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FINITE SUMS OF COMMUTATORS IN C^* -ALGEBRAS

by Thierry FACK

Introduction.

Let A be a C^* -algebra and put

$$A_0 = \left\{ x \in A \mid x = \sum_{n \geq 1} x_n x_n^* - x_n^* x_n ; \text{norm convergence} \right\}.$$

By [4] (theorem 2.6), A_0 is exactly the null space of all finite traces on the self-adjoint part of A .

For von Neumann algebras, A_0 is spanned by finite sums of the above type (see for example [6]). This is not always true for C^* -algebras, as it is shown by Pedersen and Petersen ([8], lemma 3.5) for a very natural algebra. A reasonable question is then : when can this happen for C^* -algebras ?

The aim of this paper is to show that A_0 is spanned by finite sums for stable algebras and C^* -algebras with "sufficiently many projections" like infinite simple C^* -algebras or simple A.F.-algebras (with unit).

We use the usual terminology of C^* -algebras as in [7]. A commutator of the form $[x, x^*] = xx^* - x^*x$ is called a self-adjoint commutator.

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I'd like to thank G. Skandalis for fruitful discussions and G.K. Pedersen who originally asked this question.

1. Stable C^* -algebras.

Recall that a C^* -algebra A is stable if $A \approx A \otimes \mathcal{K}$, where \mathcal{K} is the C^* -algebra of compact operators. We have

THEOREM 1.1. — *Let A be a stable C^* -algebra. Then, every hermitian element of A is the sum of five self-adjoint commutators.*

Every simple A - F -algebra A without non zero finite trace being stable, it follows that A_0 is spanned by finite sums of self-adjoint commutators.

The proof of theorem 1.1 is based on the following lemmas.

LEMMA 1.2. — *Let A be a C^* -algebra and $x = x^* \in A$. Let p be a projection in $M(A)$. Then, there exists $v \in A$ such that*

$$x = p x p + (1 - p) x (1 - p) + [v, v^*].$$

Proof. — Put

$$v = 1/2 |(1 - p) x p|^{1/2} - |(1 - p) x p|^{1/2} u^* + u |(1 - p) x p|^{1/2}$$

where u is the phase of $(1 - p) x p$. As $p \in M(A)$, we have $v \in A$. By direct calculation, we have $p x (1 - p) + (1 - p) x p = [v, v^*]$.

LEMMA 1.3. — *Let A be a C^* -algebra with unit and $x = x^* \in A$. Let $(\lambda_1, \dots, \lambda_n)$ be a sequence of real numbers satisfying*

$$0 \leq \sum_{i=1}^k \lambda_i \leq 1 \quad (k = 1, \dots, n - 1)$$

and

$$\sum_{i=1}^n \lambda_i = 0.$$

Then, there exists $u \in M_n(A)$, $\|u\| \leq \|x\|^{1/2}$, such that

$$\begin{bmatrix} \lambda_1 x & & & 0 \\ & \ddots & & \\ 0 & & \ddots & \\ & & & \lambda_n x \end{bmatrix} = [u, u^*].$$

Proof. — Write $x = x_+ - x_-$ and put

$$\mu_k^+ = \left(\sum_{i=1}^k \lambda_i \right)^{1/2} x_+^{1/2}$$

$$\mu_k^- = \left(\sum_{i=1}^k \lambda_i \right)^{1/2} x_-^{1/2} \quad (k = 1, \dots, n-1).$$

Take $u = \sum_{k=1}^{n-1} (\mu_k^+ \otimes e_{k,k+1} + \mu_k^- \otimes e_{k+1,k})$, where $(e_{ij})_{1 \leq i, j \leq n}$ is the canonical system of matrix units. As $x_+ x_- = 0$, we get the result by direct calculation. \square

Let e be a rank one projection in \mathcal{K} .

LEMMA 1.4. — *Let A be a C^* -algebra and $x = x^* \in A$. Then, $x \otimes e$ is the sum of two self-adjoint commutators of $A \otimes \mathcal{K}$.*

Proof. — Write $x \otimes e = \begin{bmatrix} x & & & \\ & \lambda_1 x & & \\ & & \lambda_2 x & \\ \circ & & & \ddots \end{bmatrix} - \begin{bmatrix} \circ & & & \\ & \lambda_1 x & & \\ & & \lambda_2 x & \\ \circ & & & \ddots \end{bmatrix},$

where $(\lambda_n)_{n \geq 1}$ is the sequence

$$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \underbrace{-\frac{1}{8}, \dots, -\frac{1}{8}}_{8 \text{ terms}}, \dots \right).$$

The result follows from lemma 1.3.

Proof of theorem 1.1. — Let x be a hermitian element of $A \otimes \mathcal{K}$. Take a projection $p \in M(\mathcal{K})$ with $p \sim 1 - p \sim 1$.

