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WHITNEY REGULARITY AND GENERIC WINGS

by V. NAVARRO AZNAR and D. J. A. TROTMAN

Introduction.

In his Arcata lectures, « Introduction to equisingularity problems » [10], B. Teissier points out that one of the desirable properties for a condition of equisingularity of a stratification along a linear stratum Y is that it be preserved after intersection with generic linear spaces (wings) containing Y. He defined in [10] an equisingularity condition (*c*), for families of complex hypersurfaces with isolated singularities, which has the required property. Moreover it is a generic condition (and implies Whitney regularity). Proofs of these results, now published in the proceedings of the 1976 symposium on singularities at Oslo (§ 2 of [20]), circulated privately in 1974 in handwritten notes which supplemented [10]. Inspired by these proofs, J. Briançon and J.-P. Speder [2] proved, again for families of complex hypersurfaces with isolated singularities, that Whitney regularity also has the required property of stability after intersection with generic linear wings. As a consequence they deduce that Whitney regularity is actually equivalent to Teissier's condition (*c*) in this special case.

In [8] V. Navarro Aznar showed that Whitney regularity is preserved after intersection with generic wings when the strata have analytic closures (of arbitrary codimension) and the base stratum is 1-dimensional. Several consequences of this result are described in [8]. Using a quite different method B. Teissier has recently removed the hypothesis that the base stratum be 1-dimensional in the complex analytic case (see [11]).

Theorem 3.14 of this paper covers the general case of subanalytic incident strata of arbitrary dimensions for two equisingularity conditions : Kuo's ratio test [6], and Verdier's condition (*w*) [16]. Because the ratio test is equivalent to the Whitney conditions when the base stratum is 1-dimensional it now makes good sense to talk of the *level of Whitney regularity* of a subanalytic stratum X along a 1-dimensional stratum Y at a point y in Y : this is the smallest integer k such that for an open dense subset of the codimension k wings W containing Y, the pair $(X \cap W, Y)$ is Whitney regular at y. For Y of dimension two or higher the method is no longer valid without further hypotheses on the pair (X,Y); we give a semialgebraic counterexample (3.20).

Now Whitney regularity decomposes into two independent conditions, (a)-regularity and (b^π)-regularity, where π is a retraction onto the base stratum [18]. Each condition says that certain limits of secant vectors are contained in corresponding limits of tangent spaces to X. When X is subanalytic the set Λ^π of « bad » limits of secant vectors for (b^π) (i.e. those not contained in the corresponding limiting tangent space) is also subanalytic, and hence has a well-defined dimension, namely the maximal dimension of strata of a stratification of Λ^π into smooth submanifolds. Theorem 3.17 says that the dimension of Λ^π is precisely one less than the level of (b^π)-regularity for the pair (X,Y).

For smooth stratified sets, where the strata are not necessarily subanalytic, the method of proof of [8], which uses the curve selection lemma, breaks down. However when an extra hypothesis on the Hausdorff dimension of the space of limits of tangent spaces to X at 0 is satisfied, a general position argument is enough to imply that Whitney regularity is preserved after intersection with generic wings. We give examples showing that this extra hypothesis cannot be dropped.

1. Definitions.

Let X, Y be disjoint C^1 submanifolds of $\mathbf{R}^n$. We carry out a local study which will apply equally well to the case of submanifolds of a manifold. Let $0 \in Y \cap \bar{X}$, and let π be a C^1 retraction onto Y induced by a C^1 tubular neighbourhood of Y.

The regularity conditions introduced by H. Whitney 15 years ago in [18] and [19] are as follows. Suppose we are given sequences $\{x_i\}$ in X and $\{y_i\}$ in Y tending to 0 such that $\{T_{x_i}X\}$ tends to a limit τ, $\left\{\frac{x_i y_i}{|x_i y_i|}\right\}$ tends to a limit λ_1, and $\left\{\frac{x_i \pi(x_i)}{|x_i \pi(x_i)|}\right\}$ tends to a limit λ_2, each in the appropriate grassmannians. If $T_0Y \subset \tau$ for all such sequences we say that the pair (X,Y) is (*a*)-regular at 0. Similarly if always $\lambda_1 \in \tau$ we say (X,Y) is (*b*)-regular at 0, and if merely every $\lambda_2 \in \tau$ we shall say (X,Y) is (b^π)-regular at 0. The conditions (*a*) and (b^π) were those in the first paper [18] of Whitney; (*b*) was defined in [19]. A simple argument, given in [17] and [15], shows that (*b*) is equivalent to the combination of (*a*) and (b^π). An equally simple argument, which we have not seen before, shows that if (b^π) is satisfied for all linear retractions π then (*a*) and hence (*b*) follow. For if v is any unit vector in T_0Y, then any sequence $\{x_i\}$ defines some limit vector λ_2 associated to a given linear retraction π as above, and we can choose another linear retraction π' so that if λ'_2 is the associated limit vector, then the vector subspace containing λ_2 and λ'_2 contains v. If (b^π) and ($b^{\pi'}$) hold we have that

$$\lim T_{x_i}X \supset \langle\lambda_2\rangle \oplus \langle\lambda'_2\rangle \ni v.$$

Repeating the argument for each unit vector in T_0Y we deduce (*a*)-regularity. We take this implication as justification for concentrating on (b^π) in this paper. Notice that if Y is 1-dimensional it is enough that (b^π) be satisfied for precisely n distinct linear retractions whose fibres have unit normal vectors corresponding to a basis of $\mathbf{R}^n$.

We call a C^1 submanifold W of $\mathbf{R}^n$ of codimension k with $Y \subset W$, a *wing of codimension k attached to* Y (extending the terminology of Whitney in [19]). We say that a pair (X,Y) is ($E_{\mathrm{cod}\,k}$)-regular if, for an open dense subset of the space of wings of codimension k attached to Y (in the topology induced by that on G^m_{m-k}, where $m = n - \dim Y$, by taking tangent spaces at 0), the pair $(X \cap W, Y)$ is an E-regular pair at 0, where E is some equisingularity condition. An important problem (see the introduction) is to determine when (*b*)-regularity implies ($b_{\mathrm{cod}\,k}$)-regularity. Our answers to this problem hinge on the following elementary fact (compare Thom [12]) :

LEMMA 1.1. – *Let* $\{H_i\}$ *and* $\{T_j\}$ *be sequences in* G^n_{n-k} *and* G^n_p *respectively. If* $\lim_{i\to\infty} H_i$ *and* $\lim_{j\to\infty} T_j$ *are transverse as vector subspaces of* $\mathbf{R}^n$, *then* $(\lim_{i\to\infty} H_i) \cap (\lim_{j\to\infty} T_j) = \lim_{i\to\infty} (H_i \cap T_i)$.

