ANNALES DE L'INSTITUT FOURIER

T. R. RAMADAS

# On the space of maps inducing isomorphic connections

*Annales de l'institut Fourier*, tome 32, nº 1 (1982), p. 263-276

<http://www.numdam.org/item?id=AIF_1982__32_1_263_0>

Ann. Inst. Fourier, Grenoble
32, 1 (1982), 263-276

# ON THE SPACE OF MAPS INDUCING ISOMORPHIC CONNECTIONS

by T.R. RAMADAS

## 1. Introduction.

In this paper we prove the following

THEOREM. – *Let* M *be a smooth compact manifold,* P *a principal bundle on* M *with the unitary group* $U(k)$ *as structure group,* A *a smooth connection on* P, *and* Aut A *the group of gauge transformations* [i.e., *automorphisms of* P *which act trivially on* M] *which leave* A *invariant. Let* B *be the Grassmanian of k-planes in a separable Hilbert space* $\mathcal{H}$, E *the Stiefel bundle of orthonormal k frames in* $\mathcal{H}$, *and* $\omega$ *the canonical universal connection on* E. *Denote by* $\Sigma(A)$ *the space of maps* $p : M \longrightarrow B$ *such that the pull-back bundle* $p^*(E)$, *with the connection* $p^*\omega$, *is isomorphic to* (P, A).

*Then the space* $\Sigma(A)$, *with the* $C^\infty$ *topology, has the homotopy type of* $B_{(\mathrm{Aut}\,A)}$ *where* $B_{(\mathrm{Aut}\,A)}$ *is the base-space of a universal bundle for* Aut A.

The connectedness of $\Sigma(A)$ is shown in [6]. We use some ideas from this paper.

To motivate this result, consider the case when P is a principal G-bundle with G a compact Lie group. Let Aut P denote the group of gauge transformations of P. Denote by $\mathcal{C}$ the space of $C^\infty$ connections on P. The group Aut P acts on $\mathcal{C}$, though not freely in general. Denote by $\underset{\sim}{\mathcal{C}}$ the quotient.

By [4] there exists a finite dimensional principal G-bundle $E(G, M) \longrightarrow B(G, M)$ with connection such that the following diagram commutes, and the map $\varphi$ is onto:

$$\begin{array}{ccc} \mathrm{Mor}_G(P, E(G,M)) & \xrightarrow{\ \varphi\ } & \mathcal{C} \\ {\scriptstyle \mathrm{Aut}\,P}\downarrow & & \downarrow \\ \mathrm{Mor}_P(M, B(G,M)) & \xrightarrow[\ \underset{\sim}{\varphi}\ ]{} & \underset{\sim}{\mathcal{C}} \end{array}$$

Here $\mathrm{Mor}_G(P, E(G,M))$ is the space of G-morphisms of P into E and $\mathrm{Mor}_P(M, B(G,M))$ is the component of $C^\infty(M, B(G,M))$ which induces pull-back bundles isomorphic to P. $\underset{\sim}{\varphi}$ is the map given by pulling back the universal connection on $E(G,M)$.

We wish to investigate the fibres of the map $\underset{\sim}{\varphi}$. It is possible to do so when we consider instead of $E(G,M)$ a universal bundle $E_G$ with connection such that $E_G$ is contractible. Suppose then, that in the above diagram we replace $E(G,M)$ by $E_G$ and $B(G,M)$ by $B_G$. Let $A \in \mathcal{C}$ and $\underset{\sim}{A}$ its class in $\underset{\sim}{\mathcal{C}}$. We argue heuristically:

The spaces $\mathcal{C}$ and $\mathrm{Mor}_G(P, E_G)$ are both contractible. This would imply that $\varphi^{-1}(A)$ is contractible (all the mappings being assumed to be good fibrations). The group Aut A acts on $\varphi^{-1}(A)$ to give $\underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$. If all goes well this implies

a) $\varphi^{-1}(A) \longrightarrow \underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$ is a universal Aut A bundle. The fibre over A of the map $\underset{\sim}{\varphi}$ has the same homotopy type as $B_{(\mathrm{Aut}\,A)}$.

b) If G has trivial centre and all connections are generic (i.e. Aut P acts freely on $\mathcal{C}$) $\underset{\sim}{\varphi}$ has a section.

The quotient space $\underset{\sim}{\mathcal{C}}$ is relevant in studies of Yang-Mills theories, at present very popular in Physics. It has been pointed out [1] that the Universal Connection Theorem could possibly provide connections between Yang-Mills theories and so-called $\sigma$-models which concern themselves with the space $\mathrm{Mor}(M, B)$. Also in the cases when $\underset{\sim}{\varphi}$ has a section, it could give an alternative to "gauge-fixing" which has been shown to be impossible in general [3, 7, 5].

The paper is organized as follows. In § 2 we imbed E and B as closed submanifolds of Hilbert spaces. In § 3 we describe a one parameter family of isometries $A_t : \mathcal{H} \longrightarrow \mathcal{H}$, and also give the

$C^\infty$ topology to be used on the function spaces $\mathrm{Mor}_{U(k)}(P, E)$ and $\mathrm{Mor}_P(M, B)$. In § 4 we prove that $\varphi^{-1}(A)$ is contractible [Proposition 4.1] using the isometries $A_t$. Then we prove [Proposition 4.3] that $\varphi^{-1}(A) \longrightarrow \underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$ is a locally trivial principal fibre space with Aut A as structure group. This involves, among other things, proving that the above projection is closed [Lemma 4.4], which is done by studying a certain differential equation. The completeness of the $C^\infty$ topology is crucial, and the imbeddings obtained in § 2 simplify proofs throughout.