By lemma 1.2, there exists $v \in A \otimes \mathcal{K}$ such that

$$x = p x p + (1 - p) x (1 - p) + [v, v^*].$$

By lemma 1.4, $p x p$ and $(1 - p) x (1 - p)$ are both sums of two self-adjoint commutators. \square

2. Infinite simple C^* -algebras.

The main result of this section is the following

THEOREM 2.1. — *Let A be a C^* -algebra with unit. Suppose that there exist two orthogonal projections e and f such that $e \sim f \sim 1$ in A . Then, each hermitian element of A is the sum of five self-adjoint commutators.*

Recall that a simple C^* -algebra with unit is said to be *infinite* if it contains an element x such that $x^*x = 1$ and $xx^* \neq 1$. From theorem 2.1, we deduce

COROLLARY 2.2. — *Let A be an infinite simple C^* -algebra with unit. Then each hermitian element of A is the sum of five self-adjoint commutators.*

Apply theorem 2.1 and proposition 2.2 of [1]. The proof of theorem 2.1 is based on the following lemma :

LEMMA 2.3. — *Let A , e and f be as in theorem 2.1. Let p be a rank one projection in \mathcal{K} . Then, there exists a homomorphism*

$\varphi : A \otimes \mathcal{K} \longrightarrow A$ *such that*

$\varphi(x \otimes p) = x$ *for each $x \in (1 - f)A(1 - f)$.*

Proof. — Let u, v be partial isometries such that

$$u^*u = v^*v = 1 \quad ; \quad uu^* = e, \quad vv^* = f.$$

Put $w_1 = 1 - f + vf$ and $w_n = vu^{n-1}v$ ($n \geq 2$).

The w_n are isometries with pairwise orthogonal ranges. Let (e_{ij}) be a system of matrix units for \mathcal{K} , with $e_{11} = p$. Put then

$$\varphi(z \otimes e_{ij}) = w_i z w_j^* \quad (z \in A). \quad \square$$

Proof of the theorem 2.1. — Let $x = x^* \in A$. By lemma 1.2, there exists $y \in A$ such that $x = exe + (1 - e)x(1 - e) + [y, y^*]$. By lemmas 2.3 and 1.4, both exe and $(1 - e)x(1 - e)$ are sums of two self-adjoint commutators (note that $exe \in (1 - f)A(1 - f)$). \square

For non simple infinite C^* -algebras with unit, we may combine corollary 2.2 with the following obvious lemma :

LEMMA 2.4. — *Let $0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$ be an exact sequence of C^* -algebras. Suppose that each hermitian element of J (resp. of B) is a sum of n (resp. k) self-adjoint commutators. Then, any hermitian element of A is the sum of $n + k$ self-adjoint commutators.*

Example. — Let $A = (A(i, j))_{i, j \in \Sigma}$ be a transition matrix on a finite set Σ . Assume that A has no zero columns or rows. For $i, j \in \Sigma$, write $i \leq j$ if the transition from j to i is possible

(cf. [2]). We call i and j equivalent if $i \leq j \leq i$. Let F be the set of maximal states : $F = \{i \in \Sigma \mid \forall j \in \Sigma \ i \leq j \implies j \leq i\}$. F is an union of equivalence classes and every element of Σ is majorized by an element of F .

Assume that the restriction A_γ of A to each equivalence classe γ of F is not a permutation matrix. Then \mathcal{O}_A is defined in [2], [3] as the C^* -algebra generated by *any* system $(S_i)_{i \in \Sigma}$ of non zero partial isometries with pairwise orthogonal ranges satisfying

$$S_i^* S_i = \sum_{j \in \Sigma} A(i, j) S_j S_j^* \quad (i \in \Sigma).$$

We claim that each hermitian element of \mathcal{O}_A is the sum of ten self-adjoint commutators.

Put $A' = A_{\Sigma - F}$ and $A'' = A_F$.

As $\mathcal{O}_{A''}$ is a finite direct sum of \mathcal{O}_B with B irreducible, each hermitian element of $\mathcal{O}_{A''}$ is the sum of five self-adjoint commutators by corollary 2.2 and theorem 2.14 of [3]. But it is easy to see that there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_{A'} \otimes \mathcal{K} \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_{A''} \longrightarrow 0$$

and the result follows from lemma 2.4 and theorem 1.1.

