

ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 31, n° 3 (1981), p. 71-90

[<http://www.numdam.org/item?id=AIF_1981__31_3_71_0>](http://www.numdam.org/item?id=AIF_1981__31_3_71_0)

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RATIONAL HOMOTOPY OF SERRE FIBRATIONS

par Jean-Claude THOMAS

1. Preliminaries.

In this paper we adopt the terminology of [8] and [9].

Let A denote the Sullivan functor [16] from topological path connected spaces with base point to commutative graded differential augmented algebras over a field k of characteristic zero :

$$A : \text{Top} \longrightarrow \text{C.G.D.A.}$$

To each sequence of base point perserving continuous maps, in particular to each Serre fibration,

$$(*) \quad F \xrightarrow{j} E \xrightarrow{\pi} M$$

D. Sullivan [16] associated a commutative diagram (in C.G.D.A.)

$$(D) \quad \begin{array}{ccccc} (A(M), d_M) & \xrightarrow{A(\pi)} & (A(E), d_E) & \xrightarrow{A(j)} & (A(F), d_F) \\ \uparrow m & & \uparrow \phi & & \uparrow \bar{\phi} \\ (B, d_B) & \xrightarrow{\iota} & (B \otimes \Lambda X, d) & \xrightarrow{\rho} & (\Lambda X, \bar{d}) \end{array}$$

where :

•) ΛX is the free c.g.a over the graded space $X = \bigoplus_{i \geq 0} X^i$ and $m^* : H(B, d_B) \longrightarrow H(A(M), d_M) (\cong H(M, k))$ is an isomorphism.

••) $\iota(b) = b \otimes 1$, $\rho = \epsilon_B \otimes \text{Id}_{\Lambda X}$, where ϵ_B is the augmentation of B .

•••) $\phi^* : H(B \otimes \Lambda X, d) \longrightarrow H(A(E), d_E)$ is an isomorphism.

•v) There exists an homogeneous basis $(e_\alpha)_{\alpha \in K}$ of X indexed by a well ordered set K such that

$$d(1 \otimes e_\alpha) \in B \otimes \Lambda(X_{<\alpha})$$

where we denote by $X_{<\alpha}$ the graded vector space generated by the e_β with $\beta < \alpha$.

The sequence

$$\mathcal{E} : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \bar{d})$$

is called a K.S-extension ([8]), the pair (\mathcal{E}, ϕ) a KS-model of the sequence (*), $(e_\alpha)_{\alpha \in K}$ a KS-basis.

If there exists a K.S basis such that

$$(e_\alpha \in X^i, e_\beta \in X^j, i < j) \implies (\alpha < \beta)$$

for all α and β in K and all degrees i and j , the K.S-extension \mathcal{E} (resp. the K.S-model (\mathcal{E}, ϕ)) is called *minimal*.

When, in the diagram (D), $\bar{\phi}$ induces an isomorphism $\bar{\phi}^*$ between cohomologies, the sequence (*) is called *rational fibration*.

When the base M of (*) is a point, then $((\Lambda X, d), \Phi)$ is a K.S-model of $E = F$ (resp. $((\Lambda X, d), \phi)$ is a minimal K.S model of $E = F$ if \mathcal{E} is minimal).

For all rational fibration (*), with base M , if (\mathcal{E}, ϕ) is a minimal K.S model of (*) then $((\Lambda X, \bar{d}), \bar{\phi})$ is minimal K.S model of the fiber F .

Theorem 20.3 of [8] asserts that rational fibrations include Serre fibrations of path connected spaces when one of $H^*(M, \mathbf{k})$ or $H^*(F, \mathbf{k})$ is a graded space of finite type and $\Pi_1(M)$ acts nilpotently in each $H^p(F, \mathbf{k})$.

It can be easily deduced from definitions that if M, E, F are nilpotent spaces with $H(M, \mathbf{Q}), H(E, \mathbf{Q}), H(F, \mathbf{Q})$ graded spaces of finite type then (*) is a rational fibration if and only the rationalized sequence

$$(**) \quad F_{\mathbf{Q}} \xrightarrow{j_{\mathbf{Q}}} E_{\mathbf{Q}} \xrightarrow{\pi_{\mathbf{Q}}} M_{\mathbf{Q}}$$

is a rational fibration.

If $((\Lambda X, d), \phi)$ is a K.S minimal model of the topological space M , the graded vector space $\Pi_\psi(M) = \bigoplus_{i \geq 1} \Pi_\psi^i(M)$ of indecomposable elements of ΛX is called the ψ -homotopy of M .

Every rational fibration have a long ψ -homotopy sequence,

$$\dots \longrightarrow \Pi_{\psi}^i(M) \xrightarrow{\pi^{\#}} \Pi_{\psi}^i(E) \xrightarrow{j^{\#}} \Pi_{\psi}(F) \xrightarrow{\partial^{\#}} \Pi_{\psi}^{i+1}(M) \longrightarrow \dots$$

Following [10], if $\dim \Pi_{\psi}^*(M) < +\infty$, we call the *Euler homotopy characteristic and the rank* of the space M the integers

$$\chi_{\Pi}(M) = \sum_{i=1}^{+\infty} (-1)^i \dim \Pi_{\psi}^i(M)$$

and

$$rk(M) = \sum_{i=1}^{+\infty} \dim \Pi_{\psi}^{2i+1}(M).$$

If the spaces $\Pi_{\psi}^*(M)$ and $H^*(M, k)$ are finite dimensional, M is called a space of type F ([7]).

2. Main results.

A rational fibration $(*)$ is called *pure* if there exists a K.S-minimal model (\mathcal{E}, ϕ) such that

$$dX^{\text{even}} = 0, \quad dX^{\text{odd}} \subset B \otimes \Lambda(X^{\text{even}}).$$

In this case $(B \otimes \Lambda(X^{\text{even}}) \otimes \Lambda(X^{\text{odd}}), d)$ is a Koszul complex [12] and from [5] when $k = \mathbf{R}$, and [17] for $k = \mathbf{Q}$, we have :

THEOREM 1. — *If G is a compact connected Lie group and H a closed connected subgroup, then every fibre bundle with standard fiber G/H , associated to a G -principal bundle via the standard action of G on G/H is a pure fibration.*

In this paper we prove the following results.

