ROBERT KAUFMAN

On the weak $L^1$ space and singular measures


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ON THE WEAK L\(^1\) SPACE AND SINGULAR MEASURES

by Robert KAUFMAN

Introduction.

The class \(R\) of finite, complex measures \(\mu\) on \((-\infty, \infty)\) such that \(\hat{\mu}(\infty) = 0\), has been intensively investigated (since 1916). For this class \(o(1)\) is trivial and for absolutely continuous measures, we have the Riemann-Lebesgue Lemma. We investigate the corresponding \(o(1)\) condition for the partial-sum operators

\[
S_T(x, \mu) = \int D_T(x-t) \mu(dt),
\]

\[
D_T(t) = (\pi t)^{-1} \sin T t, \quad T > 0.
\]

The \(o(1)\) condition for \(S_T\) depends on the weak \(L^1\) norm, defined by

\[
\|u\|_1^* = \sup Y m\{|u| > Y\};
\]

\[
\|S_T(\mu)\|_1^* \leq C \|\mu\|, \quad 0 < T < +\infty.
\]

The weak estimate is an easy consequence of Kolmogorov’s estimate for the Hilbert transform [2, Chapter II]. Elementary approximations show that when \(\mu = f(x) \, dx\), then \(\lim \|S_T(\mu) - f\|_1^* = 0\). When \(\mu\) is singular and \(\lim \|S_T(\mu) - g\|_1^* = 0\) for a certain measurable \(g\), two conclusions can be obtained without great difficulty (see below):

a) \(\|S_k(\mu) - S_{k+1}(\mu)\|_1^* \longrightarrow 0\) whence \(\hat{\mu}(\infty) = 0\);

b) \(S_T(\mu) \longrightarrow 0\) in measure as \(T \longrightarrow +\infty\)

whence \(g = 0\) a.e. This leads us to define:

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$W_0$ is the class of measures $\mu$ for which $\|S_T(\mu)\|_1^* \to 0$ as $T \to +\infty$.

We present an elementary structural property of $W_0$, and then show by example that

(A) There exist $M_0$-sets $F$ carrying no measure $\mu \neq 0$ in $W_0$.

The sets $F$ are defined by a purely metrical property, and they need not be especially small. Their construction is based on an idea from the theory of divergent Fourier series [31, Chapter VIII].

(B) The set $F_{\theta}$ of all sums $\sum_0^{\infty} \pm \theta^m (0 < \theta < 1/2)$ carries a measure $\lambda \neq 0$ in $W_0$, provided $F_{\theta}$ is an $M_0$-set.

To elucidate example (B) and the next one we recall that $F_{\theta}$ fails to be an $M_0$-set (or even an $M$-set) unless $\mu_\theta \in \mathbb{R}$, where $\mu_\theta$ is the Bernoulli convolution carried by $F_{\theta}$ and that $\mu_\theta \in \mathbb{R}$ except for certain algebraic numbers $\theta$ [311, p. 147-156]. Therefore the next example is somewhat unexpected.

(C) When $0 < \theta < 1/2$, then $\mu_\theta \notin W_0$, in fact

$$\|S_T(\mu_\theta)\|_1^* \geq c(\theta) > 0$$

for large $T > 0$. We observe in passing that $\mu$ is not known to be singular for $1/2 < \theta < 1$ except when $\mu_\theta \notin \mathbb{R}$, e.g., for $\theta^{-1} = (1 + \sqrt{5})/2$.

From the weak estimate for $S_T$ it is clear that $W_0$ is norm-closed in the space of all measures. We shall prove that when $\mu \in W_0$ and $\psi \in C^1 \cap L^\infty$, then $\psi \mu \in W_0$; consequently the same is true if only $\psi \in L^1(\mu)$. We need two lemmas; the first was already used implicitly.

**Lemma 1.** Let $\mu$ be a measure such that $S_k(\mu) - S_{k+1}(\mu) \to 0$ in measure (over finite intervals). Then $\hat{\mu}(\infty) = 0$, i.e., $\mu \in \mathbb{R}$.