COROLLARY 1.2. – *Let* X, Y *be disjoint* C^1 *submanifolds of* $\mathbf{R}^n$, $0 \in Y$, *such that*

(i) (X,Y) *is* (*b*)-*regular at* 0.

(ii) W *is a wing attached to* Y *transverse at* 0 *to all elements of*

$$\{T \in G_p^n | \exists \{x_i\} \in X \cap W, \text{ so } x_i \to 0, T_{x_i}X \to T\}.$$

Then $(X \cap W, Y)$ *is* (*b*)*-regular at* 0.

Proof. – Follows immediately from lemma 1.1 and the definitions.

We shall prove in section 3 that for subanalytic (X,Y) if we suppose either that the dimension of Y is one, or that (X,Y) satisfies Kuo's ratio test at 0, then the hypothesis (ii) of corollary 1.2 is satisfied for an open dense set of wings of codimension k. It follows that for 1-dimensional Y, (b) implies ($b_{\operatorname{cod} k}$), (a) implies ($a_{\operatorname{cod} k}$), and (b^π) implies ($b^\pi_{\operatorname{cod} k}$).

In section 4 we show that if the space $\tau(X,0)$ of limits of tangent spaces to X at 0 has dimension at most ($\dim X - \dim Y - k$), then generic wings of codimension k are transverse at 0 to all limits of tangent spaces to X, so that again hypothesis (ii) of corollary 1.2 is verified.

Call $\min \{k | (b^\pi_{\operatorname{cod} k}) \text{ holds for } (X,Y)\}$ the *level* of (b^π)-regularity of the pair (X,Y) at 0, similarly for other regularity conditions. Note that if Y is 1-dimensional and (X,Y) is (a)-regular, the level of (b^π)-regularity and the level of (b)-regularity are the same.

Let $\Lambda^\pi(X,Y) = \{\lambda | \exists x_i \in X \text{ such that } \lambda = \lim \frac{x_i\pi(x_i)}{|x_i\pi(x_i)|} \not\subset \lim T_{x_i}X\}$, the *bad limit set* for (b^π)-regularity. This is subanalytic if X is subanalytic (see the proof of theorem 2.1), but may be highly pathological in general. In section 3 we derive the following characterization.

MAIN THEOREM (3.17). – *Let* X *be a subanalytic* C^1 *submanifold of* $\mathbf{R}^m \times \mathbf{R}$ *such that* $Y = 0 \times \mathbf{R} \subset \bar{X} - X$. *Then* $\dim \Lambda^\pi(X,Y)$ *is precisely one less than the level of* (b^π)*-regularity for the pair* (X,Y).

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2. The dimension of the bad limit set is less than the level of regularity.

In this section we prove the first of the two results which imply the Main theorem (3.17).

THEOREM 2.1. – *Let* $Y = 0^m \times \mathbf{R}^{n-m} \subset \mathbf{R}^n$ *and let* X *be a subanalytic* C^1 *submanifold of* $\mathbf{R}^n$, *disjoint from* Y, *with* $0 \in \bar{X}$. *Suppose that* (X,Y) *is* $(b^\pi_{\operatorname{cod} k})$*-regular at* 0. *Then* $\dim \Lambda^\pi(X,Y) < k$.

Proof. – Let G denote the (subanalytic) graph in $\mathbf{R}^n \times G^m_1 \times G^n_d$ of the map $x \longmapsto \left(\frac{\pi(x)}{|x\pi(x)|}, T_x X\right)$ where $d = \dim X$. Let p_1, p_2 denote projection from $\mathbf{R}^n \times G^m_1 \times G^n_d$ onto $\mathbf{R}^n$ and G^m_1 respectively.

Let $B = \{(0,\lambda,T) | \lambda \not\subset T\}$, and let $E = p_1^{-1}(0) \cap \bar{G}$. Write Λ for $\Lambda^\pi(X,Y)$ as defined in section 1. Then $\Lambda = p_2(E \cap B)$ and is a subanalytic subset of G^m_1 because $p_2 | p_1^{-1}(0)$ is proper.

Choose Whitney stratifications $\mathscr{G}$ of $\bar{G}$ and $\mathscr{L}$ of Λ so that E is a union of strata of $\mathscr{G}$ and p_2 maps strata of $\mathscr{G}$ submersively onto strata of $\mathscr{L}$ ($\mathscr{G}$ and $\mathscr{L}$ exist by [5], for example). Each linear wing W of codimension k attached to Y defines a smooth submanifold $L_W = \{\lambda \in G^m_1 | \lambda \in W\}$ which is of codimension k in G^m_1 (L_W is a copy of P^{m-1-k} in $P^{m-1} = G^m_1$). Suppose that the conclusion of the theorem is invalid and that $\dim \Lambda \geqslant k$. Let $\lambda_0 \in \Lambda$ be a point of a stratum Λ_0 of $\mathscr{L}$ of dimension $\geqslant k$, and let W_0 be a linear wing of codimension k such that L_{W_0} meets Λ_0 transversely at λ_0. Then there is an open neighbourhood U of W_0 such that for every linear wing W of codimension k in U, L_W meets Λ_0 transversely near λ_0.

Choose now $z_0 = (0,\lambda_0,T_0) \in E \cap B$. Such a point exists by definition of Λ. Let Z_0 denote the stratum of $\mathscr{G}$ containing z_0, then by our choice of $\mathscr{G}$, $p_2 : Z_0 \to \Lambda_0$ is a submersion. Because $E \cap B$ is open in E, we can find a smaller neighbourhood $U_1 \subset U$ of W_0 such that for every linear wing $W \in U_1$, $p_2^{-1}(L_W)$ meets Z_0 transversely in at least one point z_W in $E \cap B$. We assert that for such W, $p_2^{-1}(L_W)$ contains a sequence of points $\{z_i\}$ in G tending to z_W, i.e. that $z_W \in \overline{p_2^{-1}(L_W) \cap G}$. This is because $\mathscr{G}$ is a Whitney stratification and $Z_0 \subset \bar{G}$ (see 10.4 of Mather's Harvard notes on topological stability (1970)). Write $x_i = p_1(z_i)$, then

$\{x_i\}$ is a sequence of points in $X \cap W$ tending to 0 with the property that $\lim \frac{x_i\pi(x_i)}{|x_i\pi(x_i)|} \not\subset \lim T_{x_i}X$ (since $z_W \in B$), and hence (b^π) fails for the pair $(X \cap W, Y)$ for all linear W in the open set U_1. It follows at once that $(b^\pi_{\mathrm{cod}\,k})$ is not satisfied. This completes the proof of theorem 2.1.