THEOREM 2. — *For any rational fibration such that the fibre F is a space of type F with $\chi_{\pi}(F) = 0$ the following assertions are equivalent :*

- i) $(*)$ is totally non cohomologous to zero (T.N.C.Z)
- ii) $(*)$ is a pure fibration.

Recall that $(*)$ is called T.N.C.Z if $j^* : H^*(F, \mathbf{Q}) \longrightarrow H^*(E, \mathbf{Q})$ is surjective, which is equivalent [15] when $H^*(F, \mathbf{Q})$ and $H^*(M, \mathbf{Q})$ are of finite type and $(*)$ is Serre fibration, to :

- iii) *The Serre Spectral sequence collapses at the E_2 term ($d_r = 0 \ r \geq 2$).*

In particular the hypothesis of theorem 2 are satisfied when F is a homogeneous space G/H with $rkG = rkH$, for example if F is a real oriented or complex or quaternionic grassmann manifold, or $F = G/T$ when T is a maximal torus of G or F is a finite product of such spaces. It is proved in [10] that a space M of type F has a χ_π zero iff $H^{\text{odd}}(M, \mathbf{Q}) = 0$.

THEOREM 3. — *Every rational fibration such that the fibre F is a space of type F with $\chi_\pi(F) = 0$ and $rk(F) \leq 2$ is a pure fibration.*

This result can be applied when

$$F = S^{2n}, \mathbf{CP}^n, \mathbf{HP}^n, S^{2n} \times S^{2q}, \mathbf{CP}^q \times \mathbf{HP}^r, \text{SP}(2)/\text{U}(2), \\ \text{SO}(4)/\text{U}(2), \text{U}(2)/\text{U}(1) \times \text{U}(1), \text{SO}(5)/\text{SO}(1) \times \text{SO}(3), \dots$$

It is a particular case of a conjecture of S. Halperin.

Every rational fibration with fibre of type F and $\chi_\pi = 0$ is T.N.C.Z.

COROLLARY 4. — *If F is a path connected topological space of type F and $\chi_\pi = 0$ and if G is a compact connected Lie group operating on F then the total space F_G of the fiber bundle*

$$F \longrightarrow E_G \times_G F \longrightarrow B_G$$

associated with the operation is intrinsically formal and the Krull dimension of $H_G(F, \mathbf{Q}) = H(F_G, \mathbf{Q})$ equals the rank of G .

COROLLARY 5 (compare with [2]). — *There do not exist Serre fibrations (*) if one of the following conditions is satisfied:*

- i) $H^{\text{even}}(E, \mathbf{Q}) = 0$.
- ii) E is a connected Lie group.
- iii) $E = S^{2n}$ except for $H^*(F, \mathbf{Q}) = H^*(S^{2n}, \mathbf{Q})$

and if F is a non contractile space of type F with $\chi_\pi(F) = 0$ and $rk(F) \leq 2$.

From the Leray-Hirsh theorem we get, that if (*) is T.N.C.Z., then there exists a graded vector space isomorphism

$$f: H(M; \mathbf{Q}) \otimes H(F, \mathbf{Q}) \longrightarrow H(E, \mathbf{Q})$$

preserving base and fiber cohomology. When f can be chosen to be an algebra isomorphism the fibration $(*)$ is called *cohomologically trivial* (C.T.).

When E , F , M are nilpotent spaces, with rational cohomology algebras of finite type, the rational fibration $(*)$ is called

(•) homotopically trivial (H.T), or

(••) weakly homotopically trivial (W.H.T), or

(•••) a σ -fibration $(\sigma \cdot F)$

if the rational fibration $(**)$

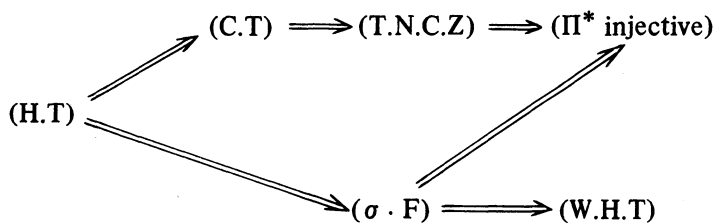
(•) is trivial or,

(••) has a long homotopy exact sequence with a connecting homomorphism $\partial^\#$ identically zero

$$(\Pi_\psi(E) = \Pi_\psi(M) \otimes \Pi_\psi(F)), \text{ or}$$

(•••) admits a section.

Naturally we have the following diagram



with all the reversed implications false. We do not know if in the general case $(C.T) \implies (W.H.T)$, but we obtain the following results. (For all fibrations $F \longrightarrow E \longrightarrow M$ the spaces are assumed to have cohomology of finite type).

PROPOSITION 6. — a) Every T.N.C.Z rational fibration with fibre F such that $H^*(F, k)$ is a free commutative graded algebra is H.T.

b) Every C.T rational fibration with fibre F a space of type F and $\chi_\pi = 0$ is H.T.

PROPOSITION 7. — a) Every σ -fibration $(*)$ such that M is ℓ -connected and $\Pi_\psi^i(F) = 0$ for $i < r$ and $i \geq r + \ell$ is H.T.

b) Every rational fibration such that $\dim H^*(F, \mathbf{k}) < +\infty$ and M is a coformal space [13], [14] with spherical cohomology zero in dimension $2p$ if $\Pi_{\psi}^{2p-1}(F) \neq 0$ is W.H.T.