I would like to thank M.S. Narasimhan for several suggestions and much encouragement. I also thank M.S. Raghunathan, S. Ramanan and V. Sunder for their help.

## 2. The bundle of orthonormal $k$-frames in a Hilbert space.

Fix an integer $k > 0$. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space over the complex numbers. Denote by E the space of orthonormal $k$-frames in $\mathcal{H}$. The group $U(k)$ acts on E on the right and the quotient is the Grassmannian B of $k$-dimensional subspaces of $\mathcal{H}$. In fact E is a universal principal bundle for $U(k)$. It also carries a natural connection, which is a universal connection for $U(k)$.

It will be useful, in the following, to have characterizations of E and B as closed submanifolds of Hilbert spaces.

We shall identify a point $p$ in B with the orthogonal projector onto the corresponding subspace, denoted by $H(p)$. Thus $H(p) = \{x \in \mathcal{H} \mid px = x\}$. For $p_0 \in B$, define

$$\mathscr{U}_0 = \{p \in B \mid H(p_0) \cap \ker p = \{0\}\}.$$

Then we have a bijection $L_0 : \mathscr{U}_0 \longrightarrow \mathscr{L}(H(p_0), \ker p_0)$ such that for $p \in \mathscr{U}_0$ its image $L \equiv L_0(p)$ has $H(p)$ as graph.

LEMMA 2.1 [2]. – *The charts* $\{(\mathscr{U}_0, L_0)\}$ *give* B *the structure of a* $C^\infty$ *Hilbert manifold.*

Let $\mathscr{I}_2$ denote the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$.

PROPOSITION 2.2. – *Let* $\psi$ *denote the injection* $B \longrightarrow \mathcal{J}_2$ *given by associating to each k-dimensional subspace its orthogonal projector. Then* $\psi$ *is a* $C^\infty$ *immersion, and a homeomorphism onto its image.*

*Proof.* – Follows from Lemmas 2.3 and 2.4.

*Remark.* – This shows that B, with the manifold structure given in Lemma 2.1 is a submanifold of $\mathcal{J}_2$.

LEMMA 2.3. – *On a chart* $(\mathcal{G}_0, L_0)$ $\psi$ *is given by* $(1-3)$. *It is a* $C^\infty$ *immersion.*

*Proof.* – Let $L \in \mathcal{L}(H(p_0), \ker p_0)$ and let $p = \psi L_0^{-1}(L)$. Write

$$p = A + LA \tag{1}$$

where $A : \mathcal{H} \longrightarrow H(p_0)$. Then we claim that A satisfies

$$A = p_0 + L^+(1-p_0) - L^+LA \tag{2}$$

which can be solved to give

$$A = \frac{1}{1+L^+L}\,(p_0 + L^+(1-p_0)). \tag{3}$$

To see that $p$ given by (2.1)-(2.3) is indeed equal to $\psi L_0^{-1}(L)$, we verify:

a) Image of $p = \{x + Lx \mid x \in H(p_0)\}$. The map is clearly into this set. In fact it is onto since A is invertible on $H(p_0)$.

b) $p^2 = p$. This follows since $Ap = p$, which in turn is clear because $Ap$ satisfies the same equation as $p$.

$$\begin{aligned} Ap = p_0 p + L^+(1-p_0)\,p - L^+LAp &= A + L^+LA - L^+LAp \\ &= p_0 + L^+(1-p_0) - L^+LAp\,. \end{aligned}$$

c) $p$ is an orthogonal projector, for

$$\ker p = \{y - L^+y \mid y \in \ker p_0\}$$

which is the orthogonal subspace to $\operatorname{Im} p$.

(i) $\psi$ is $C^\infty$ : To see this split $\psi$ into the steps:

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{c}\mathcal{L}(\mathcal{H}, \mathrm{H}(p_0))\\ \{p_0 + \mathrm{L}^+(1-p_0)\}\end{array} \arrow[r, hook] & \begin{array}{c}\mathcal{L}(\mathcal{H}, \mathcal{H})\\ \{p_0 + \mathrm{L}^+(1-p_0)\}\end{array} \arrow[dr] & \\
\begin{array}{c}\mathcal{L}(\mathrm{H}(p_0), \ker p_0)\\ \{\mathrm{L}\}\end{array} \arrow[ur] \arrow[dr, "{\left[\begin{array}{l}\text{Positive, hermitian}\\ \text{operators on } \mathrm{H}(p_0)\end{array}\right]}"'] & & & \mathcal{L}(\mathcal{H}, \mathcal{H}) \\
 & \begin{array}{c}\mathcal{L}^{+,h}(\mathrm{H}(p_0), \mathrm{H}(p_0))\\ \{\mathrm{L}^+\mathrm{L}\}\end{array} \arrow[r] & \begin{array}{c}\mathcal{L}(\mathrm{H}(p_0), \mathrm{H}(p_0))\\ \left\{\dfrac{1}{1+\mathrm{L}^+\mathrm{L}}\right\}\end{array} \arrow[r] & \begin{array}{c}\mathcal{L}(\mathcal{H}, \mathcal{H})\\ \left\{p_0\left[\dfrac{1}{1+\mathrm{L}^+\mathrm{L}}\right]p_0\right\}\end{array} \arrow[u]
\end{tikzcd}
$$

$\psi$ is in fact real-analytic.

