_n - f \circ \hat{g}\|_{\alpha_L, \alpha} &\leq \alpha_L! \sum_{\beta_L \leq \alpha_L} \|f_n - f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n\|_{\beta_L} \\ &+ \|f\|_{\alpha_L - \beta_L, \alpha} C_{\beta_L} \|\hat{g}_n(p_0) - \hat{g}(p_0)\| \end{aligned}$$

where  $C_{\beta_L}$  is a constant depending on the multiindex  $\beta_L$ .

b) Denote by  $\mathbf{G}$  the graph of the equivalence relation defined by  $\text{Aut } A$  on  $\varphi^{-1}(A)$ . Then the map  $\mathbf{G} \rightarrow \text{Aut } A$  is continuous. This follows since the map is given by  $(f_1, f_2) \mapsto f_1^+(y_0) f_2(y_0)$  which is clearly continuous.

c) The projection  $\varphi^{-1}(A) \rightarrow \varphi^{-1}(\underline{A})$  is continuous and closed. Continuity follows from lemma 3.1 and lemma 4.4 shows that it is closed. Thus  $\varphi^{-1}(\underline{A})$  has the quotient topology w.r. to the projection.

d) Thus we have shown that  $\varphi^{-1}(A) \rightarrow \varphi^{-1}(\underline{A})$  is a principal fibre space. Now note that there is a  $\text{Aut } A$ -morphism

$$\begin{array}{ccc} \varphi^{-1}(A) & \longrightarrow & E \\ \downarrow & & \downarrow \\ \varphi^{-1}(\underline{A}) & \longrightarrow & E/\text{Aut } A \end{array}$$

given by  $f \mapsto f(y_0)$ . Since  $E \rightarrow E/\text{Aut } A$  is locally trivial, the proof is complete.

LEMMA 4.4. — *The map  $\varphi^{-1}(A) \rightarrow \varphi^{-1}(\underline{A})$  is closed.*

*Proof.* — Let  $f_n \in \varphi^{-1}(A)$  s.t.  $p_n = f_n f_n^+ \rightarrow p$  in  $\varphi^{-1}(\underline{A})$ .

It is enough to prove that  $\{f_n\}$  contains a convergent subsequence. Since  $p_n(x_0) \rightarrow p(x_0)$  and  $E$  has compact fibres one

can assume  $f_n(y_0) \rightarrow g_0 \in E$  without loss of generality. Note that the  $f_n$  satisfy

$$df_n = f_n A + dp_n f_n. \quad (5)$$

We now prove that the  $f_n$  are Cauchy in the  $C^0$  norm so that  $\exists$  a  $C^0$  function  $f$  such that  $f_n \rightarrow f$ . Put  $D = f_n - f_m$ . Then from (5) we have

$$d(DD^+) = DD^+ dp_n + dp_n DD^+ + d(p_n - p_m) f_m D^+ + D f_m^+ d(p_n - p_m).$$

Evaluating on a vector field  $X_t$ , taking the trace and then absolute value of both sides we get

$$\begin{aligned} |X_t \text{Tr}(DD^+)| &\leq |\text{Tr}(DD^+ X_t p_n)| + |\text{Tr}(X_t (p_n) DD^+)| \\ &\quad + |\text{Tr}(X_t (p_n - p_m) f_m D^+)| + |\text{Tr}(D f_m^+ X_t (p_n - p_m))| \\ &\leq 2 \{ \|D\|^2 \|X_t p_n\| + \|D\| \|X_t (p_n - p_m)\| \} \end{aligned}$$

or,

$$|X_t \|D\|^2| \leq 2 \{ \|D\|^2 \|X_t p_n\| + \|X_t (p_n - p_m)\| \}. \quad (6)$$

Consider now the set  $\{X_i, Y_q\}$  which we collectively denote by  $\{Z_j\}$ . They give a map from  $P \times \mathbb{R}^N$  (where  $N = k^2 + p$ ) to the tangent bundle  $TP$  which is onto:

$$(x, (t_1 \dots t_N)) \mapsto (x, \sum_i t_i Z_i(x)).$$

Take the obvious metric on the vector bundle  $P \times \mathbb{R}^N$ . This induces a splitting of the above map as well as a Riemannian metric on  $P$ . Then we have the following obvious result: if  $X$  is a vector field on  $P$  of norm  $\leq 1$  and we express  $X = \sum a_i Z_i$  with respect to the above splitting then  $|a_i| \leq 1 \forall i$ .

Now let  $y \in P$  and let  $\Gamma(y)$  be a minimal geodesic joining  $y_0$  to  $y$  [such a geodesic exists for  $P$  compact] parametrized with respect to arc-length. Then the length of  $\Gamma(y) < T$  for some constant  $T$  independent of  $y$ . Now let  $X_t$  be the tangent vector field to  $\Gamma$  (which is necessarily of norm one). This gives

$$\begin{aligned} \|X_t (p_n - p_m)\| &= \sum_i \|p_n - p_m\|_i \text{ where } \|p\|_i = \sup_x \|Z_i p\| \\ &= \sum_{|\alpha|=1} \|p_n - p_m\|_\alpha. \end{aligned}$$