3. Some examples and counter examples.

Example 1. — Even if a rational fibration $(*)$ is pure not every KS minimal model $(*)$ need verify

$$dX^{\text{even}} = 0 \quad \text{and} \quad dX^{\text{odd}} \subset B \otimes \Lambda(X^{\text{even}}).$$

Indeed the minimal K.S-extension

$$\mathcal{E}: (\Lambda b_1, 0) \xrightarrow{\iota} (\Lambda b_1 \otimes \Lambda(x_2, x_3, x_4, x_7), d) \xrightarrow{\rho} (\Lambda(x_2, x_3, x_4, x_7), \bar{d})$$

with

$$db_1 = 0$$

$$dx_3 = x_2^2 \quad dx_7 = x_4^2 + 2b_1 x_3 x_4$$

$$dx_2 = 0 \quad dx_4 = b_1 x_2^2$$

is a K.S-minimal model of a pure fibration

$$(*) \quad S^2 \times S^4 \xrightarrow{j} E \xrightarrow{\Pi} S^1.$$

Example 2. — As a particular case of pure fibration we get the notion of pure space. Evidently in a pure fibration the fiber is a pure space; the converse however is false. In [10] it is proved that a space of type F with χ_π zero is a pure space, but the conjecture and theorem 2 fail if we replace the hypothesis “ F is a space of type F with $\chi_\pi(F) = 0$ ” by the hypothesis “ F is a pure space of type F ”. Indeed consider the rational fibration

$$F_{\mathbf{Q}} \xrightarrow{j} E \xrightarrow{\Pi} S^3$$

with $F = (S^2 VS^4)_7 \cup e^7$ where $(S^2 VS^4)_7$ is the 7th Posnikov stage of the space $\phi S^2 VS^4$ and $\phi = [S^4, [S^2, S^2]] - [S^2[S^2, S^4]]$ defined by its K.S-minimal model

$$\mathcal{E}: (\Lambda b_3, 0) \longrightarrow (\Lambda b_3 \times \Lambda(x_2, x_3, x_4, x_5, x_7), d) \longrightarrow (\Lambda(x_i), \bar{d})$$

$$db_3 = 0$$

$$dx_2 = 0 \quad dx_4 = b_3 x_2$$

$$dx_3 = x_2^2, \quad dx_5 = x_2 x_4 + b_3 x_3, \quad dx_7 = x_4^2 + 2b_3 x_5.$$

Then $\chi_\pi(F) = -1$ and $H^4(E, \mathbf{k}) = 0$, and $(*)$ is neither a pure fibration nor a T.N.C.Z. fibration.

Example 3. — There exists one (unique up to rational homotopy equivalence) Serre fibration

$$(S^2 VS^2)_{\mathbf{Q}} \xrightarrow{j} E \xrightarrow{\pi} S^3$$

which is C.T. but not H.T., as it can be easily seen from the calculations of [11].

Example 4. — The universal fiber bundle

$$S^{2n} \longrightarrow B_{SO(2n)} \longrightarrow B_{SO(2n+1)}$$

is T.N.C.Z. and W.H.T. but not C.T.

Example 5. — Let a vector bundle

$$\eta : \mathbf{R}^{2n+1} \longrightarrow E \longrightarrow M$$

and $p_n(\eta)$ its n^{th} Pontryagin class, and

$$\eta_S : S^{2n} \longrightarrow E_S \longrightarrow M$$

its associated sphere bundle. Suppose that η_S is T.N.C.Z. then $p_n(\eta) = 0$ if and only if η_S is H.T.

Example 6. — If a fibration admits a section then it is a σ fibration. The converse is false indeed, consider the σ -fibration

$$S^4 \times S^3 \xrightarrow{j} E \xrightarrow{\pi} S^5$$

of orthonormal two frames on S^5 .

4. Proof of theorem 2.

A K.S-extension $\mathcal{E} : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \bar{d})$ is called pure if there exists a K.S-extension

$$\mathcal{E}' : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X', d') \xrightarrow{\rho'} (\Lambda X', \bar{d}')$$

and an isomorphism of K.S extension $(\text{Id}_B, f, \bar{f}) \mathcal{E} \simeq \mathcal{E}'$ with $d'X'^{\text{even}} = 0$ and $d'X'^{\text{odd}} \subset B \otimes \Lambda(X'^{\text{even}})$.

In view of proposition 1.11 of [8], theorem 2 follows from the following algebraic version.

THEOREM 2'. — Let \mathcal{E} be a K.S-minimal extension with connected base B and $\dim H(\Lambda X, d) < \infty$, $\dim X^{\text{odd}} = \dim X^{\text{even}} < +\infty$ then the two assertions are equivalent :

- i) ρ^* is surjective
- ii) E is pure.

A) First suppose that \mathcal{E} is pure then $\Lambda(X^{\text{even}})$ maps into $H(B \otimes \Lambda X, d)$ and from [7] $H(\Lambda X, \bar{d}) = \Lambda(X^{\text{even}})/\bar{d}X^{\text{odd}} \cdot \Lambda(X^{\text{even}})$ so ρ^* is surjective.

B) The converse is in two steps. First we prove that \mathcal{E} is isomorphic to \mathcal{E}' with $d'X^{\text{even}} = 0$ and then we show \mathcal{E}' is isomorphic to \mathcal{E}'' with $d''X^{\text{even}} = 0$ and $d''X^{\text{odd}} \subset B \otimes \Lambda X^{\text{even}}$.

B1) *First step.* — From [10] we can suppose that \bar{d} satisfies

$$\bar{d}X^{\text{even}} = 0 \quad \text{and} \quad \bar{d}X^{\text{odd}} \subset \Lambda(X^{\text{even}}).$$

Since ρ and ρ^* are surjective for all $x \in X^{\text{even}}$ there exists $\Phi_x \in (B \otimes \Lambda X) \cap \ker d$ such that $\rho(\Phi_x) = x$.

Then

$$\Phi_x = x + \Omega_x$$

with $\Omega_x \in B^+ \otimes \Lambda X = \ker \rho$. Let x run through a K.S-minimal basis and define a linear map $g : X \longrightarrow B \otimes \Lambda X$ by

$$\begin{aligned} g(x) &= x \quad \text{if} \quad x \in X^{\text{odd}} \\ g(x) &= x + \Omega_x \quad \text{if} \quad x \in X^{\text{even}} \end{aligned}$$

g extends uniquely to a B -linear algebra isomorphism. $g : B \otimes \Lambda X \longrightarrow B \otimes \Lambda X$. It can be easily proved that g is an isomorphism.

Let $\mathcal{E}' : (B, d_B) \longrightarrow (B \otimes \Lambda X, g^{-1}dg) \longrightarrow (\Lambda X, \bar{d})$ so that $(\text{Id}_B, g, \text{Id}_{\Lambda X})$ is an isomorphism of K.S-extensions between \mathcal{E} and \mathcal{E}' and $d'(X^{\text{even}}) = g^{-1}dg(X^{\text{even}}) = 0$.