**Proof.** $|D_k(t) - D_{k+1}(t)| \ll \min(1, |t|^{-1}) \equiv K(t)$, say, and $K \in L^2(-\infty, \infty)$. Thus the functions $|S_k(\mu) - S_{k+1}(\mu)|$ have a common majorant $\int K(x - t) |\mu||dt|$ in $L^2$. The hypothesis on
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$S_k - S_{k+1}$ then yields $\|S_k - S_{k+1}\|_2 \rightarrow 0$. This means that

$$\int_k^{k+1} (|\hat{\mu}(t)|^2 + |\hat{\mu}(-t)|^2) \, dt \rightarrow 0$$

so $\hat{\mu}(\infty) = 0$, because $\hat{\mu}$ is uniformly continuous.

**Lemma 2.** Let $\mu \in \mathbb{R}$ and $\psi \in C^1 \cap L^\infty$. Then as $T \rightarrow +\infty$

$$\|S_T(x, \psi \cdot \mu) - \psi(x) S_T(x, \mu)\|_1^* \rightarrow 0.$$  

**Proof.** Since $\mu$ can be approximated in norm by measures $\mu_n \in \mathbb{R}$, each of compact support, we can suppose that $\mu$ itself has compact support, say $|t| \leq a$. Now $S_T(\psi \cdot \mu) - \psi S_T(\mu)$ converges to 0 uniformly on $[-a - 1, a + 1]$, being equal to

$$\pi^{-1} \int \sin T(t - x) \cdot \varphi(x, t) \mu(\,dt),$$

with $\varphi(x, t) = (t - x)^{-1} [\psi(t) - \psi(x)]$; $\varphi(x, t)$ is jointly continuous. This is sufficient to obtain the uniform convergence claimed.

For $|x| > a + 1$ we write

$$x S_T(x, \mu) = \pi^{-1} \int \sin T(t - x) \cdot \sigma(x, t) \mu(\,dt),$$

with $\sigma(x, t) = x (t - x)^{-1}$; now $|\sigma| \leq a + 1$ and

$$\left| \frac{\partial}{\partial t} \sigma(x, t) \right| \leq a + 1,$$

for $|t| \leq a$. Therefore $x S_T(\mu, x) \rightarrow 0$ as $T \rightarrow +0$, uniformly for $|x| \geq a + 1$. The same applies to $x S_T(x, \psi \cdot \mu)$, because $\psi \cdot \mu \in \mathbb{R}$, and these inequalities show that $\psi S_T(\mu) - S_T(\psi \cdot \mu) \rightarrow 0$.

2. Examples.

I. Let $F$ be a compact set in $(-\infty, \infty)$, $0 < \alpha < 1$, $(e_j)$ a sequence decreasing to 0; for each $j$, let $F = \bigcup F_k^j$, where

$$\text{diam}(F_k^j) \leq e_j, \quad d(F_k^j, F_k^\alpha) \geq e_j^\alpha, \quad k \neq \emptyset.$$  

Then $F$ carries no probability measure $\mu$ in $W_0$ (and hence no signed measure $\mu \neq 0$ in $W_0$).

We define the following property of a number $\beta$ in $[0, 1)$, relative to $\mu$ and the sequence of partitions $F = \bigcup F_k^j$:

$$\text{(**)}$$

The total $\mu$-measure of the sets $F_k^j$, such that $\mu(F_k^j) > e_j^\beta$, tends to 0, as $j \rightarrow +\infty$. 

Plainly \( \beta = 0 \) has property (**), because \( \mu \), being an element of \( \mathbb{R} \), can have no discontinuities. We shall prove that if \( \beta \) has property (**), and \( 0 \leq \beta < \alpha \), then \( \gamma = \beta + (1 - \alpha)/2 \) has property (**). This leads to a contradiction as soon as \( \gamma > \alpha \), since the number of sets \( F_k^l \neq \emptyset \) is \( O(e_j^{-\alpha}) \).

