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SPHERICAL SUMMATION:
A PROBLEM OF E. M. STEIN

by A. CÓRDOBA and B. LÓPEZ-MELERO

In this paper we present a proof of a conjecture formulated by E.M. Stein [1], page 5, about the spherical summation operators. We obtain a stronger version of the Carleson-Sjölin theorem [2] and, as a corollary, we obtain a.e. convergence for lacunary Bochner-Riesz means.

With $\lambda > 0$ let $T^\lambda_R$ denote the Fourier multiplier operator given by

$$(T^\lambda_R f)^*(\xi) = (1 - |\xi|^2 / R^2)^{\lambda}\hat{f}(\xi) \quad \text{for} \quad f \in \mathcal{S}(\mathbb{R}^2),$$

and let $\{R_j\}$ be any sequence of positive numbers.

**Theorem 1.** - Given $\lambda > 0$ and

$$\frac{4}{3 + 2\lambda} < p < \frac{4}{1 - 2\lambda},$$

there exists some positive constant $C_{\lambda,p}$ such that

$$\| \sum_j T^\lambda_{R_j} f_j \|_p \leq C_{\lambda,p} \left( \sum_j |f_j|^2 \right)^{1/2}. $$

Let $T^\lambda_* f = \sup_j |T^\lambda_{R_j} f_j|$. The methods developed to prove Theorem 1 yield, as an easy consequence, the following result.

**Theorem 2.** - For $\lambda > 0$ and

$$\frac{4}{3 + 2\lambda} < p < \frac{4}{1 - 2\lambda},$$

there exists some constant $C_{\lambda,p}'$ such that

$$\| T^\lambda_* f \|_p \leq C_{\lambda,p}' \| f \|_p.$$
As a result we have, for $f \in L^p(\mathbb{R}^2)$

$$f(x) = \lim_{j} T_{2j}^\lambda f(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^2.$$  

As part of the machinery in the proofs of Theorems 1 and 2 we shall make use of the two following results, whose proofs can be found in [3] and [4].

Given a real number $N > 1$ consider the family $B$ of all rectangles with eccentricity $N$ and arbitrary direction, and let $M$ be the associated maximal operator

$$Mf(x) = \sup_{x \in \mathbb{R} \in B} \frac{1}{|R|} \int_R f(x) \, dx.$$  

**Theorem 3.** — There exist constants $C, \alpha$ independent of $N$ such that

$$\|Mf\|_2 \leq C \|\log N\|^\alpha \|f\|_2.$$  

Consider a disjoint covering of $\mathbb{R}^n$ by a lattice of congruent parallelepipeds $\{Q^\nu\}_{\nu \in \mathbb{Z}^n}$ and the associated multiplier operators

$$(P^\nu f)^\prec = \chi_{Q^\nu} \hat{f}.$$  

**Theorem 4.** — For each $s > 1$ there exists a constant $C_s$ such that, for every non negative, locally integrable function $\omega$ and every $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \sum_{\nu} |P^\nu f(x)|^2 \, \omega(x) \, dx \leq C_s \int_{\mathbb{R}^n} |f(x)|^2 \, A_{\alpha} \omega(x) \, dx$$  

where $A_{\alpha} g = [M(g^s)]^{1/s}$ and $M$ denotes the strong maximal function in $\mathbb{R}^n$.

**Proof of Theorem 1.** — Suppose that $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function supported in $[-1, +1]$, and consider the family of multipliers $S^\delta_j$ defined by

$$(S^\delta_j f)^\prec (\xi) = \phi(\delta^{-1}(R_j^{-1} |\xi| - 1)) \hat{f}(\xi)$$  

and also, for a fixed $\delta > 0$, consider the family

$$(T^\delta_j f)^\prec (\xi) = \psi_n(\arg(\xi)) (S^\delta_j f)^\prec (\xi)$$  

where the $\psi_n$ are a smooth partition of the unity on the circle,
\[ 1 = \sum_{n=1}^{N} \psi_n ; \]

\( \psi_n \) is supported on \( \left| \frac{N}{2\pi} \theta - n \right| \ll 1 \) and \( N = [\delta^{-1/2}] \), so that the support of \((T^n f)^\perp\) is much like a rectangle with dimensions \( R_j \delta \times R_j \delta^{1/2} \).