B2) *Second step.* — Suppose \mathcal{E} is a K.S-minimal extension such that

$$(H_2) = \begin{cases} dX^{\text{even}} = 0 \\ dX^{\text{odd}} \subset (B \times \Lambda(X^{\text{even}})) \otimes (B^{\geq 2} \otimes (\Lambda^+ X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})) \end{cases}$$

and let $(\hat{B}_2 \otimes \Lambda X, \hat{d})$ be the quotient c.g.d.a.

$$(B \otimes \Lambda X, d)/(B^{\geq 2+1} \otimes \Lambda X, d).$$

LEMMA 1. — In $(\hat{B}_\ell \otimes \Lambda X, \hat{d})$ we obtain

- a) $(\ker \hat{d}) \cap (B^\ell \otimes \Lambda X) = (B^\ell \otimes \Lambda(X^{\text{even}}))$
 $\quad \quad \quad + (d(B^\ell \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})))$
b) $(\ker \hat{d}) \cap (B^\ell \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}))$
 $\quad \quad \quad \subset \hat{d}(B^\ell \otimes \Lambda^+(X^{\text{even}}) \otimes \Lambda(X^{\text{odd}})),$

Proof. — a) One inclusion in a) is immediate, the second results from the relation $H_+(\Lambda X, \bar{d}) = 0$ where $H_i(\Lambda X, \bar{d})$ is the homology of the Koszul complex

$$\dots \longrightarrow (\Lambda(X^{\text{even}}) \otimes \Lambda^{i+1}(X^{\text{odd}})) \xrightarrow{\bar{d}} \Lambda(X^{\text{even}}) \otimes \Lambda^i(X^{\text{odd}}) \xrightarrow{\bar{d}} \Lambda(X^{\text{even}}) \otimes \Lambda^{i-1}(X^{\text{odd}}) \longrightarrow$$

and from the relation

$$\hat{d}\phi_i = (1 \otimes \bar{d})\phi_i \quad \text{for} \quad \phi_i \in B^\ell \otimes \Lambda^i(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}).$$

b) is true for the same reason.

Clearly if \mathcal{E} satisfies the hypothesis of theorem 2' then \mathcal{E} satisfies hypothesis (H_1) , and since X is a finite dimensional vector space, theorem 2' results from

LEMMA 2. — If \mathcal{E} satisfies hypothesis (H_ℓ) there exists a minimal K.S-extension \mathcal{E}' isomorphic to \mathcal{E} which satisfies $(H_{\ell+1})$.

Proof. — 1) Suppose $\ell = 2\ell'$, so for each $x \in X^{\text{odd}}$, in $(\hat{B}_\ell \otimes \Lambda X, \hat{d})$ we have

$$\hat{d}x = \Phi_x + \sum_{s \geq 1} \phi_{x,2s}$$

with $\Phi_x \in \hat{B}_\ell \otimes \Lambda(X^{\text{even}})$, $\phi_{x,2s} \in B^\ell \otimes \Lambda^{2s}(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})$.

From relation $\hat{d} \circ \hat{d}x = 0$ we deduce

$$0 = \hat{d}\Phi_x + \hat{d}\left(\sum_{s \geq 1} \phi_{2s,x}\right) = (d_B \otimes id)\Phi_x + (id \otimes \bar{d})\left(\sum_{s \geq 1} \phi_{2s,x}\right) \in B^{\text{odd}} \otimes \Lambda X \oplus \hat{B}^2 \otimes \Lambda X.$$

Hence
$$0 = \hat{d}\Phi_x = \hat{d}\left(\sum_{s \geq 1} \phi_{2s,x}\right).$$

By lemma 1,
$$\hat{d}x = \Phi_x + \sum_{s \geq 1} \hat{d}\Psi_{x,2s+1}$$

with $\Psi_{x,2s+1} \in B^\ell \otimes \Lambda^{2s+1}(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})$.

Thus $d\left(x - \sum_{s \geq 1} \Psi_{x, 2s+1}\right) = \Phi_x + \Omega_x$ with $\Omega_x \in B^{\geq \ell+1} \otimes \Lambda X$.
 The linear map $g: X \rightarrow B \otimes \Lambda X$ defined by $g(x) = x$ if $x \in X^{\text{even}}$ $g(x_\alpha) = x_\alpha - \sum_{s \geq 1} \psi_{\alpha, 2s+1}$ if (x_α) is a minimal K.S basis of X^{odd} uniquely extends to a c.g.d.a isomorphism $B \otimes \Lambda X \xrightarrow{\cong} B \otimes \Lambda X$ with $g/B = \text{Id}_B$.

Define \mathcal{E}' by $(B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, g^{-1}dg) \xrightarrow{\iota} (\Lambda X, \bar{d})$ then \mathcal{E}' satisfies hypothesis $(H_{\ell+1})$.

2) Suppose $\ell = 2\ell' + 1$. In the same way as in the preceding case we get a K.S minimal extension \mathcal{E}_1 and an isomorphism $(\text{Id}_B, g_1, \text{Id}_{\Lambda X})$ between \mathcal{E} and \mathcal{E}_1 such that,

$$\mathcal{E}_1: (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d_1) \xrightarrow{\rho} (\Lambda X, \bar{d})$$

with

$$\begin{cases} d_1(x) = 0 & \text{if } x \in X^{\text{even}} \\ d_1(x) \in (B^{\text{even}} \otimes \Lambda(X^{\text{even}})) \oplus (B^\ell \otimes \Lambda^1 X^{\text{odd}} \otimes \Lambda X^{\text{even}}) \\ \quad \oplus (B^{\geq \ell+1} \otimes \Lambda X) & \text{if } x \in X^{\text{odd}}. \end{cases}$$

We put

$$\begin{aligned} B^\ell &= K^\ell \oplus dB^{\ell-1} & \text{if } \ell \geq 2 \\ B^1 &= K^1 & \text{if } \ell = 1. \end{aligned}$$