(ii) It is enough to check the differential at $\mathrm{L} = 0$. Here $\delta p = \delta \mathrm{L}^+(1 - p_0) + p_0 \delta \mathrm{L}$ which is clearly injective. Also the image, being defined by $p_0 \delta p p_0 = (1 - p_0)\, \delta p (1 - p_0) = 0$ and $\delta p^+ = \delta p$, is closed, and hence admits a supplement.

LEMMA 2.4. – *The inverse* $\psi^{-1}$ *is given by (4) and is continuous.*

*Proof.* – Consider a chart $(\mathcal{P}_0, \mathrm{L}_0)$. Let $p \in \mathcal{P}_0$ and let $\mathrm{Q} = (p_0|_{\mathrm{H}(p)})^{-1}$. Then for $x \in \mathrm{H}(p)$, $\mathrm{Q}x = x + (1 - p_0) p \mathrm{Q} x$. This gives, for $\mathrm{L} = (1 - p_0)\mathrm{Q}$, $\mathrm{L} = (1 - p_0) p (1 + \mathrm{L})$.

This can be solved to give $p \overset{}{\underset{\psi^{-1}}{\longmapsto}} \mathrm{L}$ such that

$$\mathrm{L}x = (1 - p_0) \frac{1}{1 - (1 - p_0) p} x , x \in \mathrm{H}(p_0). \tag{4}$$

The continuity of $\psi^{-1}$ follows easily.

We turn now to E. This can be identified with a closed subset of $\mathcal{L}(\mathbf{C}^k, \mathcal{H})$: $\mathrm{E} = \{\mathrm{U} : \mathbf{C}^k \longrightarrow \mathcal{H} \mid \mathrm{U}^+\mathrm{U} = 1\}$. Standard arguments show:

LEMMA 2.5. – E *is a closed submanifold of* $\mathcal{L}(\mathbf{C}^k, \mathcal{H})$. *It is a principal* $\mathrm{U}(k)$ *bundle on* B. *The* $u(k)$*-valued one-form* $\mathrm{U}^+ d\mathrm{U}$ *is a connection on* E.

LEMMA 2.6. – E *is contractible and hence a universal* U($k$) *bundle. The connection is a universal* U($k$) *connection.*

*Proof.* – Both statements are well-known. The first follows also from the remarks after Lemma 4.2. The second is a consequence of the Universal Connection Theorem.

## 3. Some preliminary remarks and definitions.

(i) A one-parameter-family of isometries on $\mathcal{H}$.

Following [6], we introduce, on $\mathcal{H}$, a one-parameter family of isometries which we will use later. Define, for $t \in [0\,,1]$ an isometry $A_t : \mathcal{H} \longrightarrow \mathcal{H}$ as follows. Fix an orthonormal basis, so that $\mathcal{H} \approx$ {square-summable sequences in $\mathbf{C}$}. Then let $A_0 =$ Identity

$$A_t(a_0\,, a_1\,, a_2\,,\dots) = (a_0\,, a_1\ \dots\ a_{n-2}\,, a_{n-1}\cos\theta_n(t)\,, a_{n-1}\sin\theta_n(t)$$
$$a_n\cos\theta_n(t)\,, a_n\sin\theta_n(t)\ a_{n+1}\cos\theta_n(t)\,, a_{n+1}\sin\theta_n(t)\dots)$$

for $\dfrac{1}{n+1} \leqslant t \leqslant \dfrac{1}{n}$ where $\theta_n(t) = \dfrac{\pi}{2}\, n[(n+1)\,t - 1]$.

The $A_t$ are continuous in $t$ w.r. to the strong operator topology. Note that

$$A\left(\frac{1}{2}\right)(a_0\,, a_1\,,\dots) = (a_0\,, 0\,, a_1 0\,,\dots) \in \mathcal{H}_{\text{even}}$$
$$A(1)\,(a_0\,, a_1\,,\dots) \quad = (0\,, a_0\,, 0\,, a_1\dots) \in \mathcal{H}_{\text{odd}}$$

where $\mathcal{H}_{\text{even}}$ and $\mathcal{H}_{\text{odd}}$ denote obvious subspaces of $\mathcal{H}$.

(ii) The topology of the function spaces $\mathrm{Mor}_{U(k)}(P\,,E)$ $\mathrm{Mor}(M\,,B)$.

