

ANNALES DE L'INSTITUT FOURIER

H. G. DALES

W. K. HAYMAN

**Esterlè's proof of the tauberian theorem
for Beurling algebras**

Annales de l'institut Fourier, tome 31, n° 4 (1981), p. 141-150

http://www.numdam.org/item?id=AIF_1981__31_4_141_0

© Annales de l'institut Fourier, 1981, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

ESTERLE'S PROOF OF THE TAUBERIAN THEOREM FOR BEURLING ALGEBRAS

by H. G. DALES and W. K. HAYMAN

1. Introduction.

In [5], J. Esterle gave a new proof of the Wiener Tauberian theorem for the algebra $L^1(\mathbf{R})$ by using some results from complex analysis and from the theory of radical Banach algebras. In this note, we show that a proof with the same idea also establishes the analogous result for Beurling algebras.

We first give the basic properties of the algebras of Beurling that we are considering.

Let φ be a non-negative, measurable function on \mathbf{R} , and set

$$L_\varphi^1 = \left\{ f : \|f\| = \int_{-\infty}^{\infty} |f(t)|e^{\varphi(t)} dt < \infty \right\}.$$

Then L_φ^1 is a Banach space : as usual, we equate functions equal almost everywhere. If

$$(1) \quad \varphi(s+t) \leq \varphi(s) + \varphi(t) \quad (s, t \in \mathbf{R}),$$

then L_φ^1 is a commutative Banach algebra with respect to convolution multiplication defined by the equation

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s) ds \quad (f, g \in L_\varphi^1).$$

These algebras were introduced by Beurling in 1938 [1].

Condition (1) ensures the existence of the finite limits $\alpha = \lim_{t \rightarrow \infty} \varphi(t)/t$ and $\beta = \lim_{t \rightarrow -\infty} \varphi(t)/t$. Let Π be the open strip $\{-\alpha < \operatorname{Re} z < -\beta\}$, and let $\bar{\Pi}$ be the closed strip $\{-\alpha \leq \operatorname{Re} z \leq -\beta\}$ of \mathbf{C} : if $\alpha = \beta$, then $\bar{\Pi}$ is a line. For $f \in L_\varphi^1$, we define the Laplace transform, \hat{f} , of f on $\bar{\Pi}$ by

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(t)e^{-zt} dt \quad (z \in \bar{\Pi}).$$

The integral converges absolutely for $z \in \bar{\Pi}$. Let $A_0(\bar{\Pi})$ denote the uniform algebra of functions which are continuous on $\bar{\Pi}$, analytic on Π , and which converge uniformly to zero as $z \rightarrow \infty$ with $z \in \bar{\Pi}$. Then $\hat{f} \in A_0(\bar{\Pi})$. It is well known (for example, see [6], §18) that the character space, or space of maximal modular ideals, of L_φ^1 can be identified with $\bar{\Pi}$, and that the map $f \mapsto \hat{f}$ is a monomorphism of L_φ^1 into $A_0(\bar{\Pi})$.

Let I be a closed ideal of L_φ^1 . We are interested in conditions on I which ensure that $I = L_\varphi^1$. Let

$$Z(I) = \{z \in \bar{\Pi} : \hat{f}(z) = 0 \quad (f \in I)\}.$$

Clearly, a necessary condition for the equality $I = L_\varphi^1$ is that $Z(I) = \emptyset$. Wiener posed the problem for the algebra $L^1(\mathbf{R})$ (for which $\varphi = 0$), and he proved that, if $Z(I) = \emptyset$, then $I = L^1(\mathbf{R})$. This is Wiener's Tauberian theorem; of course, the formulation of Wiener was different.

DEFINITION. — *Let L_φ^1 be a Beurling algebra. Then spectral analysis holds for L_φ^1 if each proper closed ideal of L_φ^1 is contained in a maximal modular ideal of L_φ^1 .*

Clearly, spectral analysis holds for L_φ^1 if and only if $I = L_\varphi^1$ for each I with $Z(I) = \emptyset$, and Wiener's theorem is that spectral analysis holds for $L^1(\mathbf{R})$.

It was shown by Beurling in [1] that spectral analysis holds for the algebra L_φ^1 provided that the weight φ satisfies (1) and the additional condition that

$$(2) \quad \int_{-\infty}^{\infty} \frac{\varphi(t)}{1+t^2} dt < \infty.$$

(Note that this condition implies that $\alpha = \beta = 0$, and so in this case we are identifying the character space of L_φ^1 with the imaginary axis.)