Remark 2.2. — It seems plausible that theorem 2.1 remains valid if we remove the hypothesis that X be subanalytic, at least when X is locally connected at 0, however we have no complete proof as yet.

3. The dimension of the bad limit set is greater than or equal to the level of regularity minus one.

LEMMA 3.1. — *Let* Z *be a subanalytic* C^1 *submanifold of* $\mathbf{R}^n$ *and let* A *be a linear subspace of* $\mathbf{R}^n$, $A \subset \bar{Z} - Z$. *Then there exists an open dense set* U *of* A *such that for each* $a \in U$, *there is a constant* $C_a < \infty$, *and a neighbourhood* V_a *of* a *in* $\mathbf{R}^n$ *such that*

$$\frac{|\pi_{N_z Z}(t)|}{|z - \pi_A(z)|\,|t|} \leqslant C_a$$

for all $t \in T_aA - \{0\}$, *and* $z \in Z \cap V_a$.

Proof. — By theorem 2.2 of [16] there is a stratification of $\bar{Z}$, $\cup X_\alpha$, compatible with A, which verifies the (w)-regularity condition of Verdier. Let $\{X_\alpha\}_{\alpha\in B}$ be the strata contained in A of dimension strictly less than A, and let $U = A - \bigcup_{\alpha\in B} X_\alpha$, then U is open and dense in A. Let $a \in U$ and let $X_1, \ldots, X_r$ be the strata of Z containing a in their closure. Because the pairs (X_i, A) are (w)-regular at a there exist constants C_i and neighbourhoods V_i of a in $\mathbf{R}^n$ such that

$$\frac{|\pi_{N_z X_i}(t)|}{|z - \pi_A(z)|\,|t|} \leqslant C_i < \infty$$

for all nonzero $t \in T_aA$ and all $z \in X_i \cap V_i$ $(i = 1,\ldots,r)$. As we have that $|\pi_{N_z Z}(t)| \leqslant |\pi_{N_z X_i}(t)|$ for all $t \in T_aA$ and $z \in X_i$, it suffices to take $C_a = \inf\{C_i | i = 1,\ldots,r\}$ and $V_a = \bigcap_{i=1}^{r} V_i$. This proves lemma 3.1.

Notation. – In this section we frequently study an analytic arc $a(t) = (p(t),q(t)) \in \mathbf{R}^m \times \mathbf{R}^s$. We shall write

$$\alpha = \inf\{v(p_i(t)) | i = 1,\ldots,m\}$$
$$\beta = \inf\{v(q_i(t)) | i = 1,\ldots,s\}$$

where $v(\ )$ denotes the usual valuation on real analytic functions.

LEMMA 3.2. – *Let* X *be a subanalytic* C^1 *submanifold of* $\mathbf{R}^m \times \mathbf{R}^s$, *such that* $Y = 0 \times \mathbf{R}^s \hookrightarrow \bar{X} - X$. *For each* k, $0 \leqslant k \leqslant m$, *there exists an open dense subset* U_0^k *of the grassmannian of* $(m+s-k)$*-planes containing* Y *such that if* $H \in U_0^k$ *and* $a(t) = (p(t), q(t)) \in \mathbf{R}^m \times \mathbf{R}^s$ *and* $H(t) \in U_0^k$ *are real analytic curves with* $a(0) = 0$, $H(0) = H$, $a(t) \in X \cap H(t)$ *if* $t \neq 0$ *and* $\alpha \leqslant \beta$, *then* H *is transverse to* $\lim_{t\to 0} T_{a(t)}X$.

Proof. – We can restrict our attention to the open set of planes of codimension k containing Y defined by sets of equations,

$$x_1 = \sum_{i=k+1}^{m} a_{i1}x_i, \ldots, x_k = \sum_{i=k+1}^{m} a_{ik}x_i.$$

Because X is subanalytic, if we define

$$f : \mathbf{R}^{m-k} \times \mathbf{R}^s \times \mathbf{R}^{k(m-k)} \to \mathbf{R}^m \times \mathbf{R}^s$$

by

$$f(x_{k+1},\ldots,x_m,y_1,\ldots,y_s,a_{k+1,1},\ldots,a_{mk})$$
$$= \left(\sum_{i=k+1}^{m} a_{i1}x_i,\ldots,\sum_{i=k+1}^{m} a_{ik}x_i, x_{k+1},\ldots,x_m,y_1,\ldots,y_s\right)$$

then $Z = f^{-1}(X)$ is subanalytic by [5, 3.8]. Also Z is a C^1 submanifold since f is a submersion at (x,y,a) whenever

$$x = (x_{k+1},\ldots,x_m) \neq (0,\ldots,0),$$

as is easily checked. Because $A = 0 \times 0 \times \mathbf{R}^{k(m-k)}$ is contained in $\bar{Z} - Z$ we can now apply lemma 3.1. Thus there exists an open dense subset U_0^k of A such that for each $a \in U_0^k$ there is a constant $C_a < \infty$ and a neighbourhood V_a of a in $\mathbf{R}^{m-k} \times \mathbf{R}^s \times \mathbf{R}^{k(m-k)}$ such that

$$\frac{|\pi_{N_z Z}(b-a)|}{|z-\pi_A(z)|\,|b-a|} \leqslant C_a \tag{3.3}$$

for all $b \in A - \{a\}$ and $z \in (Z - A) \cap V_a$.

We shall show that U_0^k satisfies the conclusion of the theorem in an equivalent form : let H be the $(m+s-k)$-plane corresponding to a point $a \in U_0^k$ and let h be a unit normal vector to H, and let $n(t)$ be a field of unit normal vectors to X along the real analytic curve $a(t)$, then $\lim_{t\to 0} |\langle h,n(t)\rangle| < 1$.