We topologize $\mathrm{Mor}_{U(k)}(P, E)$ as a (closed) subset of

$$C^\infty(P\,, \mathcal{L}(\mathbf{C}^k\,,\mathcal{H}))\,,$$

and $\mathrm{Mor}(M\,,B)$ as a (closed) subset of $C^\infty(M\,,\mathcal{G}_2)$. The $C^\infty$ topology is described below:

Let X be a compact manifold and $\mathcal{J}$ a Hilbert space. Let $X_1\,,\dots,X_q$ be a set of vector fields on X which together span the tangent space at each point of X. For a multi index $\alpha = (\alpha_1\,,\dots,\alpha_2)$

set $D^\alpha = X_1^{\alpha_1}, \dots, X_q^{\alpha_q}$. We make $C^\infty(X, \mathcal{J})$ a Frechet space w.r. to the seminorms $\|f\|_\alpha = \sup\limits_x \|D^\alpha f\|$ where the heavy bars $\| \ \|$ denote the Hilbert space norm. The topology is clearly independent of the choice of $X_1, \dots, X_q$. If $N \subset \mathcal{J}$ is a closed submanifold then $C^\infty(X, N)$ is a closed subset of $C^\infty(X, \mathcal{J})$ and we give it the relative topology, which makes it a complete metric space.

We choose now, once and for all, a set of vector fields $X_1, \dots, X_p$ spanning the tangent space of $M$ at each point. Let $\hat{X}_1, \dots \hat{X}_p$ be their lifts to $P$ w.r. to some connection, and let $\hat{Y}_1, \dots, \hat{Y}_{k^2}$ be vertical vector fields on $P$, the images of a fixed basis $Y_1, \dots, Y_{k^2}$ in $u(k)$ by the group action. We will use these to determine the seminorms. Note that $[\hat{X}_i, \hat{Y}_\ell] = 0 \ \forall X_i$ and $Y_\ell$. We will let let $\alpha_L = (\alpha_1, \dots, \alpha_{k^2})$ and $\alpha = (\alpha_1, \dots, \alpha_p)$, and write the seminorms as $\|f\|_{\alpha_L, \alpha} = \sup\limits_{x \in P} \|D^{\alpha_L} D^{\alpha} f\|$.

When there is no need to distinguish between the vertical and horizontal vectors we simply denote $(\alpha_L, \alpha)$ by $\gamma$.

LEMMA 3.1. – $\mathrm{Mor}_{U(k)}(P, E)$ *and* $\mathrm{Mor}(M, B)$ *are closed subsets of* $C^\infty(P, \mathcal{L}(\mathbf{C}^k, \mathcal{H}))$ *and* $C^\infty(M, \mathcal{J}_2)$ *respectively. The map* $\mathrm{Mor}_{U(k)}(P, E) \longrightarrow \mathrm{Mor}(M, B)$ *is continuous.*

*Proof.* – For $g \in U(k)$ the map $C^\infty(P, E) \longrightarrow C^\infty(P, E)$ given by $f \underset{g}{\longmapsto} f^g$, $f^g(x) \equiv f(xg)g^{-1}$ $(x \in P)$, is continuous. This follows since

$$\begin{aligned}
\|f_1^g - f_2^g\|_{\alpha_L, \alpha} &= \sup_{x \in P} \|D_x^{\alpha_L} D_x^{\alpha}(f_1(xg)g^{-1} - f_2(xg)g^{-1})\| \\
&= \sup_{x \in P} \|D_x^{\alpha_L} D_x^{\alpha}(f_1(xg) - f_2(xg))\| \\
&= \sup_{xg \in P} \|D_{xg}^{[\alpha_L, g]} D_{xg}^{\alpha}(f_1(xg) - f_2(xg))\| \\
&= \|f_1 - f_2\|_{[\alpha_2, g], \alpha}
\end{aligned}$$

where $D^{[\alpha_L, g]}$ denotes the differential operator

$$D^{[\alpha_L, g]} = (\widehat{g^{-1} Y_1 g})^{\alpha_1} \dots (\widehat{g^{-1} Y_{k^2} g})^{\alpha_k}.$$

Here $\widehat{g^{-1} Y_i g}$ is the image of the Lie algebra element $g^{-1} Y_i g$. This proves the first statement. To prove the second statement, let $f_n \longrightarrow f$ in $\mathrm{Mor}_{U(k)}(P, E)$ and let $p_n = f_n f_n^+$. Then

$$\begin{aligned}
\|p_n - p\|_\alpha &= \sup_{x\in B} \|D^\alpha(p_n - p)\| \qquad (\text{where} \quad D^\alpha = X_1^{\alpha_1}\dots X_n^{\alpha_n}) \\
&= \sup_{x\in P} \|D^\alpha(p_n - p)\| \qquad (\text{where} \quad D^\alpha = \hat{X}_1^{\alpha_1}\dots \hat{X}_n^{\alpha_n}) \\
&= \sup_{x\in P} \|\sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} f_n D^\beta f_n^+ - D^{\alpha-\beta} f D^\beta f^+)\| \\
&\leqslant \alpha \mid \sum_{\beta<\alpha} \|f_n\|_\beta \, \|f_n - f\|_{\alpha-\beta} + \|f\|_{\alpha-\beta} \, \|f_n - f\|_\beta \,.
\end{aligned}$$

This proves $p_n \longrightarrow p$ in $\mathrm{Mor}(M, B)$.

## 4. The topology of the fibres.

We will be interested in the fibres of the map $\underset{\sim}{\varphi}$. Consider first a fibre of $\varphi$.

PROPOSITION 4.1. – *Let* $A \in \mathcal{C}$. *Then* $\varphi^{-1}(A)$ *is contractible. In other words th space of morphisms* $P \longrightarrow E$ *which induce a fixed connection on* $P$ *is contractible.*

*Proof.* – The proof proceeds in two steps.