Assuming that \( \beta \) has property (**), we form \( \lambda = \lambda_j \), by omitting from \( F_k \) the intervals \( F_k^l \) of \( \mu \)-measure \( > e_j^\beta \). By Kolmogorov's estimate, \( \| S_T(\lambda_j) \|_1^* \to 0 \), as \( j \to +\infty \) and \( T \to +\infty \), independently. Let now \( \int^* \) denote an integral over the domain \( |x - t| > e_j^\alpha/2 \). Then

\[
\int^* |x - t|^{-1} \lambda_j(dt) = 0(e_j^{-\alpha}), \quad \text{if } \beta = 0,
\]

\[
\int^* |x - t|^{-1} \lambda_j(dt) = 0(e_j^{-\beta - \alpha})(\log e_j), \quad 0 < \beta < \alpha.
\]

The first of these is obvious; the second is obtained by packing the subsets \( F_k^l \) as close to \( x \) as is consistent with the condition \( d(F_k^l, F_q^l) \geq e_j^\alpha \).

For each \( k \) such that \( \lambda_j(F_k^l) > e_j^\gamma \), we let \( \xi_k \) belong to \( F_k^l \) and consider the set defined by

\[
(S_k^l) : \frac{1}{2} \lambda(F_k^l) e_j^\sigma < |x - \xi_k| < \lambda(F_k^l) e_j^\sigma,
\]

\[
|\sin e_j^{-\tau}(x - \xi_k)| > \frac{1}{2}
\]

where \( \sigma = -\beta + 3\alpha/4 + 1/4 \), \( \tau = (1 + \gamma + \sigma)/2 \).

The number \( \lambda(F_k^l) e_j^\sigma \) lies between \( e_j^{\beta + \sigma} \) and \( e_j^{\gamma + \sigma} \); we note that \( \beta + \sigma > \alpha \), and \( \gamma + \sigma = 3/4 + \alpha/4 < 1 \). Moreover \( e_j^{-\tau} e_j = o(1) \), while \( e_j^{-\tau} \lambda(F_k^l) e_j^\sigma \to +\infty \).

For each \( k \) in question, the Lebesgue measure of \( S_k^l \) is asymptotically \( c\lambda(F_k^l) e_j^\sigma \), and the different sets are disjoint, because \( \lambda(F_k^l) e_j^\sigma = o(e_j^\sigma) \). We shall prove that \( |S_T(\lambda_j)| > c' e_j^{-\sigma} \) for a certain \( c' > 0 \), with \( T = e_j^{-\tau} \to +\infty \). This will prove that the total \( \mu \)-measure of the subsets \( F_k^l \), such that \( e_j^\gamma < e_j < e_j^\beta \), is \( o(1) \).

When \( x \in S_k^l \),

\[
|S_T(x) - \int_{F_k^l} D_T(x - t) \lambda(dt)| < \int^* |x - t|^{-1} \lambda(dt),
\]

and the error term on the right is \( o(e_j^{-\sigma}) \), because \( \sigma > \alpha - \beta \).
When \( t \in F_k \), \( t - \xi_k = o(x - \xi_k) \) because \( \gamma + \sigma < 1 \), and

\[
\sin T(t - x) = \sin T(\xi_k - x) + o(1)
\]

because \( \tau < 1 \). This easily leads to the lower bound on \( |S_T(x)| \).

Our construction is adapted from Kolmogorov's divergent Fourier series [31, Chapter VIII].

To complete our example, we must present a set \( F \) that is also an \( M_0 \)-set. This is known for various \( M_\beta \)-sets, but seems to occur explicitly in [1]: there exists a closed set \( E \subseteq [0,1] \) and a sequence of integers \( N_k \to +\infty \) such that

1. \( |N_k x| < N_k^{-1} \) (modulo 1) for \( x \in E \), \( k \geq 1 \),
2. The mapping \( y = e^x \) transforms \( E \) onto an \( M_0 \)-set.