There are three main steps in our proof.

a) The same argument of ref. [3] allows us to reduce theorem 1 to prove the following inequality

\[ \left\| \sum_j |S_j f_j|^2 \right\|_{1/2}^4 \leq C |\log \delta|^{\delta} \left\| \sum_j |f_j|^2 \right\|_{1/2}^4. \quad (1) \]

b) With adequate decompositions of the multipliers and geometric arguments, we prove

\[ \left\| \sum_j |S_j f_j|^2 \right\|_{1/2}^4 \leq C' |\log \delta| \left\| \sum_{j,n} |T^n f_j|^2 \right\|_{1/2}^4. \quad (2) \]

c) An estimate of the kernels of \( T^n_j \), together with theorems 3 and 4 yields,

\[ \left\| \sum_{j,n} |T^n f_j|^2 \right\|_{1/2}^4 \leq C'' |\log \delta|^{\delta} \left\| \sum_j |f_j|^2 \right\|_{1/2}^4. \quad (3) \]

We refer to [3] for a) and begin with part b).

Fixed \( \delta > 0 \), we select just one dyadic interval \( 2^k < R \leq 2^{k+1} \) out of each \( |\log \delta| \) correlative intervals, and we allow in the left hand side of (2) only those indices \( j \) for which \( R_j \) lays in a selected interval. Also we only take one \( T^n_j \) for each \( 4 \) correlative indices \( n \), and only those supported in the angular sector \( |\sin \theta| \leq 1/2 \). All these operations will contribute with the factor \( 2^4 |\log \delta| \) to the inequality (2).

The left hand side of (2) is less than the 4th rooth of twice

\[ \sum_{R_j \leq R_k} \int \left( \sum_n |T^n_j f_j|^2 \right) \left( \sum_m |T^m_k f_k|^2 \right)^2 \quad (4) \]

and now we only have two kinds of pairs \((j, k)\) : either \( R_j \leq R_k \leq 2R_j \) or \( R_j \leq \delta R_k \). Let's denote \( \Sigma^I \) and \( \Sigma^{II} \) the two corresponding halves of (4). We have
\[ \Sigma^1 = \Sigma^1 \int \left| \sum_{n,m} (T^n_j f_j)^* (T^m_k f_k)^* \right|^2 \leq 4 \Sigma^1 \int \left| \sum_{n \leq m} (T^n_j f_j)^* (T^m_k f_k)^* \right|^2. \]

Now an easy geometric argument shows that, for fixed \( j, k \), the supports of \((T^n_j f_j)^* (T^m_k f_k)^*\) are disjoint for different pairs \( n \leq m \), so that we have

\[ \Sigma^1 \leq 4 \int \Sigma^1 \sum_{n < m} |(T^n_j f_j)^* (T^m_k f_k)^*|^2 \leq 4A \] (5)

with

\[ A = \left\| \sum_{j,n} |T^n_j f_j|^2 \right\|_4^{1/2}. \]

For the pairs \((j, k)\) in \( \Sigma^\Pi \) we have

\[ \phi = \text{supp } |(T^n_j f_j)^* (T^m_k f_k)^*| \cap \text{supp } |(T^n_j f_j)^* (T^m_k f_k)^*| \]

if \( m_1 \neq m_2 \), because \( R_j \leq \delta R_k \), so that

\[ \Sigma^\Pi = \Sigma^\Pi \int \sum_m \left| \left( \sum_n T^n_j f_j \right) T^m_k f_k \right|^2 \]

\[ \leq \left| \int \left( \sum_n T^n_j f_j \right)^2 \right|^{1/2} \left| \int \left( \sum_{k,m} T^m_k f_k \right)^2 \right|^{1/2} \leq \sqrt{2} \left| \Sigma^1 + \Sigma^\Pi \right|^{1/2} A^{1/2}. \] (6)

From (5) and (6) we obtain (2).