3. Simple A.F-algebras.

In this section, we shall prove the following result :

THEOREM 3.1. — *Let A be a simple approximately finite dimensional C^* -algebra with unit. Then, each element of A_0 is the sum of seven self-adjoint commutators.*

The proof is based on the following technical lemmas :

LEMMA 3.2. — *Let A be a C^* -algebra and $x = x^* \in A$. Let p, q, r be orthogonal projections in A with $p + q + r = 1$. Then, there exists $u \in A$, $\|u\| \leq 2\sqrt{2}\|x\|^{1/2}$, such that*

$$x - p x p - q x q - r x r = [u, u^*].$$

Proof. — Put

$$u = p - r - \frac{1}{2} (pxq - qxp) - \frac{1}{4} (pxr - rxp) - \frac{1}{2} (qxr - rxq).$$

We have $x - pxp - qxq - rxr = [u, u^*]$ by direct calculation. Moreover, $\|x\| \leq 2$ implies $\|u\| \leq 4$. The lemma follows. \square

LEMMA 3.3. — Let A be a C^* -algebra and $x = x^* \in A$. Let p, q, r be orthogonal projections in A with $p + q + r = 1$ and $p \preceq q \preceq r$. Then, there exists $u \in A$, $\|u\| \leq 3 \|x\|^{1/2}$ and $y \in A$ such that

$$x = [u, u^*] + y$$

$$pyp = qyq = 0$$

$$\|ryr\| \leq 3 \|x\|.$$

Proof. — Let v and w be partial isometries such that $vv^* = p$, $v^*v \leq q$, $ww^* = q$, $w^*w \leq r$. Put

$$u = \sqrt{(pxp)_+} v + v^* \sqrt{(pxp)_-} + \sqrt{(qxq + v^*xv)_+} w + w^* \sqrt{(qxq + v^*xv)_-}$$

and $y = x - [u, u^*]$. We have $\|u\| \leq 3 \|x\|^{1/2}$, $pyp = qyq = 0$ and $\|ryr\| \leq 3 \|x\|$ by direct calculation. \square

LEMMA 3.4. — Let A be a C^* -algebra and $x = x^* \in A$. Let p, q, r be orthogonal projections in A with $p + q + r = 1$ and $p \preceq q \preceq r$. Then, there exist $u, v \in A$; $\|u\| \leq 3 \|x\|^{1/2}$, $\|v\| \leq 13 \|x\|^{1/2}$ such that $x - [u, u^*] - [v, v^*] \in rAr$ and $\|x - [u, u^*] - [v, v^*]\| \leq 3 \|x\|$.

Proof. — By lemma 3.3, we have $x = [u, u^*] + y$ with $\|u\| \leq 3 \|x\|^{1/2}$, $pyp = qyq = 0$ and $\|ryr\| \leq 3 \|x\|$. We deduce $\|y\| \leq 19 \|x\|$, and the result follows from lemma 3.2. \square

LEMMA 3.5. — Let B be a finite dimensional C^* -algebra and $x \in B_0$. Then, there exists $u \in B$, $\|u\| \leq \sqrt{2} \|x\|^{1/2}$, such that $x = [u, u^*]$.

Proof. — Using the decomposition of B into simple components, we can assume that $B = M_n(\mathbb{C})$. One may also suppose x is diagonal. The proper values of x are real numbers $\lambda_1, \dots, \lambda_n$

with $\sum_{i=1}^n \lambda_i = 0$. As there exists a permutation τ of $\{1, \dots, n\}$ such that $0 \leq \sum_{i=1}^k \lambda_{\tau(i)} \leq 2 \sup_{1 \leq i \leq n} |\lambda_i|$ for $k = 1, \dots, n$, we can assume that $x = \sum_{i=1}^n \lambda_i e_{ii}$ and $0 \leq \sum_{i=1}^k \lambda_i \leq 2 \|x\|$ ($k = 1, \dots, n$), where $(e_{ij})_{1 \leq i, j \leq n}$ is some system of matrix units. Apply then lemma 1.3. \square

LEMMA 3.6. — Let A be a simple $A.F.$ -algebra with unit. Suppose that A is non isomorphic to $M_n(\mathbb{C})$. Then, there exist sequences $(p_n)_{n \geq 1}$, $(q_n)_{n \geq 1}$ and $(r_n)_{n \geq 1}$ of projections such that

- i) $p_1 + q_1 + r_1 = 1$
- ii) $p_n \lesssim q_n \lesssim r_n$ ($n \geq 1$)
- iii) the r_n are mutually orthogonal,
- iv) $r_{n-1} = p_n + q_n$ ($n \geq 2$).

Proof. — It suffices to show that there exists, for each projection $p \neq 0$, an element $q \in K_0(A)_+$ such that $2q \leq p \leq 3q$. Passing to pAp , we may assume that $p = 1$. By [5] (lemma A.4.3), $K_0(A)$ is the limit of a system $Z^{r(1)} \xrightarrow{\varphi_1} Z^{r(2)} \xrightarrow{\varphi_2} \dots$ having the following properties :

- i) the φ_n are strictly positive, i.e. $\varphi_n = (\alpha_{ij}^n)$ with $\alpha_{ij}^n > 0$,
- ii) there exist order units $u_n \in Z^{r(n)}$ such that

$$u_1 \longrightarrow u_2 \longrightarrow \dots \longrightarrow 1.$$

One then may choose $q \in K_0(A)_+$ such that $2q \leq 1 \leq 3q$. \square

Proof of theorem 3.1. — The case $A = M_n(\mathbb{C})$ is trivial, so that we can assume $A \not\cong M_n(\mathbb{C})$. Let x be in A_0 . Let $(p_n)_{n \geq 1}$, $(q_n)_{n \geq 1}$ and $(r_n)_{n \geq 1}$ be sequences of projections as in lemma 3.6.