Then \( y(E) \) is covered by intervals of length \( \leq 2eN_k^{-2} \), whose distances are at least \( (N_k^{-1} - 2N_k^{-2}) \).

In the remaining examples it is occasionally convenient to write \( S_T(y) \) in place of \( S_T(y, \mu) \), when \( \mu = \mu_\theta \).

II. We present example (C) first, because (B) is based on an improvement in one of the inequalities used in (C). For each \( n = 0, 1, 2, 3, \ldots, F_\theta \) is a union of \( 2^{n+1} \) sets \( E_k \) of diameter \( 2\theta^{n+1}(1 - \theta)^{-1} \), and mutual distances at least

\[
2\theta^{n+1}(1 - 2\theta)(1 - \theta)^{-1} \equiv c_1 \theta^{n+1}; \mu(E_k) = 2^{-n-1}.
\]

The lower bound on the mutual distances gives a Hölder condition on \( \mu : \mu(B) \leq c_2 (\text{diam } B)^\alpha \), where \( \alpha = -\log 2/\log \theta < 1 \). If \( \xi_k \) is the center of \( E_k \), we have an identity

\[
\int_{E_k} f(t) \mu(dt) = 2^{-n-1} \int f(\xi_k + \theta^{n+1} t) \mu(dt).
\]

For each set \( E_k \), we define the set \( E_k^- \) by the inequality

\[
d(x, E_k) < c_1 \theta^{n+1}/3,
\]

so the sets \( E_k^- \) have distances at least \( 2c_1 \theta^{n+1}/3 \). If \( x \in E_k^- \), then

\[
|S_T(x, \mu) - \int_{E_k} D_T(x - t) \mu(dt)| < \int_{R - E_k} |x - t|^{-1} \mu(dt),
\]

and in the last integral, \( |x - t| \geq 2c_1 \theta^{n+1}/3 \). Hence, by the Hölder condition, the integral is \( \leq c_3 (\theta^n)^{\alpha-1} = c_3 2^{-n}\theta^{-n} \). The principal term can be evaluated by the identity above, and simplified to the form

\[
2^{-n}\theta^{-n-1} S_{T_\theta^{n+1}}(\theta^{-n-1} x - \theta^{-n-1} \xi_k).
\]
We observe that
\[ \lim \int S_T(x, \mu) f(x) dx = \int f(x) \mu(dx), \]
for suitable test functions \( f \); for example, this is true if \( f \) and \( \int f(x) \mu(dx) \) are integrable. Since \( \mu \) is singular, we can find a test function \( f \), such that \( \|f\|_1 < 1 \) and \( \int f(x) \mu(dx) > 2c_3 + 2c_1^{-1} \). Hence \( \max |D_T(\mu)| > 2c_3 + 2c_1^{-1} \) for large \( T \), say for \( T > T_0 \).

Let \( T > \theta^{-1} T_0 \), and let \( n \geq 0 \) be chosen so that \( T^* = \theta^{n+1} T \) satisfies the inequalities \( T_0 \leq T^* \leq \theta^{-1} T_0 \). Suppose that
\[ |D_T(\theta^{-n-1} x - \theta^{-n-1} \xi_k)| > c_3 + c_1^{-1}. \]
Then \( d(\theta^{-n-1} x - \theta^{-n-1} \xi_k, F_\theta) < c_1/3 \), since \( \pi > 3 \), or \( d(x, \xi_k + \theta^{n+1} F_\theta) < c_1 \theta^{n+1}/3 \), so \( x \in E_k^- \). Hence
\[ |D_T(x, \mu)| > c_3 \cdot 2^{-n} \theta^{-n+1} - c_3 2^{-n} \theta^{-n} = c_4 2^{-n} \theta^{-n}. \]
But it is easy to see that the set of \( x \)'s in question has measure at least \( c_s 2^n \theta^n \), because \( T_0 \leq T^* \leq \theta^{-1} T_0 \), and the functions \( D_T \) have derivatives bounded by \( \theta^{-2} T_0^2 \). Hence \( \|D_T(\mu)\|_1 \geq c_4 c_5 \).