(i) Define a map

$$\xi : \varphi^{-1}(A) \times [0, 1/2] \longrightarrow \varphi^{-1}(A)$$

by

$$\xi_t(f)(x) = A_t \circ f(x) \quad \begin{cases} f \in \varphi^{-1}(A) \\ x \in P \\ t \in [0, 1/2]. \end{cases}$$

The map is into $\varphi^{-1}(A)$ since,

a) $\xi_t(f)(xg) = A_t \circ f(xg)\ (g \in U(k)) = A_t \circ f(x) \circ g$

$$= \xi_t(f)(x) \circ gU$$

b) $\xi_t(f)^+ \, d\xi_t(f) = f^+ df = A$.

By lemma 4.2 below $\xi$ is continuous.

(ii) There exists a $f_0 \in \varphi^{-1}(A)$ s.t. $\forall x \in P, f_0(x)$ maps $\mathbf{C}^k$ into $\mathcal{H}_{\mathrm{odd}}$ [Apply $A_1$ to any $f \in \varphi^{-1}(A)$ to get such an $f_0$]. Define for $t \in [1/2, 1]$ a map $\eta : \varphi^{-1}(A) \times [1/2, 1] \longrightarrow \varphi^{-1}(A)$ by

$$\eta_t(f)(x)v = (\sin t\pi)\, A_{1/2}\, f(x)\, v - \cos t\pi\, f_0(x)\, v\,.$$

Again the map is into $\varphi^{-1}(A)$. Note that $A_{1/2}f$ maps into $\mathcal{H}_{\text{even}}$. This means that $\forall\ (x, t)$, $\eta_t f(x)$ defines an isometry of $\mathbf{C}^k$ into $\mathcal{H}$, for, given $v, v' \in \mathbf{C}^k$,

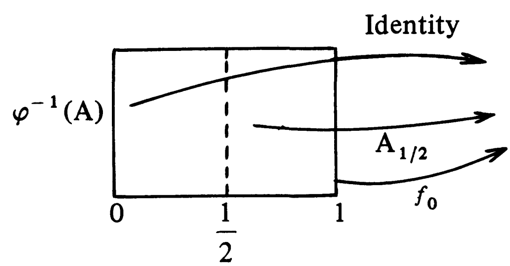
$$(\eta_t f(x) v, \eta_t f(x) v') = \sin^2 t\pi\, (A_{1/2} f(x) v, A_{1/2} f(x) v') + (\cos^2 t\pi)\, (f_0(x) v, f_0(x) v') = (v, v')$$

where ( , ) denotes the inner product.

The points a), b) above can be checked easily. Lemma 4.2 gives continuity.

(iii) Compose $\xi$ and $\eta$ to get the contraction

$\psi : \varphi^{-1}(A) \times [0,1] \longrightarrow \varphi^{-1}(A)$. (See diagram)




LEMMA 4.2. – *The maps $\xi$, $\eta$ constructed in the proof of Proposition 4.1 are continuous (in the product topology).*

*Proof.* – Consider the map $\xi$. Let $(f_n, t_n)$ be a sequence in $\varphi^{-1}(A) \times [0, 1/2]$. Then

$$\begin{aligned} \|\xi_{t_n}(f_n) - \xi_t(f)\|_\gamma &= \sup_{x \in P} \|A_{tn} \circ D^\gamma f_n - A_t \circ D^\gamma f\| \\ &= \sup_{x \in P} \|A_{tn} \circ D^\gamma (f_n - f) + (A_{t_n} - A_t) \circ D^\gamma f\| \\ &\leqslant \|f_n - f\|_\gamma + \|(A_{t_n} - A_t) f\|_\gamma . \end{aligned}$$

This shows continuity of $\xi$. The continuity of $\eta$ follows similarly.

*Remark.* – The proof of Proposition 4.1 can be extended to prove contractivility of $\mathrm{Mor}_{U(k)}(P, E)$. In particular, taking $P = U(k)$, we see that E itself is contractible.

We turn now to the fibres of the map $\underset{\sim}{\varphi}$. Note that if $A \in \mathcal{C}$ and $\underset{\sim}{A} \in \underset{\sim}{\mathcal{C}}$ is its class, then $\varphi^{-1}(A)$ projects onto $\underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$. Also if $\operatorname{Aut} A$ is the subgroup of $\operatorname{Aut}$ that leaves A fixed $\operatorname{Aut}(A)$ acts freely on $\varphi^{-1}(A)$, the quotient being in bijection with $\underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$.

Aut A is the space of maps $\hat{g} : P \longrightarrow U(k)$ such that

(i) $\hat{g}(xh) = h^{-1} g(x) h \quad x \in P,\ h \in U(k)$

(ii) $A = \hat{g}^{-1} A \hat{g} + \hat{g}^{-1} d\hat{g}$.

Since $\hat{g} \in \operatorname{Aut} A$ is determined by its value at a fixed point in P, we shall, fixing $y_0 \in P$ (projecting onto $x_0 \in M$) identify $\operatorname{Aut} A \ni \hat{g} \sim \hat{g}(y_0) \in U(k)$.