Using only degree argument, we prove that there exists a minimal K.S-extension \mathcal{E}_2 and an isomorphism $(\text{Id}_B, g_2, \text{Id}_{\Lambda X})$ between \mathcal{E}_1 and \mathcal{E}_2 such that

$$\mathcal{E}_2: (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d_2) \xrightarrow{\rho} (\Lambda X, \bar{d})$$

with

$$\begin{cases} d_2(x) = 0, & \text{if } x \in X^{\text{even}} \\ d_2(x) \in (B \otimes \Lambda X^{\text{even}}) \oplus (K^\ell \otimes \Lambda^1(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})) \\ \quad \oplus (B^{\geq \ell+1} \otimes \Lambda^+ X^{\text{odd}} \otimes \Lambda X^{\text{even}}), & \text{if } x \in X^{\text{odd}} \end{cases}$$

so that in the quotient algebra $(\hat{B}_\ell \otimes \Lambda X, \hat{d}_2)$, we write

$$\hat{d}_2 x_\alpha = \bar{d} x_\alpha + \sum_{r \geq 1} \Phi_{\alpha, 2r} + \sum_{s=1}^{\alpha-1} \phi_{\alpha, s} x_s$$

with (x_α) a K.S-minimal basis of \mathcal{E}_2 and $x_\alpha \in X^{\text{odd}}$,

$$\Phi_{\alpha, 2r} \in B^{2r} \otimes \Lambda(X^{\text{even}}) \quad \phi_{\alpha, s} \in K^{\mathbb{Q}} \otimes \Lambda(X^{\text{even}}).$$

From the relation $\hat{d} \circ \hat{d} x_{\alpha} = 0$ and lemma 1 we obtain for each α ,

$$d_2(x_{\alpha} - \theta_{\alpha}) = \bar{d} x_{\alpha} + \sum_{r \geq 1} \Phi_{\alpha, 2r} + \Omega_{\alpha}$$

with

$$\begin{aligned} \Omega_{\alpha} &\in B^{\geq \mathbb{Q}+1} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}) \\ \theta_{\alpha} &\in B^{\mathbb{Q}} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}) \end{aligned}$$

and so there exists a minimal K.S-extension \mathcal{E}' and an isomorphism $(\text{Id}_B, g', \text{Id}_{\Lambda X})$ between \mathcal{E}_2 and \mathcal{E}' such that

$$\mathcal{E}' : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d') \xrightarrow{\rho} (\Lambda X, \bar{d})$$

and \mathcal{E}' satisfies $H_{\mathbb{Q}+1}$. This ends the proof of lemma 2.

5. Derivations in Poincaré duality algebras and proof of theorem 3.

Let $(A, d) = (\Lambda(x_1, \dots, x_n, y_1, \dots, y_n), d)$ a K.S complex such that the y_i and x_j have respectively even degree $|y_i|$ and odd degree $|x_j|$ and

$$\begin{aligned} |y_1| &\leq |y_2| \leq \dots \leq |y_n| \\ |x_1| &\leq |x_2| \leq \dots \leq |x_n|. \end{aligned}$$

Suppose

$$\begin{aligned} dy_i &= 0 \quad i = 1, \dots, n \\ dx_j &= f_j \in \Lambda(y_1, \dots, y_n) \quad j = 1, \dots, n \end{aligned}$$

then (A, d) is a pure K.S complex and from [10] if $\dim H(A, d) < +\infty$ then $H(A, d_A) = \Lambda(y_1, \dots, y_n)/(f_1, \dots, f_n)$ is a Poincaré duality algebra of formal dimension

$$N = |f_1| + \dots + |f_n| - |y_1| - \dots - |y_n|$$

(i.e.)

- i) $H^i(A, d) = 0$ if $i > N$
- ii) $H^N(A, d) = ke$
- iii) the bilinear form $\langle, \rangle : H^p(A, d) \times H^{N-p}(A, d) \longrightarrow k$

defined by $\langle a, b \rangle e = a \cdot b$ is non degenerate.

Since $\dim H(A, d) < +\infty$ and $H^0(A, d) = \mathbf{k}$, one verifies immediately:

LEMMA 1. — Any derivation $\theta \in \text{Der}_{\leq 0}(H(A), d)$ satisfies $I_m \theta \cap H^0(A, d) = 0$ and hence maps $H^+(A, d)$ to itself.

We put \bar{y}_i the class of y_i in $H(A, d)$ and we say that a derivation $\tilde{\theta}$ of $H(A, d_A)$ is nilpotent with respect to $(\bar{y}_1, \dots, \bar{y}_n)$ if $\tilde{\theta}(y_i)$ is polynomial in $\bar{y}_1, \dots, \bar{y}_{i-1}$. We denote by $\tilde{\text{Der}}_{\leq 0}(H(A), d)$ the subspace of $\text{Der}_{\leq 0}(H(A), d)$ of such derivations.

LEMMA 2. — Any derivation $\tilde{\theta} \in \tilde{\text{Der}}_{\leq 0}(H(A), d)$ satisfies $\tilde{\theta}(H^N(A, d)) = 0$.

Proof. — Let m_1 be the largest integer such that $\bar{y}_1^{m_1} \neq 0$ and $\bar{y}_1^{m_1+1} = 0$.

Let m_i be the largest integer such that $(\bar{y}_1^{m_1}, \dots, \bar{y}_{i-1}^{m_{i-1}}) \bar{y}_i^{m_i} \neq 0$ and $(\bar{y}_1^{m_1} \dots \bar{y}_{i-1}^{m_{i-1}}) \bar{y}_i^{m_i+1} = 0$, then we obtain an element $\Phi = \bar{y}_1^{m_1} \bar{y}_2^{m_2} \dots \bar{y}_n^{m_n}$ such that for every $a \in H^+(A, d)$ $a \cdot \Phi = 0$.

Necessarily $|\Phi| = N$ and we may put $e = \bar{y}_1^{m_1} \dots \bar{y}_n^{m_n}$. Then $\tilde{\theta}(e) = 0$, since $\tilde{\theta}$ is nilpotent with respect to $(\bar{y}_1, \dots, \bar{y}_n)$.