For simplicity of notation let $a = 0 \in U_0^k$, so that $h = (h_1,\ldots,h_k,0)$ and if $n(t) = (n_1(t),\ldots,n_{m+s}(t))$ we must show that

$$\inf\{v(n_i(t))|1 \leqslant i \leqslant k\} \geqslant \inf\{v(n_i(t))|k+1 \leqslant i \leqslant m+s\}.$$

Consider the analytic curve

$$\tilde{a}(t) = (p_{k+1}(t),\ldots,p_m(t), q_1(t),\ldots,q_s(t), a_{k+1,1}(t),\ldots,a_{mk}(t))$$

on Z, where the $\{a_{ij}(t)\}$ define $H(t)$, and the field of nonzero normal vectors $\tilde{n}(t) = (\tilde{n}_{k+1}(t),\ldots,\tilde{n}_{m+s}(t), \tilde{n}_{k+1,1}(t),\ldots,\tilde{n}_{mk}(t))$ at $\tilde{a}(t)$ which is the lift of $n(t)$ by f, i.e.

$$\begin{aligned} \tilde{n}_i &= n_i + \sum_{j=1}^{k} a_{ij}n_j \quad (k+1 \leqslant i \leqslant m) \\ \tilde{n}_i &= n_i \quad (m+1 \leqslant i \leqslant m+s) \\ \tilde{n}_{ij} &= p_i n_j \quad (k+1 \leqslant i \leqslant m, 1 \leqslant j \leqslant k). \end{aligned} \tag{3.4}$$

By (3.3) we have for $k+1 \leqslant i \leqslant m$, and $1 \leqslant j \leqslant k$, that

$$\lim_{t\to 0} \frac{|p_i(t)n_j(t)|}{|(p(t),q(t))|\,|(\tilde{n}_{k+1}(t),\ldots,\tilde{n}_{m+s}(t),\ldots,p_m(t)n_k(t))|} < \infty$$

and hence that

$$\lim_{t\to 0} \frac{|p_i(t)n_j(t)|}{|(p(t),q(t))|\,|(\tilde{n}_{k+1}(t),\ldots,\tilde{n}_{m+s}(t))|} < \infty.$$

By hypothesis $\alpha = \inf\{v(p_i)\} \leqslant \beta = \inf\{v(q_i)\}$, so that

$$\lim_{t\to 0} \frac{|n_j(t)|}{|\tilde{n}_{k+1}(t),\ldots,\tilde{n}_{m+s}(t)|} < \infty,$$

and hence

$$\begin{aligned}\inf\{v(n_j)|1\leqslant j\leqslant k\} &\geqslant \inf\{v(\tilde{n}_i)|k+1\leqslant i\leqslant m+s\}\\ &\geqslant \inf\{v(n_i)|k+1\leqslant i\leqslant m+s\} \quad \text{by (3.4).}\end{aligned}$$

This completes the proof of lemma 3.2.

LEMMA 3.5. — *Let* X *be a subanalytic* C^1 *submanifold of* $\mathbf{R}^m \times \mathbf{R}^s$ *such that* $Y = 0 \times \mathbf{R}^s \subset \bar{X} - X$. *Let* H *be an* $(m+s-k)$*-plane containing* Y, *and* $a(t) = (p(t), q(t))$ *in* $\mathbf{R}^m \times \mathbf{R}^s$ *and* H(t) *in* G^{m+s}_{m+s-k} *analytic curves with* $a(0) = 0$, $H(0) = H$, $Y \subset H(t)$, *and* $a(t) \in X \cap H(t)$ *for* $t \neq 0$. *If* (*b*)*-regularity fails for* (X,Y) *along* $a(t)$, *then* H *is transverse to* $\lim_{t\to 0} T_{a(t)}X$.

Proof. — With notation as in lemma 3.3, if (*a*)-regularity fails for (X,Y) along $a(t)$, then

$$\begin{aligned}\inf\{v(n_{m+i})|1\leqslant i\leqslant s\} &\leqslant \inf\{v(n_j)|1\leqslant j\leqslant m\}\\ &\leqslant \inf\{v(n_j)|1\leqslant j\leqslant k\}\end{aligned}$$

so that the required transversality follows.

Let $\pi : \mathbf{R}^m \times \mathbf{R}^s \to \mathbf{R}^s$ be projection. If (b^π) fails for (X,Y) along $a(t) \in X \cap H(t)$, then

$$\lim_{t\to 0} \frac{\sum_{i=1}^{m} p_i(t)n_i(t)}{|p(t)|\,.\,|(n_1(t),\ldots,n_{m+s}(t))|} \neq 0.$$

Assuming (*a*)-regularity we find that

$$v\left(\sum_{i=1}^{m} p_i n_i\right) \leqslant v(p) + \inf\{v(n_i)|1\leqslant i\leqslant m\}, \tag{3.6}$$

where we write $v(p)$ for $\inf\{v(p_i)|1\leqslant i\leqslant m\}$ by abuse of notation.

Now we may assume that H is given by $x_1 = \ldots = x_k = 0$. Then for $1 \leqslant i \leqslant k$, we can write $p_i(t) = \sum_{j=k+1}^{m} a_{ji}(t)p_j(t)$ where $a_{ji}(0) = 0$, since

$H(0) = H$ and $a(t) \in H(t)$. Hence

$$\sum_{i=1}^{m} p_i n_i = \sum_{j=k+1}^{m} p_j (n_j + \sum_{i=1}^{k} a_{ji} n_i). \tag{3.7}$$

Write

$$v(n_1) = \inf \{v(n_i) | 1 \leqslant i \leqslant k\},$$

and

$$v(n_m) = \inf \{v(n_i) | k+1 \leqslant i \leqslant m\}.$$

Then

$$v(p) + v(n_1) \geqslant v\left(\sum_{i=1}^{m} p_i n_i\right)$$

by (3.6)

$$\geqslant \inf \left(\left\{v(p_j) + v\left(n_j + \sum_{i=1}^{k} a_{ji} n_i\right) \middle| k + 1 \leqslant j \leqslant m\right\}\right)$$

by (3.7),

$$\geqslant v(p) + \inf (v(n_m), \inf \{v(a_{ji} n_i) | k+1 \leqslant j \leqslant m, 1 \leqslant i \leqslant k\}).$$

Thus either $v(n_1) \geqslant v(n_m)$, which is what we wish to prove, or if not,

$$v(n_1) > \{\inf v(n_i) | 1 \leqslant i \leqslant k\} = v(n_1),$$

which is patently absurd. This finishes the proof of lemma 3.5.

LEMMA 3.8. – *Let* X *be a subanalytic* C^1 *submanifold of* $\mathbf{R}^m \times \mathbf{R}^s$ *such that* $Y = 0 \times \mathbf{R}^s \subset \bar{X} - X$. *For each* k, $1 \leqslant k \leqslant m$ *and for each odd positive integer* r *there exists an open dense subset* U_r^k *of the grassmannian of* $(m+s-k)$*-planes containing* Y *such that if* $H \in U_r^k$ *and* $a(t) = (p(t), q(t))$ *in* $\mathbf{R}^m \times \mathbf{R}^s$ *and* $H(t)$ *in* U_r^k *are analytic curves with* $a(0) = 0$, $H(0) = H$, $a(t) \in X \cap H(t)$ *if* $t \neq 0$, *and* $\beta < \alpha < \beta + r$, *and so that the Kuo ratio test is satisfied by the curve* $a(t)$ *for the pair* (X,Y) *at* 0, *then* H *is transverse to* $\lim_{t \to 0} T_{a(t)} X$.