Thus Aut A is a closed subgroup of $U(k)$ [This is seen either from the equation (ii) above, or noting the fact that under the above identification Aut A is the centralizer of the holonomy group at $y_0$] and hence a Lie subgroup.

From now on we assume that the vector fields $\hat{X}_1 \ldots \hat{X}_p$ have been lifted to P w.r. to A. Note that then $\hat{X}_i(\hat{g}) = 0$ for $\hat{g} \in \operatorname{Aut} A$.

PROPOSITION 4.3. – $\varphi^{-1}(A) \longrightarrow \underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$ *is a locally trivial principal fibre space with* Aut A *as structure group.*

*Proof.* – The proof proceeds in four steps.

a) Aut (A) acts continuously on $\varphi^{-1}(A)$. For suppose $(f_n, \hat{g}_n) \in \varphi^{-1}(A) \times \operatorname{Aut} A$ and $(f_n, \hat{g}_n) \longrightarrow (f, \hat{g})$. Then for any $\alpha_L, \alpha$

$$\begin{aligned}
\|f_n \circ \hat{g}_n - f \circ \hat{g}\|_{\alpha_L,\alpha} &\leqslant \|(f_n - f) \circ \hat{g}_n\|_{\alpha_L,\alpha} + \|f \circ (\hat{g}_n - \hat{g})\|_{\alpha_L,\alpha} \\
&= \sup_x \|D^{\alpha_L}([D^\alpha (f_n - f)]\, \hat{g}_n)\| + \sup_x \|D^{\alpha_L}([D^\alpha f]\, (\hat{g}_n - \hat{g}))\| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{since } D^\alpha \hat{g} = 0) \\
&= \sup_x \Big\| \sum_{\beta_L \leqslant \alpha_L} \binom{\alpha_L}{\beta_L} D^{\alpha_L - \beta_L} D^\alpha (f_n - f)\, D^{\beta_L} \hat{g}_n \Big\| \\
&\quad + \sup_x \Big\| \sum_{\beta_L \leqslant \alpha_L} \binom{\alpha_L}{\beta_L} D^{\alpha_L - \beta_L}\, D^\alpha f\, D^{\beta_L} (\hat{g}_n - \hat{g}) \Big\| \\
&\leqslant \alpha_L ! \Big\| \sum_{\beta_L \leqslant \alpha_L} \|f_n - f\|_{\alpha_L - \beta_L, \alpha}\, \|\hat{g}_n\|_{\beta_L} + \|f\|_{\alpha_L - \beta_L,\alpha}\, \|\hat{g}_n - \hat{g}\|_{\beta_L} .
\end{aligned}$$

Now, for any $\hat{Y}_i$, $\hat{g} \in \operatorname{Aut} A$

$$\hat{Y}_i(\hat{g}) = \lim_{t \to 0} \frac{\hat{g}(x \exp t\, Y_i) - \hat{g}(x)}{t} = [\hat{g}(x), Y_i].$$

Also, if $\hat{g}_1, \hat{g}_2$ are in $\operatorname{Aut} A$, $d(\operatorname{Tr}(\hat{g}_1 - \hat{g}_2)^+ (\hat{g}_1 - \hat{g}_2)) = 0$, so that $\|\hat{g}_1(x) - \hat{g}_2(x)\| = \|\hat{g}_1(y_0) - \hat{g}_2(y_0)\|$.

So, we have

$$\|f_n \circ \hat{g}_n - f \circ \hat{g}\|_{\alpha_L, \alpha} \leqslant \alpha_L! \sum_{\beta_L \leqslant \alpha_L} \|f_n - f\|_{\alpha_L - \beta_L, \alpha} \|\hat{g}_n\|_{\beta_L}$$
$$+ \|f\|_{\alpha_L - \beta_L, \alpha} C_{\beta_L} \|\hat{g}_n(p_0) - \hat{g}(p_0)\|$$

where $C_{\beta_L}$ is a constant depending on the multiindex $\beta_L$.

b) Denote by **G** the graph of the equivalence relation defined by $\operatorname{Aut} A$ on $\varphi^{-1}(A)$. Then the map $\mathbf{G} \longrightarrow \operatorname{Aut} A$ is continuous. This follows since the map is given by $(f_1, f_2) \longmapsto f_1^+(y_0) f_2(y_0)$ which is clearly continuous.

c) The projection $\varphi^{-1}(A) \longrightarrow \underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$ is continuous and closed. Continuity follows from lemma 3.1 and lemma 4.4 shows that it is closed. Thus $\underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$ has the quotient topology w.r. to the projection.

d) Thus we have shown that $\varphi^{-1}(A) \longrightarrow \underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$ is a principal fibre space. Now note that there is a $\operatorname{Aut} A$-morphism

$$\begin{array}{ccc}
\varphi^{-1}(A) & \longrightarrow & E \\
\big\downarrow & & \big\downarrow \\
\varphi^{-1}(\underset{\sim}{A}) & \longrightarrow & E/\operatorname{Aut} A
\end{array}$$

given by $f \longmapsto f(y_0)$. Since $E \longrightarrow E/\operatorname{Aut} A$ is locally trivial, the proof is complete.