From lemmas 1 and 2 we deduce,

COROLLARY. — If $\tilde{\theta} \in \tilde{\text{Der}}_{\leq 0}(H(A), d)$ then

$$\text{i) } \langle \tilde{\theta}(a), b \rangle = -\langle a, \tilde{\theta}(b) \rangle$$

$$\text{ii) } \text{Im } \tilde{\theta} \subset \bigoplus_{i=1}^{N-1} H^i.$$

LEMMA 3. — If $\tilde{\theta} \in \tilde{\text{Der}}_{\leq 0}(H(A), d)$ then

$$(\tilde{\theta}(\bar{y}_1) = \tilde{\theta}(\bar{y}_2) = \dots = \tilde{\theta}(\bar{y}_{n-1}) = 0) \implies (\tilde{\theta} \equiv 0).$$

Proof. — Suppose that $\tilde{\theta}(\bar{y}_n) = \Phi' \neq 0$ and let

P_1 be the largest integer such that $\Phi' \bar{y}_1^{-P_1} \neq 0$ and $\Phi' \bar{y}_1^{-P_1+1} = 0$

P_i be the largest integer such that $\Phi' \bar{y}_1^{-P_1} \dots \bar{y}_i^{-P_i} \neq 0$ and $\Phi' \bar{y}_1^{-P_1} \dots \bar{y}_i^{-P_i+1} = 0$.

So we obtain an element $\Psi = \Phi' \bar{y}_1^{P_1} \dots \bar{y}_n^{P_n}$ such that $\Psi \in H^N(A, d)$ and $\tilde{\theta}\left(\frac{1}{P_n + 1} \bar{y}_1^{P_1} \dots \bar{y}_{n-1}^{P_{n-1}}, \bar{y}_n^{P_n+1}\right) = \Psi$ which contradicts part (ii) of the corollary above.

In particular if $n = 2$ since $\hat{\theta}(y_1)$ is always zero,

$$\text{Der}_{\leq 0}(H(A, d)) = 0.$$

This is what we will need to prove theorem 3.

Proof of Theorem 3.

A) Suppose $\dim \Pi_\psi(F) = 2$ then theorem 3 is equivalent to the following.

THEOREM 3'. — *Let \mathcal{E} a K.S-minimal extension*

$$(B, d_B) \xrightarrow{\iota} (B \otimes \Lambda(x, y), d) \xrightarrow{\rho} (\Lambda(x, y), \bar{d})$$

such that B is a connected algebra, $\dim H((x, y), \bar{d}) < +\infty$ and $|x|$ odd, $|y|$ even then ρ^ is surjective.*

Proof. — Since $\dim H(\Lambda(x, y), \bar{d}) < +\infty$, we have $\bar{d}x = \lambda y^m$ with $\lambda \in k - \{0\}$ and $m \geq 2$. Thus

$$dx = \lambda y^m + b_1 y^{m-1} + \dots + b_m$$

with $|b_i| = i |y|$, whence

$$d\left(y + \frac{1}{m\lambda} b_1\right) = 0$$

$$\rho\left(y + \frac{1}{m\lambda} b_1\right) = y$$

and $\rho^*: H(B \otimes \Lambda(x, y)) \longrightarrow \Lambda(y)/(y^m) = H(\Lambda X, \bar{d})$ is surjective.

B) Suppose $\dim \Pi_\psi(F) = 4$ then theorem 3 is equivalent to the following.

THEOREM 3''. — *Let \mathcal{E} a K.S-minimal extension*

$$(B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, d)$$

such that B is a connected algebra, $\dim H(\Lambda X, d) < +\infty$, $\dim X^{\text{odd}} = \dim X^{\text{even}} = 2$ then \mathcal{E} is pure.

We prove theorem 3'' by induction on ℓ , in the following manner

$$H_\ell^1 \implies H_\ell^2 \implies H_\ell^3 \implies H_{\ell+1}^1$$

where the hypothesis H_ℓ^i are defined by :

$$\begin{aligned} H_\ell^1 &= \begin{cases} dx \in B^{\geq \ell} \otimes \Lambda X, & \text{if } x \in X^{\text{even}} \\ dx \in (B \otimes \Lambda(X^{\text{even}})) \oplus (B^{\geq \ell} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{odd}} \end{cases} \\ H_\ell^2 &= \begin{cases} dx \in (B^{\geq \ell} \otimes \Lambda(X^{\text{even}})) \oplus (B^{\geq \ell+1} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{even}} \\ dx \in (B \otimes \Lambda(X^{\text{even}})) \oplus (B^{\geq \ell} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{odd}} \end{cases} \\ H_\ell^3 &= \begin{cases} dx \in (B^{\geq \ell} \otimes \Lambda(X^{\text{even}})) \oplus (B^{\geq \ell+1} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})), & \text{if } x \in X^{\text{even}} \\ dx \in (B \otimes \Lambda(X^{\text{even}})) \oplus (B^\ell \otimes \Lambda^1(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}) \\ \quad \oplus B^{\geq \ell+1} \otimes \Lambda X), & \text{if } x \in X^{\text{odd}} \end{cases} \end{aligned}$$

To prove $H_\ell^1 \implies H_\ell^2$ and $H_\ell^2 \implies H_\ell^3$, we use lemma 1 of IV which again follows from the relation $d \circ d = 0$.

In the case $\ell = 2\ell'$ for degree reasons $H_\ell^3 = H_{\ell+1}^1$. When $\ell = 2\ell' + 1$ we prove $H_\ell^3 \implies H_{\ell+1}^1$.