III. The example (B) requires a complicated construction, but relies in essence on small improvements on estimates already used. To estimate \( S_T(\mu, x) \) we divide the range of integration into the subsets \( \{|x - t| < T^{-1}\} \) and \( \{|x - t| > T^{-1}\} \). The second yields an integral \( O(T^{1-\alpha}) \), by the Hölder condition, and the first yields \( T \cdot O(T^{-\alpha}) = O(T^{1-\alpha}) \) for the same reason (and the inequality \( |D_T| < T \)).

We give another estimate on \( S_T(x, \mu) \) for large \( T \), supposing that \( \mu \in R \).

**Lemma 3.** — To each \( \epsilon > 0 \) there is a \( T_0 \) such that
\[ |S_T(x, \mu)| < \epsilon d(x, F_\theta)^{-1} \]
whenever \( T \geq T_0 \) and \( d \equiv d(x, F_\theta) \geq \epsilon \).

**Proof.** — Let \( \delta = d(x, F) \) and observe that
\[ \delta S_T(x, \mu) = \pi^{-1} \int \sin T(x - t) \cdot \delta \cdot (x - t)^{-1} \mu(dt). \]
The function \( g(t) = \delta \cdot (x - t)^{-1} \) is bounded by 1 on \( F \), and
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\[ |g(t_1) - g(t_2)| \leq \delta^{-1} |t_1 - t_2| \] for numbers $t_1, t_2$ in $F_\theta$. Hence the conclusion follows from our assumption that $\mu \in \mathbb{R}$ and the Tietze extension theorem.

The inequality of the Lemma can be written in a more useful way. When $t \in F_\theta$, then $|x - t| \leq d + 2 \leq d(1 + 2e^{-1})$. Hence $d(x, F_\theta)^{-1} \leq (1 + 2e^{-1}) \int |x - t|^{-1} \mu(dt)$. Suppose now that $x \notin E_k^\infty$ so that $d(\theta^{-n-1}x - \theta^{-n-1}\xi_k, F_0) \geq c_1 \theta^{n+1}/3$. Using the identity for integrals over $E_k$, we find the following estimate:

If $x \notin E_k^\infty$ and $T\theta^{n+1} > T_{oo}$, then

\[ \left| \int_{E_k} D_T(x - t) \mu(dt) \right| < \epsilon \int_{E_k} |x - t|^{-1} \mu(dt). \]

Consequently, when $x \in E_k^\infty$ and $T\theta^{n+1}$ is sufficiently large (depending on $\epsilon > 0$)

\[ |S(x, \mu) - 2^{-n-1}\theta^{-n-1}S_{T\theta^{n+1}}(\theta^{-n-1}x - \theta^{-n-1}\xi_\ell)| < \epsilon \theta^{n(n-1)}. \]

**Lemma 4.** To each $\epsilon > 0$ there is a $\delta > 0$ so that, when $\theta^{-1} < Y < \delta T^{1-\alpha}$ then $Ym\{|S(x, \mu) > Y\} < \epsilon$.

**Proof.** We choose $n \geq 0$ so that $1 < \theta^{n+1}Y^{1/1-\alpha} < \theta^{-1}$; this leads to the inequalities $\theta^{n(n-1)} > Y$, and $T\theta^{n+1} > \delta^{-1}$. For fixed $\ell$, we must estimate the Lebesgue measure of the set defined by

\[ |S_{T\theta^{n+1}}(\mu, \theta^{-n-1}x - \theta^{-n-1}\xi_\ell)| > \frac{1}{2} \cdot 2^{n+1} \theta^{n+1} Y. \]

The right hand side exceeds $\frac{1}{2} \theta^{-1}$; when $T\theta^{n+1}$ is large, the measure of the set is at most $\epsilon \theta^{n+1}$; the total for all $\ell$ is at most $\epsilon 2^{n+1} \theta^{n+1} < eY^{-1}$. Hence $Ym\{|S_T(x, \mu) > Y\} < \epsilon$.