Proof. – We begin as in lemma 3.2 but taking

$$f(x_{k+1}, \ldots, x_m, y_1, \ldots, y_s, a_{k+1,1}, \ldots, a_{mk}) = \left(\sum_{i=k+1}^{m} a_{i1} x_i^r, \ldots, \sum_{i=k+1}^{m} a_{im} x_m^r, y_1^r, \ldots, y_s^r\right).$$

The statement of lemma 3.8 discusses only curves tangent to Y so we may ignore the fact that df may drop in rank when $y = 0$.

Applying lemma 3.1 again we find an open dense subset U_r^k of $A = 0 \times 0 \times \mathbf{R}^{k(m-k)}$ such that if $a \in U_r^k$ there is a constant $C_a < +\infty$ and a neighbourhood V_a of a in $\mathbf{R}^{m-k} \times \mathbf{R}^s \times \mathbf{R}^{k(m-k)}$ such that (3.3) holds.

As in lemma 3.2 we shall show that

$$\inf\{v(n_i(t)) | 1 \leqslant i \leqslant k\} \geqslant \inf\{v(n_i(t)) | k+1 \leqslant i \leqslant m+s\} \quad (3.9).$$

Consider the continuous path

$$a(t) = (p_{k+1}^{1/r}(t), \ldots, p_m^{1/r}(t), q_1^{1/r}(t), \ldots, q_s^{1/r}(t),\ a_{k+1,1}(t), \ldots, a_{mk}(t))$$

on Z, where the $\{a_{ij}(t)\}$ define $H(t)$, and the field of nonzero normal vectors along this path,

$$\tilde{n}(t) = (\tilde{n}_{k+1}, \ldots, \tilde{n}_{m+s}, \tilde{n}_{k+1,1}, \ldots, \tilde{n}_{mk}),$$

where

$$\tilde{n}_i(t) = r(p_i(t))^{\frac{r-1}{r}} n_i(t) + r \sum_{j=1}^{k} (p_i)^{\frac{r-1}{r}} a_{ij} n_j \qquad (k+1 \leqslant i \leqslant m)$$

$$\tilde{n}_{m+i}(t) = r(q_i(t))^{\frac{r-1}{r}} n_{m+i}(t) \qquad (1 \leqslant i \leqslant s)$$

$$\tilde{n}_{ij}(t) = p_i(t) n_j(t). \qquad (k+1 \leqslant i \leqslant m, 1 \leqslant j \leqslant s)$$

By (3.3) we have for $k+1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant k$, that

$$\lim_{t \to 0} \frac{|p_i n_j|}{|(p_{k+1}^{1/r}, \ldots, q_s^{1/r})| \cdot |(\tilde{n}_{k+1}, \ldots, \tilde{n}_{m+s})|} < \infty \qquad (3.10).$$

Suppose that

$$\beta = v(q_1) \leqslant v(q_j) \qquad (2 \leqslant j \leqslant s),$$

$$\alpha = v(p_m) \leqslant v(p_i) \qquad (k+1 \leqslant i \leqslant m-1),$$

and that

$$v(n_1) \leqslant v(n_i) \qquad (2 \leqslant i \leqslant k).$$

Then,

$$\lim_{t\to 0} \frac{|(p_{k+1}^{1/r},\ldots,q_s^{1/r})|\cdot|((q_1)^{\frac{r-1}{r}} n_{m+1},\ldots,(q_s)^{\frac{r-1}{r}} n_{m+s})|}{|p_m n_1|}$$

$$= \lim_{t\to 0} \frac{|q_1|\cdot|(n_{m+1},\ldots,(q_s/q_1)^{\frac{r-1}{r}} n_{m+s})|}{|p_m n_1|}, \quad \text{since} \quad \beta < \alpha,$$

$$\leqslant \lim_{t\to 0} \frac{|q_1|\cdot|(n_{m+1},\ldots,n_{m+s})|}{|p_m n_1|} \quad \text{since} \quad v(q_1) \leqslant v(q_j), 2 \leqslant j,$$

$$\leqslant \lim_{t\to 0} \frac{|a(t)|\langle v.n\rangle}{|p(t)|} \quad \text{with} \quad v = (0,\ldots,0,\underbrace{1/\sqrt{s},\ldots,1/\sqrt{s}}_{s \text{ terms}}) \in T_0Y,$$

assuming that $n(t)$ is a unit vector and that (3.9) is *not* satisfied,

$= 0$, by Kuo's ratio test [6], assumed to hold along $a(t)$.

Hence we obtain from (3.10) that

$$\lim_{t\to 0} \frac{|p_m n_1|}{|q_1^{1/r}|\cdot|(\tilde{n}_{k+1},\ldots,\tilde{n}_m)|} < \infty.$$

Choose j, $k+1 \leqslant j \leqslant m$, such that $\lim_{t\to 0} \frac{|\tilde{n}_i|}{|\tilde{n}_j|} < \infty$ for $k+1 \leqslant i \leqslant m$.

Then $\lim_{t\to 0} \frac{|p_m n_1|}{|q_1^{1/r}|\cdot|\tilde{n}_j|} < \infty$, in other words,

$$\text{(3.11)} \quad \lim_{t\to 0} \frac{|p_m n_1|}{|q_1^{1/r}|\cdot|(p_j)^{(r-1)/r} n_j + \sum_{i=1}^{k} (p_j)^{(r-1)/r} a_{ji} n_i|} < \infty.$$

Suppose first that $v(n_j) > v\left(\sum_{i=1}^{k} a_{ji} n_i\right)$. Then (3.11) implies

$$v(p_m) + v(n_1) \geqslant \frac{v(q_1)}{r} + \frac{(r-1)}{r} v(p_j) + v\left(\sum_{i=1}^{k} a_{ji} n_i\right)$$

$$\geqslant \frac{v(q_1)}{r} + \frac{(r-1)}{r} v(p_m) + v(n_1) + 1,$$

and so $v(p_m) - v(q_1) \geqslant r$, i.e. $\alpha - \beta \geqslant r$, contradicting the hypothesis of the lemma. We are left with the possibility that $v(n_j) \leqslant v\left(\sum_{i=1}^{k} a_{ji} n_i\right)$. Then (3.11) implies that

$$\begin{aligned} v(p_m) + v(n_1) &\geqslant \frac{1}{r} v(q_1) + \frac{(r-1)}{r} v(p_j) + v(n_j) \\ &\geqslant \frac{1}{r} v(q_1) + \frac{(r-1)}{r} v(p_m) + v(n_1) + 1, \end{aligned}$$

if (3.9) fails to hold. Again this implies a contradiction to the hypothesis of the lemma. We deduce that our assumption that (3.9) does not hold was false. This completes the proof of lemma 3.8.