Modern proofs of the theorem of Beurling use only the fact, ensured by (2), that the Banach algebra L_ϕ^1 is regular, in the sense that, given $y_0 \in \mathbf{R}$ and a neighbourhood U of y_0 , there exists $f \in L_\phi^1$ with $\hat{f}(iy_0) = 1$ and $\hat{f}(iy) = 0$ ($y \notin U$): see [6], § 40, for example, for a proof of the theorem given that L_ϕ^1 is regular. Indeed, Gurarii ([7], page 24) states, « all proofs of Wiener's theorem known to us make essential use of this fact of regularity, and... it is hardly possible to manage without it. » Following the ideas of Esterle in [5], we shall prove Beurling's result without using the regularity of L_ϕ^1 . It is not claimed that the present proof is any shorter than the usual one.

It is perhaps worth recalling how the regularity of L_ϕ^1 follows from condition (2). The starting point is a result which is essentially Theorem XII of [10]: if ϕ is a non-negative, measurable function on \mathbf{R} , then a necessary and sufficient condition that there exists a function f which is bounded and analytic in the open upper half-plane Π^+ and which is such that $\lim_{y \rightarrow 0^+} |f(x+iy)| = \exp(-\phi(x))$ for almost all x is that ϕ satisfies (2).

To show the sufficiency of (2), suppose that ϕ satisfies this condition, and define u on Π^+ by

$$u(x+iy) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\phi(t) dt}{(x-t)^2 + y^2}.$$

Then u is harmonic on Π^+ and has non-tangential limits agreeing with ϕ at almost every point of \mathbf{R} . Let v be the harmonic conjugate of u , and set $f = \exp(-u-iv)$. This function f has the required properties.

To conclude the proof that L_ϕ^1 is regular if ϕ satisfies condition (2), take $y_0 \in (a,b) \subset \mathbf{R}$. Construct a function f_0 which is analytic and bounded in Π^+ and which is such that

$$|f_0(x)| < \frac{e^{-\phi(x)}}{1+x^2} \quad (x \in \mathbf{R}).$$

Let $f_1(z) = f_0(z)/(z+i)$, so that $f_1|_{\mathbf{R}} \in L_\phi^1$. Also, $|f_1(z)| \rightarrow 0$ as $z \rightarrow \infty$ in Π^+ , and so $\hat{f}_1(iy) = 0$ for $y \leq 0$. We can clearly choose $\alpha \in \mathbf{R}$ so that, if $g_1(x) = f_1(x)e^{i\alpha x}$, then $\hat{g}_1(iy_0) \neq 0$ and $\hat{g}_1(iy) = 0$ ($y < a$). Similarly, there exists $g_2 \in L_\phi^1$ with $\hat{g}_2(iy_0) \neq 0$ and $\hat{g}_2(iy) = 0$ ($y > b$). If $h = g_1 * g_2$, then $h \in L_\phi^1$, $h(iy_0) \neq 0$, and $h(iy) = 0$ ($y \notin (a,b)$). This shows that L_ϕ^1 is regular.

In fact, the Banach algebra L_ϕ^1 is regular if and only if condition (2)

holds. The strongest result of this type is the famous theorem of Beurling and Malliavin [2] which shows that, if φ is a non-negative, measurable function on \mathbf{R} , then the following two conditions on φ are equivalent :

- (i) for each $a > 0$, the Banach space L_φ^1 contains a non-zero element whose Fourier transform has support in $[-ia, ia]$;
- (ii) φ satisfies (2) and the condition that

$$\text{ess sup } \{|\varphi(s+t) - \varphi(s)| : s \in \mathbf{R}\} < \infty \quad (t \in \mathbf{R}).$$

Let φ be a function satisfying (1), and let α and β be the limits defined above. The algebra L_φ^1 is termed *analytic* if $\beta > \alpha$. If $\alpha = \beta = 0$, then L_φ^1 is *quasi-analytic* if the integral in (2) diverges, and L_φ^1 is *non-quasi-analytic* if condition (2) holds. Thus, our theorem is that spectral analysis holds in the non-quasi-analytic case.