In view of the inequality $|S_T(\mu, x)| = O(T^{-1-\alpha})$, the conclusion of the last lemma holds when $Y > \delta^{-1}T^{1-\alpha}$, $T > 1$, for a certain $\delta > 0$.

In preparation for the next lemma, we recall the identity

\[ (n = 1, 2, 3, \ldots) \]

\[ \int f(t) \mu(dt) = 2^{-n} \sum_{k=1}^{2^n} \int f(\xi_k + \theta^n t) \mu(dt). \]

We define

\[ \int f(t) \sigma_n(dt) = 2^{-n} \sum_{k} \int f(\xi_k + \theta^{n+k} t) \mu(dt). \]
\[ \sigma_n = g_n \cdot \mu, \text{ where } g_n \geq 0, \text{ and } g_n \text{ is continuous on } F_\theta \text{ and takes the values 0 and } 2^k (1 \leq k \leq 2^n). \text{ Using the formula for } \sigma_n \text{ we get an identity} \]

\[ S_T(x, \sigma_n) = 2^{-n} \theta^{-n} \sum_k \theta^{-k} S_{T^n 2^k}(\theta^{-n-k} x - \theta^{-n-k} \xi_k). \]

**Lemma 5.** - To each \( \varepsilon > 0 \), there is an \( N > 1 \) such that \( \limsup \sup_{T \to +\infty} ||S_T(\sigma_n)||^*_1 < \varepsilon \), if \( n \geq N \).

**Proof.** - In calculating \( \limsup \sup_{T \to +\infty} ||S_T(\sigma_n)||^*_1 \) we can omit \( x \)'s outside \((-3,3)\), because \( \sigma_n \in R \). In an obvious notation we write \( \sigma_n = \sum_k \sigma_{n,k} \), and observe that, for \( T > T_n, \varepsilon \)

\[ ||S_T(\sigma_n)|| < \max_k ||S_T(\sigma_{n,k})|| + \varepsilon/12. \]

When \( Y > \varepsilon/6 \) (the others are trivial, since we suppose that \( |x| < 6 \)),

\[ m\{|S_T(\sigma_n)| > 2Y\} \leq \sum_k m\{|S_T(\sigma_{n,k})| > Y\} < \sum_k \theta^n m\{|S_{T^n 2^k}(x, \mu)| > 2^n \theta^{n+k} Y\}. \]

Each summand is \( O(2^{-n} Y^{-1}) \) by Kolmogorov's inequality; if \( T \theta^{n+k} > 1 \), then the \( k \)-th term exceeds \( \varepsilon 2^{-n} Y \) only if

\[ \delta (T \theta^{n+k})^{1-\alpha} < Y < \delta^{-1} (T \theta^{n+k})^{1-\alpha}, \]

by Lemma 4 and the remark after it, and this inequality occurs for at most \( 2(1-\alpha)^{-1} \cdot \log \delta/\log \theta \) indices \( k = 1, \ldots, 2^n \). (We assume that \( Y > \theta^{-1} \), since \( S_T(\sigma_n) \to 0 \) almost everywhere as \( T \to +\infty \).) This proves our lemma.

A further property of \( \sigma_n \), obtained simply by increasing \( n \), is the inequality \( |\sigma_n(I) - \mu(I)| < \varepsilon \) for all intervals \( I \).

The next lemma establishes a property of the functional \( || ||^*_1 \) to simplify the remaining calculations.