Now we come into part c).

First we observe that for each fixed \( j \) it is possible to choose two grids of parallelepipeds as the one in theorem 3 and such that each of the multipliers \( T^n_j \) is supported within one of the parallelepipeds, let's call it \( Q^n_j \). If \( (P^n_j f)^* = \chi_{Q^n_j} \hat{f} \) is the corresponding multiplier operator, we have

\[ T^n_j f_j = T^n_j P^n_j f_j. \]

Furthermore, an integration by parts arguments shows that each of the kernels of the \( T^n_j \) is majorized by a sum

\[ C \sum_{\nu=0}^{\infty} 2^{-\nu} \frac{1}{|R^n_{\nu,j}|} \chi_{R^n_{\nu,j}} \]

where the \( R^n_{\nu,j} \) are rectangles with dimensions \( 2^\nu \delta^{-1} \times 2^\nu \delta^{-1/2} \) and \( C \) is independent of \( n, j \) or \( \delta > 0 \). Therefore in order to
estimate $A$ we only have to estimate uniformly in $v$ the $L^4$-norm of

$$\left| \sum_{j,n} \frac{1}{|R_{v_j}^n|} \chi_{R_{v_j}^n} \ast (P^n f_j) \right|^2 \left( \sum_{j,n} \frac{1}{|R_{v_j}^n|} \chi_{R_{v_j}^n} \ast \omega \right)^{1/2}.$$

Or, what amounts to the same, the $L^2$-norm of its square. If $\omega \geq 0$ is in $L^2(\mathbb{R}^2)$ we have

$$\begin{align*}
\sum_{j,n} \int \frac{1}{|R_{v_j}^n|} \chi_{R_{v_j}^n} \ast (P^n f_j) (x) \omega(x) dx &\leq \sum_{j,n} \int |P^n f_j(y)|^2 \left( \frac{1}{|R_{v_j}^n|} \chi_{R_{v_j}^n} \ast \omega \right)(y) dy \\
&\leq 2C |\log \delta|^\alpha C_4^2 \left( \sum_{j,n} \| f_j \|_2 \right)^2 \| \omega \|_2,
\end{align*}$$

by successive applications of theorems 4 and 3. This estimate proves (3).

Proof of Theorem 2. — With the same notations of the preceding proof, let now $R_{v_j} = 2^{\gamma}$. We have

$$\begin{align*}
T^\lambda f(x) &\leq \sup_{j} |T_j^{\lambda} f(x)| + \sup_{j} |(T_j^{\lambda} - T_j^{\gamma}) f(x)| \\
&\leq \left| \sum_{j} |T_j^{\lambda} f(x)| \right|^{1/2} + C f^*(x)
\end{align*}$$

where $T_j^{\lambda} - T_j^{\gamma}$ stands for a $C^\infty$ central core of the multiplier $T_j^{\lambda}$ and $f^*$ is the Hardy-Littlewood maximal function.

By the same arguments of part a) in the preceding proof we may reduce ourselves to prove

$$\left\| \sum_{j} \left| S^\delta_j f \right|^{1/2} \right\|_4 \leq C |\log \delta|^\alpha \| f \|_4$$

for some constants $C, \alpha$, independent of $\delta > 0$. 
We define the operators $U_j$ by
$$U_j f(x, y) = x \{2^{j-1} \leq x \leq 2^j\} \hat{f}(x, y),$$
and apply the methods in parts b) and c) above to obtain the inequality
$$\left\| \sum_j |S_j f|^2 \right\|_4^{1/2} \leq C |\log \delta|^\alpha \left\| \sum_j |U_j f|^2 \right\|_4^{1/2},$$
which yields (7) by the classical Littlewood-Paley theory.

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