Thus we have, from (6)

$$|X_t \|D\|^2| = 2 \{ a \|D\|^2 + b \|D\| \}$$

$$\text{with} \quad a = \sum_{|\alpha|=1} \|p\|_{\alpha} + c, \quad c > 0$$

$$\text{and} \quad b = \sum_{\alpha} \|p_n - p_m\|_{\alpha}.$$

Consider the ordinary differential equation

$$\frac{du^2}{dt^2} = 2(au^2 + bu)$$

$$u(0) = D(y_0).$$

The solution is clearly:

$$u(t) = D(y_0) e^{at} + \frac{(e^{at} - 1)}{a} b.$$

Consider the set  $K = \{t \geq 0 \mid \|D(t)\| > u(t)\}$ .  $K$  is open, and hence a union of disjoint open intervals. Let  $t_0$  be its least boundary point. Clearly  $D(t_0) = u(t_0)$ . From the polygonal approximations to  $\|D(t_0)\|^2$  and  $u^2(t)$  it is clear that in an interval  $(t_0, t_0 + \epsilon)$  we have  $\|D(t)\| \leq u(t)$ . Thus  $K = \emptyset$ . We have finally,

$$\|D(y)\| \leq D(y_0) e^{aT} + \frac{(e^{aT} - 1)}{a} b$$

which clearly shows that  $\{f_n\}$  are Cauchy in the  $C^0$  norm.

Let  $f$  be the  $C^0$  limit. We now turn back to (5) and 'bootstrap' the above result to show that  $f$  is  $C^\infty$  and  $f_n \rightarrow f$  in the  $C^\infty$  topology. Assume, therefore, that  $f$  is  $C^k$  and  $f_n \rightarrow f$  in  $C^k$ . For any multi-index  $\gamma$  ( $|\gamma| \geq 1$ ) define  $\gamma'$  and  $X^{(\gamma)}$  [here  $X^{(\gamma)}$  is one of the vector fields  $Z_i$ ] by  $D^\gamma = D^{\gamma'} X^{(\gamma)}$  so that  $D^{\gamma'}$  is of order  $|\gamma| - 1$ . Let  $|\gamma| = k + 1$ . Then

$$\begin{aligned} D^\gamma f_n &= D^{\gamma'} X^{(\gamma)}(f_n) = D^{\gamma'}(f_n A(X^{(\gamma)}) + X^{(\gamma)}(p_n) f_n) \\ &= \sum_{\delta \leq \gamma'} \binom{\gamma'}{\delta} [D^{\gamma'-\delta} f_n D^\delta A(X^{(\gamma)}) + D^{\gamma'-\delta} X^{(\gamma)}(p_n) D^\delta f_n]. \end{aligned}$$

Then

$$\begin{aligned} \|Df_n - \sum_{\delta \leq \gamma'} \binom{\gamma'}{\delta} [D^{\gamma'-\delta} f D^\delta A(X^{(\gamma)}) + D^{\gamma'-\delta} X^{(\gamma)}(p) D^\delta f]\| \\ \leq \gamma! \sum_{\delta \leq \gamma'} \|f_n - f\|_{\gamma-\delta} \|A(X^{(\gamma)})\|_\delta + \|p_n\|_{\gamma'-\delta, X^{(\gamma)}} \|f_n - f\|_\delta \\ + \|p_n - p\|_{\gamma'-\delta, X^{(\gamma)}} \|f\|_\delta \end{aligned}$$

where  $\|f\|_{\gamma'-\delta, X^{(\gamma)}} \equiv \sup_x \|D^{\gamma'-\delta} X^{(\gamma)} f\|$ .

This shows  $D^\gamma f_n$  tends uniformly to a  $C^0$  function, and hence  $f$  is  $C^{k+1}$ . By induction  $f$  is  $C^\infty$  and  $f_n \rightarrow f$  in  $C^\infty(P, E)$ . The proof also shows  $df = fA + pf$ .

Since  $\text{Mor}_{U(k)}(P, E)$  is closed,  $f \in \text{Mor}_{U(k)}(P, E)$  and  $p = ff^+$  by continuity of the projection  $\text{Mor}_{U(k)}(P, E) \rightarrow \text{Mor}_p(M, B)$ . (One can now easily show that  $f^+df = A$ , thus showing that the fibre  $\varphi^{-1}(A)$  is closed. This is because we have nowhere in the proof used the fact that  $p \in \varphi^{-1}(A)$ ).

The Theorem stated in the Introduction now follows.

### BIBLIOGRAPHY

- [1] M. DUBOIS-VIOLETTE and Y. GEORGELIN, Gauge Theory in terms of projector valued fields, *Physics Letters*, 82B, 251 (1979).
- [2] A. DOUADY, Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, séminaire, Collège de France (1964-65).
- [3] V.N. GRIBOV, Quantization of nonabelian gauge theories, *Nuclear Physics*, B139 (1978), 1.
- [4] M.S. NARASIMHAN and S. RAMANAN, Existence of universal connections, *Amer. J. Math.*, 83 (1961), 573-572.
- [5] M.S. NARASIMHAN and T.R. RAMADAS, Geometry of  $SU(2)$  gauge-fields, *Commun. Math. Phys.*, 67 (1979), 121-136.
- [6] R. SCHLAFLY, Universal Connections, *Inventiones Math.*, 59 (1980), 59-65.
- [7] I.M. SINGER, Some remarks on the Gribov ambiguity, *Commun. Math. Phys.*, 60 (1978), 7-12.

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