First, we can assume that

$$\begin{cases} dx \in (K^\ell \otimes \Lambda(X^{\text{even}}) \oplus (B^{\geq \ell+1} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}))), & \text{if } x \in X^{\text{even}} \\ dx \in (B \otimes \Lambda(X^{\text{even}}) \oplus (K^\ell \otimes \Lambda^1(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}})) \\ \quad \oplus (B^{\geq \ell+1} \otimes \Lambda^+(X^{\text{odd}}) \otimes \Lambda(X^{\text{even}}))), & \text{if } x \in X^{\text{odd}} \end{cases}$$

with

$$\begin{aligned} B^\ell &= K^\ell \oplus dB^{\ell-1} & \text{if } \ell > 1 \\ B^1 &= K^1 & \text{if } \ell = 1. \end{aligned}$$

In the quotient algebra $(\hat{B} \otimes \Lambda X, \hat{d})$ ($4, B_2$), we have

$$\begin{cases} dy_i = \psi_i & i = 1, 2 \\ dx_j = \bar{d}x_j + \sum_r \Phi_{j,2r} + \sum_{s=1}^{j-1} \phi_{j,s} x_s & j = 1, 2 \end{cases}$$

for a K.S.-minimal basis (y_i, x_j) of X with $|y_i|$ even and $|x_j|$ odd, with

$$\begin{aligned}\psi_1 &\in K^\ell, \quad \psi_2 \in K^\ell \otimes \Lambda(y_1) \\ \Phi_{j,2r} &\in B^{2r} \otimes \Lambda(y_1, y_2) \\ \phi_{j,s} &\in K^\ell \otimes \Lambda(y_1, y_2).\end{aligned}$$

And from the relation $\hat{d} \circ \hat{d} = 0$ we obtain

$$\hat{d}(\bar{d}x_j) = \sum_{s=1}^{j-1} \phi_{js} dx_s.$$

Let (b_ϵ) a base of K^ℓ and put for each $\Phi \in \Lambda(y_1, y_2)$

$$\hat{d}(\Phi) = \sum_{\epsilon} b_{\epsilon} \otimes \theta^{\epsilon}(\Phi).$$

This defines a degree 1- ℓ derivation θ^{ϵ} on $\Lambda(y_1, y_2)$ which respects the ideal $(\bar{d}x_1, \bar{d}x_2)$. So θ^{ϵ} induces a derivation $\tilde{\theta}^{\epsilon}$ on $\Lambda(y_1, y_2)/(\bar{d}x_1, \bar{d}x_2) = H(\Lambda X, \bar{d})$ which is nilpotent with respect to (\bar{y}_1, \bar{y}_2) . From our results on such derivations, $\tilde{\theta}^{\epsilon} \equiv 0$ and necessarily

$$\theta^{\epsilon}(y_1) = 0 \quad \theta^{\epsilon}(y_2) = \bar{d}\Phi^{\epsilon} \quad \Phi^{\epsilon} \in \Lambda X^{\text{even}} \otimes \Lambda^1 X^{\text{odd}}$$

thus,

$$\begin{cases} \hat{d}_3(y_2 + \sum_{\epsilon} b_{\epsilon} \otimes \Phi^{\epsilon}) = 0 \\ \hat{d}_3(y_1) = 0. \end{cases}$$

A standard argument now ends the proof.

6. Proof of the corollaries 4 and 5.

A) COROLLARY 4. — Since $H^{\text{odd}}(F, k) = H^{\text{odd}}(B_G, k) = 0$, the Serre spectral sequence collapses at the E_2 term so that the fibration

$$(*) \quad F \longrightarrow E_G \times_G F \longrightarrow B_G$$

is T.N.C.Z. By [1], $H(B_G, \mathbb{Q}) = \Lambda Z$, $Z = Z^{\text{even}}$ and so $(\Lambda Z, 0)$ is the minimal model for B_G . From theorem 2 there exists a K.S.-minimal model of $(*)$

$$\mathcal{E} : (\Lambda Z, 0) \xrightarrow{\iota} (\Lambda Z \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \bar{d})$$

with

$$\begin{aligned} dX^{\text{even}} &= 0 \\ dX^{\text{odd}} &\subset \Lambda Z \otimes \Lambda X^{\text{even}}. \end{aligned}$$

So we have the Koszul complex,

$$\begin{aligned} \longrightarrow \Lambda(Z \oplus X^{\text{even}}) \otimes \Lambda^{i+1}(X^{\text{odd}}) &\xrightarrow{d} \Lambda(Z \oplus X^{\text{even}}) \otimes \Lambda^i X^{\text{odd}} \\ &\xrightarrow{d} \Lambda(Z \oplus X^{\text{even}}) \otimes \Lambda^{i-1} X^{\text{odd}} \longrightarrow \end{aligned}$$

and we easily verify that $H_+(\Lambda(Z \oplus X), d) = 0$. Thus if x_i is a homogeneous basis of X^{odd} and if we put $dx_i = g_i$ then

$$H(\Lambda(Z \oplus X), d) = H_0(\Lambda(Z \oplus X), d) = \Lambda(Z \oplus X^{\text{even}})/(g_1, \dots, g_n)$$

where (g_1, \dots, g_n) is a regular sequence of $\Lambda(Z \oplus X^{\text{even}})$. This proves directly from commutative algebra that $H(F_G, k)$ is a Cohen Macaulay ring of Krull dimension $\dim Z$ equal to the rank of G and minimalizing $(\Lambda(Z \oplus X), d)$ we obtain the brigaded model of $H(F_G, k)$ in the sense of [11]. This is two stage, and so F_G is intrinsically formal (i.e. F_G is formal and there is no space $M \not\cong F_G$ such that $H(F_G, k) = H(M, k)$).

B) COROLLARY 5. — i) Since $H^{\text{even}}(F, k)$ and $H^{\text{even}}(E, k) = 0$ the condition j^* surjective is impossible.

ii) From the long exact sequence of ψ -homotopy we deduce that in a pure fibration we have

$$rk(\Pi_{2n}(F)) \leq rk(\Pi_{2n}(E))$$

which is impossible if F non contractible and E a Lie group.

iii) A fibration satisfying the hypothesis is pure by Theorem 3 and hence has a K.S minimal model of the form

$$(B, d_B) \longrightarrow (B \otimes \Lambda X, d) \longrightarrow (\Lambda X, \bar{d})$$

with

$$\begin{aligned} dX^{\text{even}} &= 0 & \dim X^{\text{even}} &= \dim X^{\text{odd}} \\ dX^{\text{odd}} &= B \otimes \Lambda X^{\text{even}}. \end{aligned}$$

Necessarily $\dim X^{\text{even}} = 1$ and if we choice $x \in X^{\text{odd}} - \{0\}$
 $dx = y^p + b_1 y^{p-1} + \dots + b_p$ with $p \geq 2$, $y \in X^{\text{even}} - \{0\}$.
 Since j^* is surjective $p = 2$ then $F_Q \sim S^{2n}$.