In fact, spectral analysis fails in both the analytic and in the quasi-analytic cases. This was first proved by Vretblad in [11] provided that φ satisfies some slight extra conditions. We are grateful to Professor Yngve Domar for pointing out that the proof of Theorem 4 in [4] implicitly shows this result without any extra conditions on φ . Thus, spectral analysis holds for the Beurling algebra L_φ^1 if and only if φ satisfies condition (2).

In the special case that $\varphi(t) = \alpha|t|$ for a positive constant α , the family of all proper closed ideals of L_φ^1 which are not contained in any maximal modular ideal was described by Korenblum ([9]). The family does not seem to have been fully described in more general cases : see [7] and [11] for the best partial results.

2. The proof.

THEOREM. — *Let φ be a non-negative, measurable function on \mathbf{R} which satisfies (1) and (2). Then spectral analysis holds for the Banach algebra L_φ^1 .*

The proof of this theorem depends heavily on a recent result given in [8] which we first describe. We write Δ for the open unit disc, and, for each $\sigma \in \mathbf{R}$, we write Π_σ for the open right half-plane $\{(x,y) : x > \sigma\}$.

LEMMA 1. — *Let k be a positive, continuous, increasing function on $[0,1)$. Let f be analytic on Δ and satisfy the condition that*

$$(3) \quad \log |f(re^{i\theta})| \leq k(r) \quad (re^{i\theta} \in \Delta).$$

If

$$(4) \quad \int_0^1 \left(\frac{k(r)}{1-r} \right)^{\frac{1}{2}} dr < \infty,$$

then either $f = 0$, or $\limsup_{r \rightarrow 1^-} (1-r) \log |f(r)| > -\infty$.

Proof. – Theorem 5 of [8] shows that, under the hypotheses (3) and (4), there exists an analytic function g on Δ such that :

- (i) g is real and increasing on $[0,1)$, with $g(r) \rightarrow 1$ as $r \rightarrow 1 -$;
- (ii) $g(\Delta) \subset \Delta$;
- (iii) $\sup \{ |1-g(r)|/|1-r| : r \in [0,1) \} < \infty$;
- (iv) $f \circ g$ has bounded (Nevanlinna) characteristic in Δ .

It follows from (ii) and (iii) by the theory of the angular derivative that

$$(5) \quad \lim_{r \rightarrow 1^-} \frac{1-g(r)}{1-r}$$

exists in $(0, \infty)$. (The existence of this limit can also be seen from the explicit construction of g in [8], pp. 192-193.)

Suppose that $f \neq 0$. By (iv), there exist bounded, non-zero, analytic functions, say h_1 and h_2 , on Δ such that $f \circ g = h_1/h_2$ on Δ . If $\limsup_{r \rightarrow 1^-} (1-r) \log |(f \circ g)(r)| = -\infty$, then $\limsup_{r \rightarrow 1^-} (1-r) \log |h_1(r)| = -\infty$, and so, by a result of Phragmén-Lindelöf type ([3], 1.4.3, transferred from Π_0 to Δ), $h_1 = 0$, a contradiction. It follows that $\limsup_{r \rightarrow 1^-} (1-r) \log |(f \circ g)(r)| > -\infty$.

The lemma follows from the existence of the finite non-zero limit given by (5).

Condition (4) in the above lemma is necessary in the sense that, if the integral in (4) diverges, then there exists a non-zero analytic function f on Δ satisfying (3) and such that $(1-r) \log |f(r)| \rightarrow -\infty$ as $r \rightarrow 1 -$: see [8], Theorem 4.

We transform this result to the half-plane Π_1 . Throughout, if K is a positive, continuous function on $[1, \infty)$, we set

$$J(K) = \int_1^\infty \left(\frac{K(R)}{R^3} \right)^{\frac{1}{2}} dR.$$

LEMMA 2. — Let K be a positive, continuous, increasing function on $[1, \infty)$ such that $J(K) < \infty$.

Let F be analytic on Π_1 , and let F satisfy the condition that

$$\log |F(\rho e^{i\psi})| \leq K\left(\frac{\rho}{\cos \psi}\right) \quad (\rho e^{i\psi} \in \Pi_1).$$

Then either $F = 0$, or $\limsup_{\rho \rightarrow \infty} \rho^{-1} \log |F(\rho)| > -\infty$.