Apply first lemma 3.4 to get $x_1 \in r_1 A r_1$, $\|x_1\| \leq 3 \|x\|$, and $u, v \in A$ such that $x = [u, u^*] + [v, v^*] + x_1$. As r_1 is an order unit in $K_0(A)_+$, any finite trace on $r_1 A r_1$ extends uniquely to a finite trace on A , so that $x_1 \in (r_1 A r_1)_0$.

Starting from x_1 , we are going to construct sequences $(x_n)_{n \geq 1}$, $(u_n)_{n \geq 1}$, $(v_n)_{n \geq 1}$ and $(w_n)_{n \geq 1}$ satisfying

$$\alpha) x_n = [u_n, u_n^*] + [v_n, v_n^*] + [w_n, w_n^*] + x_{n+1},$$

$$\beta) u_n \in r_n A r_n; \quad v_n, w_n \in (r_n + r_{n+1}) A (r_n + r_{n+1}),$$

$$\gamma) x_n \in (r_n A r_n)_0,$$

$$\delta) \|x_n\| \leq \frac{3 \|x\|}{n}$$

$$\epsilon) \|u_n\| \leq 2 \|x_n\|^{1/2} \quad \text{and} \quad v_n, w_n \rightarrow 0 \quad (n \rightarrow \infty).$$

Suppose $(x_1, \dots, x_{n-1}, x_n)$, (u_1, \dots, u_{n-1}) , (v_1, \dots, v_{n-1}) and (w_1, \dots, w_{n-1}) constructed.

$$\text{Put } \alpha = \frac{\|x\|}{n+1}. \text{ As } x_n \in (r_n A r_n)_0, \text{ we have}$$

$$x_n = \sum_{p \geq 1} [c_p, c_p^*]$$

where $c_p \in r_n A r_n$ and the sum being norm convergent. By approximation, we can find a finite dimensional subalgebra B of $r_n A r_n$ and $y \in B_0$ such that $\|y\| \leq 2 \|x_n\|$ and $\|x_n - y\| \leq \alpha$.

By lemma 3.5, there exists $u_n \in r_n A r_n$,

$$\|u_n\| \leq \sqrt{2} \|y\|^{1/2} \leq 2 \|x_n\|^{1/2}$$

such that $x_n = [u_n, u_n^*] + z$, where $z = x_n - y$.

Note that $z \in ((r_n + r_{n+1}) A (r_n + r_{n+1}))_0$.

By lemma 3.4, there exist $v_n, w_n \in (r_n + r_{n+1}) A (r_n + r_{n+1})$ such that $z = [v_n, v_n^*] + [w_n, w_n^*] + x_{n+1}$ where $x_{n+1} \in r_{n+1} A r_{n+1}$ and

$$\|v_n\| \leq 3 \|z\|^{1/2} \leq 3\alpha^{1/2}$$

$$\|w_n\| \leq 13 \|z\|^{1/2} \leq 13\alpha^{1/2}.$$

We have

$$x_n = [u_n, u_n^*] + [v_n, v_n^*] + [w_n, w_n^*] + x_{n+1}$$

and hence $x_{n+1} \in (r_{n+1} A r_{n+1})_0$. Moreover,

$$\|x_{n+1}\| \leq 3 \|z\| \leq 3\alpha \leq \frac{3 \|x\|}{n+1}.$$

By induction, the existence of four sequences satisfying $\alpha)$, $\beta)$, $\gamma)$, $\delta)$ and $\epsilon)$ is then proved.

Put

$$U = \sum_{n \geq 1} u_n$$

$$V_{ev} = \sum_{n \geq 1} v_{2n} ; \quad V_{od} = \sum_{n \geq 0} v_{2n+1} ,$$

$$W_{ev} = \sum_{n \geq 1} w_{2n} ; \quad W_{od} = \sum_{n \geq 0} w_{2n+1} .$$

These sums make sense because they involve elements with disjoint support and norm converging to zero. Moreover, we have

$$x = [u, u^*] + [v, v^*] + [U, U^*] + [V_{ev}, V_{ev}^*] + [V_{od}, V_{od}^*] \\ + [W_{ev}, W_{ev}^*] + [W_{od}, W_{od}^*] .$$

The proof of theorem 3.1 is complete. \square

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