LEMMA 4.4. – *The map* $\varphi^{-1}(A) \longrightarrow \underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$ *is closed.*

*Proof.* – Let $f_n \in \varphi^{-1}(A)$ s.t. $p_n = f_n f_n^+ \longrightarrow p$ in $\underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$.

It is enough to prove that $\{f_n\}$ contains a convergent subsequence. Since $p_n(x_0) \longrightarrow p(x_0)$ and E has compact fibres one can assume $f_n(y_0) \longrightarrow g_0 \in \mathrm{E}$ without loss of generality. Note that the $f_n$ satisfy

$$df_n = f_n \mathrm{A} + dp_n f_n \,. \tag{5}$$

We now prove that the $f_n$ are Cauchy in the $\mathrm{C}^0$ norm so that $\exists$ a $\mathrm{C}^0$ function $f$ such that $f_n \longrightarrow f$. Put $\mathrm{D} = f_n - f_m$. Then from (5) we have

$$d(\mathrm{DD}^+) = \mathrm{DD}^+ dp_n + dp_n \mathrm{DD}^+ + d(p_n - p_m)\, f_m \mathrm{D}^+ + \mathrm{D} f_m^+ \, d(p_n - p_m).$$

Evaluating on a vector field $\mathrm{X}_t$, taking the trace and then absolute value of both sides we get

$$\begin{aligned} |\mathrm{X}_t \operatorname{Tr}(\mathrm{DD}^+)| &\leqslant |\operatorname{Tr}(\mathrm{DD}^+ \mathrm{X}_t p_n)| + |\operatorname{Tr}(\mathrm{X}_t(p_n)\, \mathrm{DD}^+)| \\ &\quad + |\operatorname{Tr}(\mathrm{X}_t(p_n - p_m)\, f_m \mathrm{D}^+)| + |\operatorname{Tr}(\mathrm{D} f_m^+ \mathrm{X}_t(p_n - p_m))| \\ &\leqslant 2\{\|\mathrm{D}\|^2 \; \|\mathrm{X}_t p_n\| + \|\mathrm{D}\| \; \|\mathrm{X}_t(p_n - p_m)\|\} \end{aligned}$$

or,

$$|\mathrm{X}_t \, \|\mathrm{D}\|^2| \leqslant 2\, \{\|\mathrm{D}\|^2 \; \|\mathrm{X}_t p_n\| + \|\mathrm{X}_t(p_n - p_m)\|\}. \tag{6}$$

Consider now the set $\{\mathrm{X}_i, \mathrm{Y}_\ell\}$ which we collectively denote by $\{\mathrm{Z}_j\}$. They give a map from $\mathrm{P} \times \mathbf{R}^{\mathrm{N}}$ (where $\mathrm{N} = k^2 + p$) to the tangent bundle TP which is onto:

$$(x\,, (t_1 \ldots t_{\mathrm{N}})) \longmapsto (x\,, \sum_i t_i \mathrm{Z}_i(x))\,.$$

Take the obvious metric on the vector bundle $\mathrm{P} \times \mathbf{R}^n$. This induces a splitting of the above map as well as a Riemannian metric on P. Then we have the following obvious result: if X is a vector field on P of norm $\leqslant 1$ and we express $\mathrm{X} = \Sigma\, a_i \mathrm{Z}_i$ with respect to the above splitting then $|a_i| \leqslant 1 \;\; \forall i$.

Now let $y \in \mathrm{P}$ and let $\Gamma(y)$ be a minimal geodesic joining $y_0$ to $y$ [such a geodesic exists for P compact] parametrized with respect to arc-length. Then the length of $\Gamma(y) < \mathrm{T}$ for some constant T independent of $y$. Now let $\mathrm{X}_t$ be the tangent vector field to $\Gamma$ (which is necessarily of norm one). This gives

$$\|\mathrm{X}_t(p_n - p_m)\| = \sum_i \|p_n - p_m\|_i \quad \text{where} \quad \|p\|_i = \sup_x \|\mathrm{Z}_i p\| = \sum_{|\alpha|=1} \|p_n - p_m\|_\alpha \,.$$

Thus we have, from (6)

$$|\mathrm{X}_t \, \|\mathrm{D}\|^2| = 2\, \{a\, \|\mathrm{D}\|^2 + b\, \|\mathrm{D}\|\}$$

with $$a = \sum_{|\alpha|=1} \|p\|_\alpha + c\,,\ c > 0$$

and $$b = \sum_{\alpha} \|p_n - p_m\|_\alpha\,.$$

Consider the ordinary differential equation

$$\frac{du^2}{dt^2} = 2(au^2 + bu)$$

$$u(0) = \mathrm{D}(y_0)\,.$$

The solution is clearly:

$$u(t) = \mathrm{D}(y_0)\, e^{at} + \frac{(e^{at} - 1)}{a}\, b\,.$$

Consider the set $\mathrm{K} = \{t \geqslant 0 \mid \|\mathrm{D}(t)\| > u(t)\}$. K is open, and hence a union of disjoint open intervals. Let $t_0$ be its least boundary point. Clearly $\mathrm{D}(t_0) = u(t_0)$. From the polygonal approximations to $\|\mathrm{D}(t_0)\|^2$ and $u^2(t)$ it is clear that in an interval $(t_0\,, t_0 + \epsilon)$ we have $\|\mathrm{D}(t)\| \leqslant u(t)$. Thus $\mathrm{K} = \phi$. We have finally,

$$\|\mathrm{D}(y)\| \leqslant \mathrm{D}(y_0)\ e^{a\mathrm{T}} + \frac{(e^{a\mathrm{T}} - 1)}{a}\ b$$

which clearly shows that $\{f_n\}$ are Cauchy in the $\mathrm{C}^0$ norm.