Proof. — Let $\zeta = \xi + i\eta = \rho e^{i\psi}$ belong to Π_1 , and let $z = (\zeta - 3)/(\zeta + 1)$ define a conformal map of Π_1 onto Δ . Then $\zeta = (3 + z)/(1 - z)$. Let $f(z) = F(\zeta)$, so that f is an analytic function on Δ . If $|z| = r < 1$, then

$$r^2 = \left| \frac{\zeta - 3}{\zeta + 1} \right|^2 = 1 - \frac{8(\xi - 1)}{(\xi + 1)^2 + \eta^2} > 1 - \frac{8\xi}{\xi^2 + \eta^2},$$

so that

$$\frac{\rho}{\cos \psi} = \frac{\xi^2 + \eta^2}{\xi} < \frac{8}{1 - r^2} < \frac{8}{1 - r}.$$

Hence, $\log |f(re^{i\theta})| \leq k(r)$ for $re^{i\theta} \in \Delta$, where

$$k(r) = K\left(\frac{8}{1 - r}\right).$$

Then k is a positive, continuous, increasing function on $[0, 1)$, and

$$\int_0^1 \left(\frac{k(r)}{1 - r}\right)^{\frac{1}{2}} dr = 8^{\frac{1}{2}} \int_8^\infty \left(\frac{K(R)}{R^3}\right)^{\frac{1}{2}} dR,$$

and so k satisfies condition (4). By Lemma 1, either $f = 0$ or $\limsup_{r \rightarrow 1^-} (1 - r) \log |f(r)| > -\infty$. In the former case, $F = 0$, and in the

latter case, $\limsup_{\rho \rightarrow \infty} \rho^{-1} \log |F(\rho)| > -\infty$, as required.

If F is an analytic function on Π_0 such that $\sup \{\exp(-|z|^\alpha) |F(z)|\} < \infty$ for some $\alpha < 1$, then, by applying Lemma 2 with $K(R) = R^\alpha$, we can deduce that either $F = 0$, or

$\limsup_{\rho \rightarrow \infty} \rho^{-1} \log |F(\rho)| > -\infty$. This is Corollary 2.2 of [5], and the theorem of Esterle followed from that Corollary. The present more general result will require the stronger Lemma 2.

Now, following [5], we introduce the functions a^ζ :

$$a^\zeta(t) = \frac{1}{\sqrt{\pi\zeta}} \exp\left(-\frac{t^2}{\zeta}\right) \quad (\zeta \in \Pi_0, t \in \mathbf{R}).$$

Since $\varphi(t) = O(|t|)$ as $|t| \rightarrow \infty$, we have $a^\zeta \in L_\varphi^1$ for each $\zeta \in \Pi_0$. It is well known and straightforward to check that the map $\zeta \mapsto a^\zeta$, $\Pi_0 \rightarrow L_\varphi^1$, is a semigroup monomorphism and an analytic map. We must calculate $\|a^\zeta\|$ in L_φ^1 . We first give a technical lemma.

LEMMA 3. — Let φ be a non-negative, measurable function on \mathbf{R} satisfying (1) and such that $\int_0^\infty (1+t^2)^{-1}\varphi(t) dt < \infty$.

(i) If $\varphi_1(t) = \max\{\varphi(s) : 0 \leq s \leq t\}$ ($t \in \mathbf{R}^+$), then φ_1 is monotone increasing on \mathbf{R}^+ , $\varphi_1(t) \geq \varphi(t)$ ($t \in \mathbf{R}^+$), and $\int_1^\infty t^{-2}\varphi_1(t) dt < \infty$.

(ii) If $\varphi_2(t) = t \max\{s^{-1}\varphi_1(s) : s \geq t\}$ ($t \in \mathbf{R}^+$), then $t^{-2}\varphi_2(t)$ is a monotone decreasing function of t on \mathbf{R}^+ , $\varphi_2(t) \geq \varphi_1(t)$ ($t \in \mathbf{R}^+$), and $\int_1^\infty t^{-2}\varphi_2(t) dt < \infty$.

Proof. — These results are obvious or are proved clearly in Lemmas 3.3 and 3.4 of [7]; they are originally due to Beurling.

LEMMA 4. — Let φ be a non-negative, measurable function on \mathbf{R} satisfying (1) and (2). Then there exists a positive, continuous, increasing function K on $[1, \infty)$ with $J(K) < \infty$ such that

$$(7) \quad \log \|a^\zeta\| \leq K\left(\frac{\rho}{\cos \psi}\right) \quad (\zeta = \rho e^{i\psi} \in \Pi_1).$$

Here, $\|a^\zeta\|$ is calculated in L_φ^1 .