7. Proof of propositions 6 and 7.

PROPOSITION 6. — *The two following lemmas are easily proved and the first is well known.*

LEMMA 1. — *A Serre fibration $(*)$ is T.N.C.Z. (resp. CT) if and only if there exists a graded vector space homomorphism (resp. a graded algebra homomorphism)*

$$\tau : H^*(F, k) \longrightarrow H^*(E, k)$$

such that

$$j^* \tau = \text{Id}_{H^*(F, k)}.$$

LEMMA 2. — *A rational fibration $(*)$ is H.T. if and only if there exists a K.S-minimal model (\mathcal{E}, ϕ) and a graded differential algebra homomorphism*

$$\sigma : (\Lambda X, \bar{d}) \longrightarrow (A(M) \otimes \Lambda(X), d)$$

such that

$$\rho \circ \sigma = \text{Id}_{\Lambda X}.$$

Remarks. — i) These two lemmas prove in particular that the notions of T.N.C.Z, C.T or H.T fibration are invariant by pull back.

ii) Every T.N.C.Z. Serre fibration is a rational fibration, when base or fibre has finite type.

Proof of a). — Since $H(F, k) = \Lambda X$, the fibration $(*)$ admits a K.S-minimal model

$$\mathcal{E} : (A(M), d_M) \xrightarrow{\iota} (A(M) \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, 0)$$

with ρ^* surjective. Choose a homogeneous basis of X , $(x_\alpha)_\alpha$ and for each α , an element $c_\alpha \in (A(M) \otimes \Lambda X) \cap \ker d$ such that $\rho^*([c_\alpha]) = x_\alpha$ so that σ in lemma 2 is defined by $\sigma(x_\alpha) = c_\alpha$.

Proof of b). — By Theorem 2 there is a K.S minimal model \mathcal{E} of $(*)$:

$$\mathcal{E} : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \bar{d})$$

with $\dim X^{\text{odd}} = \dim X^{\text{even}}$, $dX^{\text{even}} = 0$, $dX^{\text{odd}} \subset B \otimes \Lambda X^{\text{even}}$. From [10], we have $H(\Lambda X, \bar{d}) = \Lambda X^{\text{even}} / \bar{d}(X^{\text{odd}}) \cdot (\Lambda(X^{\text{even}}))$. Let τ be as in lemma 1; then for each $y \in X^{\text{even}}$, there exists $\alpha_y \in (B \otimes \Lambda X) \cap \ker d$ such that

$$\tau([y]) = [\alpha_y].$$

One verifies that

$$\rho(\alpha_y) = y + \bar{d}\beta_y^+ \quad \text{with} \quad \beta_y^+ \in \Lambda X^{\text{even}} \otimes \Lambda^{\geq 1} X^{\text{odd}}.$$

Hence

$$\alpha_y = y + d\beta_y^+ + \Omega_y \quad \text{with} \quad \Omega_y \in B^+ \otimes \Lambda X.$$

Put

$$\sigma(y) = \alpha_y - d\beta_y^+$$

then

$$\rho \circ \sigma = \text{Id} \mid_{\Lambda(X^{\text{even}})} \quad \text{and} \quad \sigma^* = \tau.$$

On the other hand, from the formulas

$$\tau[\bar{d}x] = [\sigma(\bar{d}x)] = 0 \quad \text{and} \quad \rho(\sigma \bar{d}x) = \bar{d}x, \quad x \in X^{\text{odd}},$$

we deduce

$$\sigma(\bar{d}x) = \bar{d}x + \Omega_x^+ = d\beta_x$$

with

$$\Omega_x^+ \in B^+ \otimes \Lambda X \quad \text{and} \quad \beta_x \in B \otimes \Lambda X.$$

Thus

$$\sigma(\bar{d}x) = dx + d\hat{\Omega}_x^+.$$

with $\hat{\Omega}_x^+ \in B^+ \otimes \Lambda X$ so we put

$$\sigma(x) = x + \hat{\Omega}_x^+.$$

This defines σ as required in lemma 2.

PROPOSITION 7. — *The next lemma is straightforward.*

LEMMA 3. — *A rational fibration (*) is a σ -fibration (resp. W.H.T.) if and only if there exists a K.S-minimal model*

$$\mathcal{E} : (B, d_B) \xrightarrow{\iota} (B \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \bar{d})$$

with B a connected algebra (resp. with $B = \Lambda Z$ the minimal model of M) such that :

$$\forall x \in X, \quad dx - \bar{d}x \in B^+ \otimes \Lambda^+(X)$$

(resp., $\forall x \in X, \quad dx - \bar{d}x \in (\Lambda^+ Z \cdot \Lambda^+ Z) \oplus (\Lambda^+ Z \otimes \Lambda^+ X)$).

Proof of a). — This results directly from lemma 3.

Proof of b). — Let $(\Lambda Z, d_B)$ be a K.S-minimal model of M and $\mathcal{E} : (\Lambda Z, d_B) \xrightarrow{\iota} (\Lambda Z \otimes \Lambda X, d) \xrightarrow{\rho} (\Lambda X, \bar{d})$ a K.S-minimal

model of (*). Since M is coformal $d_B Z \subset \Lambda^2 Z$ and since $\dim H(F) < +\infty$, from [6] we deduce that $\partial^\#(X^{\text{even}}) = 0$.

Suppose that there exists $x \in X^{\text{odd}}$ such that $\partial^\# x = b \neq 0$ then

$$dx = \bar{d}x + b + \Phi + \Omega$$

with

$$b \in \Lambda^1 Z, \quad \Phi \in \Lambda^1 Z \otimes \Lambda^+ X, \quad \Omega \in \Lambda^{\geq 2} Z \otimes \Lambda X.$$

We can suppose $x = e_{\alpha_0}$ where α_0 is the smallest index in a K.S-minimal basis such that $\partial^\# e_{\alpha} \neq 0$. A simple calculation from $d^2 x = 0$ and the fact that $db \in \Lambda^2 Z$ gives $db = 0$. Hence $[b]$ lives in the spherical cohomology of M and from our hypothesis, b is coboundary which is impossible. This proves $\partial^\# = 0$.

I would like to take this opportunity to thank Professors S. Halperin and D. Lehmann for their helpful advice.

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Mansucrit reçu le 29 décembre 1980.

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