Combining lemmas 3.2, 3.5 and 3.8 we obtain for $1 \leqslant k \leqslant m$ a residual subset of the $(m+s-k)$-planes H containing Y consisting of H transverse to $\lim_{t \to 0} T_{a(t)}X$ for all curves $a(t)$ in X « tangent » to H at 0 provided $a(t)$ is not of the following type : tangent to Y, (b)-regular, not satisfying Kuo's ratio test. Tzee-Char Kuo showed in [6] that such curves do not exist when the dimension of Y is one. This provides our next result.

THEOREM 3.12. – *Let* X *be a subanalytic* C^1 *submanifold of* $\mathbf{R}^m \times \mathbf{R}$ *such that* $Y = 0 \times \mathbf{R} \subset \tilde{X} - X$. *For each* k, $1 \leqslant k \leqslant m$, *there is an open dense subset* U^k *of the grassmannian of* $(m+s-k)$*-planes containing* Y *such that if* $H \in U^k$ *and if* $a(t) \in \mathbf{R}^m \times \mathbf{R}$ *and* $H(t)$ *in* U^k *are analytic curves with* $a(0) = 0$, $H(0) = H$, *and* $a(t) \in X \cap H(t)$ *when* $t \neq 0$, *then* H *is transverse to* $\lim_{t \to 0} T_{a(t)}X$.

Proof. – Let $d = \dim X$. Define $\Gamma^k = \{(x, T_xX, H) | x \in X \cap H\}$ in $\mathbf{R}^{m+1} \times G_d^{m+1} \times G_{m-k}^m$, where G_{m-k}^m denotes the $(m+s-k)$-planes containing Y. Let $E^k = \bar{\Gamma}^k \cap \{(0, T, H) | T + H \neq \mathbf{R}^{m+1}\}$. Then E^k is a compact subset of the compact manifold $\{0\} \times G_d^{m+1} \times G_{m-k}^m$. Let π_3 denote projection from $\mathbf{R}^{m+1} \times G_d^{m+1} \times G_{m-k}^m$ onto G_{m-k}^m, and define the open subset of G_{m-k}^m, $U^k = G_{m-k}^m - \pi_3(E^k)$.

We need only show that U^k contains a dense subset of G_{m-k}^m since it is easily verified that every $H \in U^k$ has the required property. We shall show that U^k contains the residual subset of G_{m-k}^m, $U_0^k \cap \left(\bigcap_{i=1}^{\infty} U_{2i-1}^k\right)$, where U_0^k is given by lemma 3.2 and U_{2i-1}^k by lemma 3.8. This is enough since a residual subset of a manifold is dense.

Let $H \in U_0^k \cap \left(\bigcap_{i=1}^{\infty} U_{2i-1}^k\right)$. If $H \notin U^k$, then $H \in \pi_3(E^k)$ and there exists $T \in G_d^{m+1}$ such that $(O,T,H) \in E^k$. By curve selection there will be an analytic curve in $\bar{\Gamma}$, $(a(t), T_{a(t)}X, H(t))$ with $a(t) \in X \cap H(t)$, $a(0) = 0$, $H(0) = H$, $\lim_{t\to 0} T_{a(t)}X = T$ and $T + H \neq \mathbf{R}^{m+1}$. Write $a(t)$ as $(p(t),q(t))$ in $\mathbf{R}^m \times \mathbf{R}$. By lemma 3.2, since $H \in U_0^k$ we have that $v(q) < \inf\{v(p_i) | 1 \leqslant i \leqslant m\}$. By lemma 3.5, (b)-regularity must hold for (X,Y) along $a(t)$ (since H and T are not transverse). In this case theorem 2 of [6] implies that Kuo's ratio test holds. Let now r be a positive odd integer such that $v(q) < \inf\{v(p_i)\} < v(q) + r$. Then lemma 3.8 shows that $H \notin U_r^k$. We have thus shown that

$U_0^k \cap \left(\bigcap_{i=1}^{\infty} U_{2i-1}^k\right) \subset U^k$, which completes the proof of theorem 3.12.

COROLLARY 3.13. — *Let* X *be a subanalytic* C^1 *submanifold of* $\mathbf{R}^m \times \mathbf{R}$ *such that* $Y = 0 \times \mathbf{R} \subset \bar{X} - X$. *If* (X,Y) *is* (b)*-regular at* 0, *then* (X,Y) *is* $(b_{\operatorname{cod} k})$*-regular at* 0 *for all* k, $1 \leqslant k \leqslant m$.

Proof. — Use theorem 3.12 and corollary 1.2.

Note I. — Jean-Pierre Henry and Michel Merle found a shorter proof of lemma 3.8 in the case of $s = 1$, also removing the restriction to curves on which Kuo's ratio test holds. Theorem 3.12 follows as before without citing lemma 3.5 or the equivalence of (b) and the ratio test.

Follow the proof of 3.8 (with $s = 1$) up until (3.10). Suppose again that $\alpha = v(p_m) \leqslant v(p_i)$ if $k + 1 \leqslant i \leqslant m - 1$, and that $v(n_1) \leqslant v(n_i)$ if $2 \leqslant i \leqslant k$. Now the hypothesis $\alpha < \beta + r$ implies that

$$v(|(p_j^{1/r}, q^{1/r})| \cdot |p_j^{(r-1)/r} n_j|) > v(p_j n_j) - 1,$$

and

$$v(|(p_j^{1/r}, q^{1/r})| \cdot |q^{(r-1)/r} n_{m+1}|) = v(q n_{m+1}).$$

Thus (3.10) implies that

$$v(p_m n_1) \geqslant \inf\left(\inf_{j>k} \{v(p_j n_j)\}, v(q n_{m+1})\right) \quad (*).$$

Now since $a(t)$ is an analytic curve in X, $a'(t)$ and $n(t)$ are orthogonal, so that $\sum_{j=1}^{m} p'_j n_j + q' n_{m+1} = 0$. Hence

$$v(q' n_{m+1}) = v\left(\sum_{j=1}^{m} p'_j n_j\right) \geqslant \inf_{j \leqslant m} \{v(p'_j n_j)\},$$

and so

$$v(q n_{m+1}) \geqslant \inf_{j \leqslant m} \{v(p_j n_j)\}.$$

From (*) we obtain in every case that

$$v(p_m) + v(n_1) \geqslant \inf_{j \leqslant m} \{v(p_j n_j)\},$$

and we may conclude as in the end of the proof of lemma 3.5 that $v(n_1) \geqslant v(n_m)$.