Proof. — Let $\zeta = \rho e^{i\psi} \in \Pi_1$. We have

$$\|a^\zeta\| = \frac{1}{\sqrt{\pi\rho}} \int_{-\infty}^\infty \exp\left(-\frac{t^2}{\rho} \cos \psi + \varphi(t)\right) dt.$$

Since $\rho \geq 1$,

$$\begin{aligned} \|a^\zeta\| &\leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{R} + \varphi(t)\right) dt \\ &= \exp K(R), \text{ say,} \end{aligned}$$

where $R = \rho/\cos \psi \geq 1$. Clearly, replacing K by $\sup\{K, 0\}$, we can suppose that K is positive, continuous, and increasing on $[1, \infty)$. To show that $J(K) < \infty$, it suffices to show that $J(\log^+ \kappa) < \infty$, where

$$\kappa(R) = \int_0^\infty \exp\left(-\frac{t^2}{R} + \varphi(t)\right) dt = R^{\frac{1}{2}} \int_0^\infty \exp(-s^2 + \varphi(R^{\frac{1}{2}}s)) ds.$$

Let φ_1 and φ_2 be as specified in Lemma 3. We can suppose that $\varphi_2(1) = 1$. For each $R \geq 1$, let

$$\mu(R) = \sup\{t : 2\varphi_2(t)R \geq t^2\}, \quad v(R) = R^{-\frac{1}{2}}\mu(R).$$

Then $v(R)$ is the supremum of the solutions of the inequality $\varphi_2(R^{\frac{1}{2}}s) \geq \frac{1}{2}s^2$. Since $\varphi(t) = O(t)$ as $t \rightarrow \infty$, $\mu(R) = O(R)$ as $R \rightarrow \infty$.

If $s \geq v(R)$, then $\varphi(R^{\frac{1}{2}}s) \leq \varphi_2(R^{\frac{1}{2}}s) \leq \frac{1}{2}s^2$, and so

$$\int_{v(R)}^\infty \exp(-s^2 + \varphi(R^{\frac{1}{2}}s)) ds \leq \int_0^\infty \exp(-\frac{1}{2}s^2) ds < \infty.$$

If $s \leq v(R)$, then $\varphi(R^{\frac{1}{2}}s) \leq \varphi_1(R^{\frac{1}{2}}s) \leq \varphi_1(\mu(R)) \leq \varphi_2(\mu(R)) \leq \frac{1}{2}R^{-1}(\mu(R))^2$, and so

$$\int_0^{v(R)} \exp(-s^2 + \varphi(R^{\frac{1}{2}}s)) ds \leq R^{-\frac{1}{2}}\mu(R) \exp\left[\frac{(\mu(R))^2}{2R}\right].$$

Thus, $\log \kappa(R) \leq \frac{1}{2}R^{-1}(\mu(R))^2 + O(\log R)$ as $R \rightarrow \infty$, and so

$$J(\log^+ \kappa) \leq \int_1^\infty \frac{\mu(R)}{R^2} dR + O(1) \text{ as } R \rightarrow \infty.$$

Using the definition of $\mu(\mathbf{R})$ and Lemma 3, we see that

$$\int_1^\infty \frac{\mu(\mathbf{R})}{\mathbf{R}^2} d\mathbf{R} - 1 = \int_1^\infty \frac{d\mu(\mathbf{R})}{\mathbf{R}} = 2 \int_1^\infty \frac{\varphi_2(t)}{t^2} dt < \infty.$$

Thus, $J(\log^+ \kappa) < \infty$, as required.

LEMMA 5. — *If A is a radical Banach algebra, and if (a^t) is a continuous semigroup in A over \mathbf{R}^+ , then $\lim_{t \rightarrow \infty} t^{-1} \log \|a^t\| = -\infty$.*

Proof. — This is [5], Lemma 2.3.

We now conclude the proof of the theorem.

Let I be a closed ideal of L_ϕ^1 . We must show that, if I is not contained in a maximal modular ideal of L_ϕ^1 , then $I = L_\phi^1$. Let $A = L_\phi^1/I$. Then the hypothesis is that A is a radical Banach algebra.