Let $f$ be the $\mathrm{C}^0$ limit. We now turn back to (5) and 'bootstrap' the above result to show that $f$ is $\mathrm{C}^\infty$ and $f_n \longrightarrow f$ in the $\mathrm{C}^\infty$ topology. Assume, therefore, that $f$ is $\mathrm{C}^k$ and $f_n \longrightarrow f$ in $\mathrm{C}^k$. For any multi-index $\gamma(|\gamma| \geqslant 1)$ define $\gamma'$ and $\mathrm{X}^{(\gamma)}$ [here $\mathrm{X}^{(\gamma)}$ is one of the vector fields $\mathrm{Z}_i$] by $\mathrm{D}^\gamma = \mathrm{D}^{\gamma'}\mathrm{X}^{(\gamma)}$ so that $\mathrm{D}^{\gamma'}$ is of order $|\gamma| - 1$. Let $|\gamma| = k + 1$. Then

$$\begin{aligned}\mathrm{D}^\gamma f_n &= \mathrm{D}^{\gamma'}\mathrm{X}^{(\gamma)}(f_n) = \mathrm{D}^{\gamma'}(f_n \mathrm{A}(\mathrm{X}^{(\gamma)}) + \mathrm{X}^{(\gamma)}(p_n) f_n)\\ &= \sum_{\delta < \gamma'} \binom{\gamma'}{\delta} [\mathrm{D}^{\gamma'-\delta} f_n \mathrm{D}^\delta \mathrm{A}(\mathrm{X}^{(\gamma)}) + \mathrm{D}^{\gamma'-\delta}\mathrm{X}^{(\gamma)}(p_n)\,\mathrm{D}^\delta f_n]\,.\end{aligned}$$

Then

$$\begin{aligned}&\|\mathrm{D}f_n - \sum_{\delta < \gamma'} \binom{\gamma'}{\delta} [\mathrm{D}^{\gamma'-\delta} f\ \mathrm{D}^\delta \mathrm{A}(\mathrm{X}^{(\gamma)}) + \mathrm{D}^{\gamma'-\delta}\mathrm{X}^{(\gamma)}(p)\,\mathrm{D}^\delta f]\|\\ &\quad \leqslant \gamma! \sum_{\delta < \gamma'} \|f_n - f\|_{\gamma-\delta}\ \|\mathrm{A}(\mathrm{X}^{(\gamma)})\|_\delta + \|p_n\|_{\gamma'-\delta,\mathrm{X}^{(\gamma)}}\ \|f_n - f\|_\delta\\ &\quad + \|p_n - p\|_{\gamma'-\delta,\mathrm{X}^{(\gamma)}}\ \|f\|_\delta\end{aligned}$$

where $\|f\|_{\gamma'-\delta,\mathrm{X}^{(\gamma)}} \equiv \sup_x \|\mathrm{D}^{\gamma'-\delta}\,\mathrm{X}^{(\gamma)} f\|\,.$

This shows $D^\gamma f_n$ tends uniformly to a $C^0$ function, and hence $f$ is $C^{k+1}$. By induction $f$ is $C^\infty$ and $f_n \longrightarrow f$ in $C^\infty(P, E)$. The proof also shows $df = fA + pf$.

Since $\mathrm{Mor}_{U(k)}(P, E)$ is closed, $f \in \mathrm{Mor}_{U(k)}(P, E)$ and $p = ff^+$ by continuity of the projection $\mathrm{Mor}_{U(k)}(P, E) \longrightarrow \mathrm{Mor}_p(M, B)$. (One can now easily show that $f^+ df = A$, thus showing that the fibre $\underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$ is closed. This is because we have nowhere in the proof used the fact that $p \in \underset{\sim}{\varphi}^{-1}(\underset{\sim}{A})$).

The Theorem stated in the Introduction now follows.

## BIBLIOGRAPHY

[1] M. DUBOIS-VIOLETTE and Y. GEORGELIN, Gauge Theory in terms of projector valued fields, *Physics Letters,* 82B, 251 (1979).

[2] A. DOUADY, Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, séminaire, Collège de France (1964-65).

[3] V.N. GRIBOV, Quantization of nonabelian gauge theories, *Nuclear Physics,* B 139 (1978), 1.

[4] M.S. NARASIMHAN and S. RAMANAN, Existence of universal connections, *Amer. J. Math.,* 83 (1961), 573-572.

[5] M.S. NARASIMHAN and T.R. RAMADAS, Geometry of SU(2) gauge-fields, *Commun. Math. Phys.,* 67 (1979), 121-136.

[6] R. SCHLAFLY, Universal Connections, *Inventiones Math.,* 59 (1980), 59-65.

[7] I.M. SINGER, Some remarks on the Gribov ambiguity, *Commun. Math. Phys.,* 60 (1978), 7-12.

Manuscrit reçu le 17 août 1981.

T.R. RAMADAS,
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400005 (India).