In this way one deduces theorem 3.12 without treating separately the cases when (*b*)-regularity holds or fails. Of course the condition

$$\sum_{j=1}^{m} p'_j n_j + q' n_{m+1} = 0$$

used above is instrumental in proving that (*b*) is equivalent to the ratio test when $\dim Y = 1$ (cf. [6]).

Note 2. – In February 1980 we had not found how to obtain an open dense set of planes for which the conclusion of 3.12 was valid, but had only the residual set $U_0^k \cap \left(\bigcap_{i=1}^{\infty} U_{2i-1}^k\right)$, and the statements of lemmas 3.2 and 3.8 had $a(t)$ in $X \cap H$ for nonzero t. The second author asked J. Giraud if one could prove by successive blowing-ups that all limits of tangent planes of the form $\lim_{t \to 0} T_{a(t)}X$ where $a(t) \in X \cap H$ could be obtained from curves $a(t)$ such that $(\alpha - \beta) < C_H$ for some large constant C_H. Giraud provided a detailed reply which made a positive answer seem promising, using in particular the constructibility of the set U^k in theorem 3.12. However the hypothesis that $a(t)$ belong to $X \cap H$ with H fixed causes problems. Shortly afterwards, M. Merle and, independently, the first author of this paper, observed that lemmas 3.2 and 3.8 worked in their present form so that obtaining an open dense set of planes as in 3.12 becomes easy.

Now we can prove that both Kuo's ratio test and Verdier's condition (w) are preserved after intersection with generic wings. Since the ratio test and (b)-regularity are equivalent for (X,Y) when $\dim Y = 1$ we obtain also that (b) implies ($b_{\mathrm{cod}\,k}$) in this case (once more).

THEOREM 3.14. – *Let* X *be a subanalytic* C^1 *submanifold of* $\mathbf{R}^m \times \mathbf{R}^s$ *such that* $Y = 0 \times \mathbf{R}^s \subset \bar{X} - X$. *Suppose that* (X,Y) *satisfies Kuo's ratio test (resp. Verdier's condition* (w)) *at* 0. *Then, for each* k, $1 \leqslant k \leqslant m$, *there is an open dense subset* U^k *of the grassmannian of* $(m+s-k)$*-planes containing* Y *such that if* W *is a wing of codimension* k *attached to* Y, *with* $T_0W \in U^k$, *then* $(X \cap W, Y)$ *satisfies Kuo's ratio test (resp. Verdier's condition* (w)) *at* 0.

Proof. – Define the open set U^k exactly as in theorem 3.12, and apply lemmas 3.2 and 3.8 to show that U^k is open and dense, noting that Verdier's condition (w) implies Kuo's ratio test [16].

Let now W be a wing of codimension k attached to Y with $T_0W \in U^k$. We must show that $(X \cap W, Y)$ satisfies Kuo's ratio test (resp. Verdier's condition (w)) at 0. Take the ratio test first. We shall work with sequences so as to allow for wings which are not subanalytic. Suppose first that W is linear, say $W = 0^k \times \mathbf{R}^{m-k} \times \mathbf{R}^s \subset \mathbf{R}^m \times \mathbf{R}^s$. Let $\{x_i\} \in X \cap W$ be a sequence tending to 0 with $x_i = (p^i, q^i) \in \mathbf{R}^m \times \mathbf{R}^s$, and take unit vectors n^i in $T_{x_i}W \cap N_{x_i}(X \cap W)$ with $\lim_{i\to\infty} n^i = n$.

Now for each i, $N_{x_i}W \oplus \langle n^i \rangle$ and $N_{x_i}X$ intersect in a 1-dimensional space since each lies in $N_{x_i}(X \cap W)$ which has dimension $n - (d-k)$ (with $n = m+s$, $d = \dim X$) and $\dim(N_{x_i}W \oplus \langle n^i \rangle) = k+1$ and $\dim N_{x_i}X = n - d$. Thus there is a unique unit vector $\hat{n}^i \in N_{x_i}X$ of the form $\hat{n}^i = (n^i_1, \ldots, n^i_k, n^i)$. Since $T_0W \in U^k$, T_0W and $\lim_{i\to\infty} \hat{n}^i$ are not orthogonal, so that $\dfrac{|(n^i_1, \ldots, n^i_k)|}{|n^i|}$ is bounded as i tends to ∞. But the ratio test for (X,Y) says that $\dfrac{|x_i| \cdot |n^i_{m+j}|}{|p_i| \cdot |\hat{n}^i|}$ tends to 0 $(1 \leqslant j \leqslant s)$ as i tends to ∞, and so we deduce that $\dfrac{|x_i| \cdot |n^i_{m+j}|}{|p_i| \cdot |n^i|}$ tends to 0 $(1 \leqslant j \leqslant s)$ as i tends to ∞, and this is precisely what is needed to show that $(X \cap W, Y)$ verifies the ratio test. A similar argument works for Verdier's condition (w) : the boundedness of $\lim_{i\to\infty} \frac{|n^i_{m+j}|}{|p_i|.|\hat{n}^i|}$ implies that of $\lim_{i\to\infty} \frac{|n^i_{m+j}|}{|p_i|.|n^i|}$.

The proof for nonlinear W is similar save that the unit vectors n^i have (small) nonzero components in the direction of N_0W. Since these vanish at 0 they do not affect the result.

This completes the proof of theorem 3.14.

Note. – To obtain $(b_{\text{cod}\,k})$ in the previous theorem, from the hypothesis that (X,Y) satisfy Kuo's ratio test, the argument is slightly easier. Take any wing W of codimension k attached to Y with the tangent space to W at 0 in U^k. Let $\{x_i\}$ be a sequence of points in $X \cap W$ tending to 0 with $T = \lim_{i\to\infty} T_{x_i}X$. Choose $(m+s-k)$-planes H_i containing Y so that the maximal distance of a unit vector in H_i to its projection on T_0W is given by the vector $Ox_i/|Ox_i|$. Because W is of class C^1 and $\{x_i\}$ in W tends to 0, it follows that H_i tends to T_0W. Then $\{(x_i,T_{x_i}X,H_i)\}$ tends to $(0,T,T_0W)$ in $\bar{\Gamma} \cap (\{0\} \times G^{m+s}_d \times G^m_{m-k})$ and since $T_0W \in U^k$ it follows that $T + T_0W = \mathbf{R}^{m+s}$, and hence that W satisfies condition (ii) of corollary 1.2. Condition (i) of corollary 1.2 is also satisfied if we assume that (X,Y) verifies the ratio test at 0, by theorem 1 of [6]. Corollary 1.2 now says that $(X \cap W, Y)$ is (b)-regular at 0.