Let (a^ζ) be the analytic semigroup in L_ϕ^1 given above, and let $[a^\zeta]$ be the coset of a^ζ in A . Let $\lambda \in A'$, the dual space of A , and set

$$\Phi(\zeta) = \langle [a^\zeta], \lambda \rangle \quad (\zeta \in \Pi_0).$$

Then Φ is an analytic function over Π_0 , and

$$|\Phi(\zeta)| \leq \|\lambda\| \|[a^\zeta]\| \leq \|\lambda\| \|a^\zeta\| \quad (\zeta \in \Pi_0).$$

By Lemma 4, there is a function K such that $J(K) < \infty$ and such that $\log |\Phi(\zeta)| \leq K(\mathbf{R})$ for $\zeta \in \Pi_1$, where $\zeta = \rho e^{i\psi}$ and $\mathbf{R} = \rho/\cos \psi$. By Lemma 5, $\lim_{\rho \rightarrow \infty} \rho^{-1} \log |\Phi(\rho)| = -\infty$, and so, by Lemma 2, $\Phi = 0$. This shows that $[a^\zeta] = 0$ in A , and hence that $a^\zeta \in I$ for $\zeta \in \Pi_0$. However, for each $f \in L_\phi^1$, $f = \lim_{\rho \rightarrow 0^+} f * a^\rho$, and so $f \in \bar{I} = I$. Thus $I = L_\phi^1$, as required.

The use of Lemma 2 in the above theorem seems to be necessary. For example, consider the case that $\varphi(t) = |t|^\beta$, where $0 < \beta < 1$, and take (a^ζ) as above. Then the best estimate of $\|a^\zeta\|$ in terms of $\rho = |\zeta|$ which we can obtain is that $\log \|a^\zeta\| = O(\rho^{2\beta/(2-\beta)})$ as $\rho \rightarrow \infty$ with $\zeta \in \Pi_1$: here we are using the fact that $1/\cos \theta \leq \rho$ for $\zeta \in \Pi_1$. We can thus apply [5], Corollary 2.2, only if $2\beta/(2-\beta) < 1$, that is, if $\beta < 2/3$, whereas the result holds if $\beta < 1$.

BIBLIOGRAPHY

- [1] A. BEURLING, Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionnelle, *Neuvième Congr. Math. Scandinaves*, (Helsinki, 1938), Tryckeri, Helsinki (1939), 199-210.
- [2] A. BEURLING and P. MALLIAVIN, The Fourier transforms of measures with compact support, *Acta Math.*, 107 (1962), 291-309.
- [3] R. P. BOAS, Jr., *Entire functions*, Academic Press, New York, 1954.
- [4] Y. DOMAR, Translation invariant subspaces of weighted l^p and L^p spaces, *Math. Scand.*, 49 (1981), to appear.
- [5] J. ESTERLE, A complex-variable proof of the Wiener Tauberian theorem, *Ann. Inst. Fourier*, Grenoble, 30 (1980), 91-96.
- [6] I. M. GELFAND, D. A. RAIKOV and G. E. SHILOV, *Commutative normed rings*, Chelsea Publishing Co., New York, 1964.
- [7] V. P. GURARII, Harmonic analysis in spaces with a weight, *Trudy Moskov. Mat. Obsc.*, 36 (1976), 21-76. = *Trans. Moscow Math. Soc.*, 35 (1979), 21-75.
- [8] W. K. HAYMAN and B. KORENBLUM, An extension of the Riesz-Herglotz formula, *Annales Academiae Scientiarum Fennicae, Series A1, Mathematica*, 2 (1976), 175-201.
- [9] B. KORENBLUM, A generalization of Wiener's Tauberian theorem and harmonic analysis of rapidly increasing functions, (Russian), *Trudy Moskov. Mat. Obsc.*, 7 (1958), 121-148.
- [10] R.E.A.C. PALEY and N. WIENER, Fourier transforms in the complex domain, *American Math. Soc. Colloquium Publications*, XIX, New York, 1934.
- [11] A. VRETBAD, Spectral analysis in weighted L^1 spaces on \mathbf{R} , *Ark. Math.*, 11 (1973), 109-138.

Manuscrit reçu le 26 janvier 1981.

H. G. DALES,
School of Mathematics
University of Leeds
Leeds LS2 9JT (England).

W. K. HAYMAN,
Department of Mathematics
Imperial College of Science and
Technology
London SW7 2BZ (England).