**Lemma 6.** - Let \( a_i = ||f_i||^*_1 \) \( 1 \leq i \leq N \). Then

\[ ||\Sigma f_i||^*_1 \leq (\Sigma a_i^{1/2})^2. \]

**Proof.** - Let \( 0 \leq t_i \leq 1 \), and \( \Sigma t_i = 1 \). Then

\[ m\{|\Sigma f_i| \geq Y\} \leq \Sigma m\{|f_i| \geq t_i Y\} \leq \Sigma t_i^{-1} Y^{-1} a_i. \]
The minimum of the sum is $Y^{-1}(\Sigma a_i^{1/2})^2$. With a little more effort, we can obtain the bound $c(1-p)^{-1}(\Sigma a_i^p)^{1/p}$, $0 < p < 1$.

We are now in a position to construct the measure $\lambda$. We shall find probability measures $\lambda_k = f_k \mu$, with $f_k \geq 0$, $\int f_k d\mu = 1$, such that $\|S_T(\lambda_k)\|_1^* < k^{-1}$ for $T > T_k > T_{k-1} \ldots$ and $|\hat{\lambda}_k(u)| < k^{-2}$ for $u > T_k$. Lemma 5 provides $\lambda_1$; let us suppose that $\lambda_k$ and $T_k$ are known. We find $\sigma_k$ so that $|\sigma_k(I) - \lambda_k(I)| < k^{-1}(1 + T_k)^{-2}$ and $\|S_T(\sigma_k)\|_1^* < k^{-4}/25$, and $|\hat{\sigma}_k(u)| < k^{-1}$, for $u > T_{k+1}^0 > T_k$. (The construction of $f_{k+1} \mu$ from $f_k \mu$ follows Lemma 5). We now set $\lambda_{k+1} = (1 - k^{-1/2}) \lambda_k + k^{-1/2} \sigma_k$; by Lemma 6, we have for $T > T_{k+1}^0$

$$\|S_T(\lambda_{k+1})\|_1^* \leq (1 - k^{-1/2})^{1/2} k^{-1/2} + k^{-2}/5.$$ 

When $k = 1$, the last bound is $1/5$, while $(k + 1)^{-1} = 1/2$. For $k \geq 2$, we need the inequality

$$(1 - k^{-1/2})^{1/2} k^{-1/2} + k^{-2}/5 < (k + 1)^{-1/2},$$

which can be verified with the aid of calculus. Clearly, we have $|\hat{\lambda}_{k+1}(u)| < (k + 1)^{-2}$ for $T > T_{k+1}^0$; we take $T_{k+1} = T_{k+1}^0 + T_{k+1}^0$.

By the construction, and integration by parts,

$$|\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| \leq k^{-3/2}(1 + T_k)^{-2} |u|;$$

consequently $|\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| \leq k^{-3/2}$ unless $|u| > 1 + T_k$.

However, if $|u| > T_{k+1}^0 > T_k$, then $|\hat{\lambda}_k(u) - \hat{\lambda}_{k+1}(u)| < 2k^{-2}$. Since $|\hat{\lambda}_k - \hat{\lambda}_{k+1}| \leq 2k^{-1/2}$, we have a limit $\varphi(u)$, with

$$|\varphi - \hat{\lambda}_k| = O(k^{-1/2}).$$

Hence $\varphi = \hat{\lambda}$, with $\lambda$ carried by $F_\theta$ and $\lambda \in \mathbb{R}$.

In verifying that $\lim \|S_T(\lambda)\|_1^* = 0$ we can calculate the weak norms over $(-3,3)$. Suppose that $T_{k-1} \leq T \leq T_k$; then

$$|S_T(\lambda_k) - S_T(\lambda)| = O(k^{-1/2}).$$

Since $T > T_{k-1}$, $\|S_T(\lambda_{k-1})\|_1^* < (k - 1)^{-1}$; and finally

$$\|S_T(\lambda_k) - S_T(\lambda_{k-1})\|_1^* = O(k^{-1/2}).$$

Hence $\|S_T(\lambda)\|_1^* = O(k^{-1/2})$ over $(-3,3)$. 


BIBLIOGRAPHY


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Robert KAUFMAN,
University of Illinois at
Urbana-Champaign
Department of Mathematics
Urbana, Ill. 61801 (U.S.A.).