Now we come to the result promised in the title of this section. First, another lemma.

LEMMA 3.15. – *Let Λ be a subset of $G^m_p(\mathbf{R})$ which is the union of countably many C^1 submanifolds of dimension $\leqslant h$, and let Ω denote $\{H \in G^m_r(\mathbf{R}) | \forall L \in \Lambda, L \not\subset H\}$. If $h < p(m-r)$, then Ω is a residual subset of $G^m_r(\mathbf{R})$.*

Proof. – Consider in $G^m_p \times G^m_r$ the subset $\Gamma = \{(L,H) | L \subset H$ and $L \in \Lambda\}$. If π is projection onto G^m_r, then Ω is the complement of $\pi(\Gamma)$.

Let $L_0 \in \Lambda$ and let U_0 be a compact neighbourhood of L_0 in the C^1 submanifold containing L_0 in Λ. Then the set $\Gamma_0 = \{(L,H) | L \subset H, L \in U_0\}$ is a C^1 submanifold of dimension at most $(r-p)(m-r) + h$, because it is fibred over U_0, which has dimension at most h, with fibres of dimension $(r-p)(m-r)$, each isomorphic with G^{m-p}_{r-p}.

Because $h < p(m-r)$ we obtain that $\dim \Gamma_0 < \dim G^m_r$, so that the complement of $\pi(\Gamma_0)$ in G^m_r is open and dense. Covering Λ by countably many neighbourhoods $\{U_i\}_{i\in I}$ we find that the complement Ω of $\pi(\Gamma)$ in G^m_r is residual, proving the lemma.

THEOREM 3.16. — *Let* X *be a subanalytic* C^1 *submanifold of* $\mathbf{R}^m \times \mathbf{R}$ *such that* $Y = 0 \times \mathbf{R} \subset \bar{X} - X$ *and* $\dim \Lambda^\pi(X,Y) = h$, *where* $\pi : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}$ *denotes projection. Then for each* k, $h < k \leqslant m$, *the pair* (X,Y) *is* $(b^\pi_{\operatorname{cod} k})$-*regular at* 0.

(See section 1 for the definitions of $\Lambda^\pi(X,Y)$ and $(b^\pi_{\operatorname{cod} k})$.)

Proof. — We apply lemma 3.15 with $p = 1$ to $\Lambda = \overline{\Lambda^\pi(X,Y)}$ which is a compact subanalytic subset of $G^{m^*}_1$ and has a finite stratification into C^1 submanifolds of dimension at most h. It follows that the set Ω of planes of codimension k containing Y which contain no elements of Λ is not only dense, because residual (by lemma 3.15), but also open, since Λ is closed.

Consider $U = \Omega \cap U^k$ where U^k is defined in the proof of theorem 3.12. Let W be a wing attached to Y with $T_0W \in U$, and let $\{x_i\} \in X \cap W$ be a sequence of points tending to 0. Because $T_0W \in U^k$, $\lim_{i\to\infty} T_{x_i}X$ and T_0W are transverse at 0. Also $\lim_{i\to\infty} \frac{x_i\pi(x_i)}{|x_i\pi(x_i)|} \in T_0W$ since x_i and $\pi(x_i)$ belong to W for each i, and so $\{x_i\}$ is a (b^π)-regular sequence since $T_0W \in \Omega$. Now apply corollary 1.2 to show that $\{x_i\}$ is a (b^π)-regular sequence for $(X \cap W, Y)$ at 0. Since $\{x_i\}$ was arbitrary $(X \cap W, Y)$ is (b^π)-regular at 0, and since this is true for each W with T_0W in the open dense subset U of G^m_{m-k} we have completed the proof of theorem 3.16.

Theorem 3.16 may be summed up by saying that « $\dim \Lambda < k$ implies $(b^\pi_{\operatorname{cod} k})$ if $\dim Y = 1$ », answering a question in {14}. In section 2 we showed that « $(b^\pi_{\operatorname{cod} k})$ implies $\dim \Lambda < k$ ». Combining these results gives,

THEOREM 3.17. — *Let* X *be a subanalytic* C^1 *submanifold of* $\mathbf{R}^m \times \mathbf{R}$ *such that* $Y = 0 \times \mathbf{R} \subset \bar{X} - X$. *Then*

$$\dim \Lambda^\pi(X,Y) = \inf\{k \mid (X,Y) \textit{ is } (b^\pi_{\operatorname{cod} k})\textit{-regular at } 0\} - 1 .$$

Proof. — Combine theorems 2.1 and 3.16.

This result has interesting corollaries for a family of complex hypersurfaces with isolated singularities.

COROLLARY 3.18. – *Let* $F : (C^{n+1} \times C, 0 \times C) \to (C, 0)$ *be a complex analytic function such that for each* t, $F_t^{-1}(0)$ *has an isolated singularity at* $0 \times t$, where $F_t(z) = F(z,t)$. *Suppose that the Milnor number of* F_t *at* 0 *is constant for* t *in a neighbourhood of* 0 *in* C. *Then* $\dim \Lambda^{\pi}(F^{-1}(0) - (0 \times C), 0 \times C) \neq 0$.

Proof. – Write $\Lambda^{\pi} = \Lambda^{\pi}(F^{-1}(0) - (0 \times C), 0 \times C)$. Suppose that $\mu(F_t)$ is constant and $\dim \Lambda^{\pi} = 0$. By theorem 3.16, $(b^{\pi}_{\mathrm{cod}\,k})$ holds for all $k \geqslant 1$. Now μ constant implies (*a*)-regularity (see [9] or [7]), and theorem 3.12 and corollary 1.2 imply that $(a_{\mathrm{cod}\,k})$ holds for all k. Thus $(b_{\mathrm{cod}\,k})$ holds for $k \geqslant 1$. Applying the Thom-Mather isotopy lemma and the topological invariance of the Milnor number (n times) we deduce that $\mu^1, \ldots, \mu^n$ are constant, so that μ^* is constant. (See [9] for the definition of the μ^i.) By [9] this implies (*b*)-regularity which in turn implies that $\Lambda^{\pi} = \varnothing$, so in particular the dimension is not zero ! This completes the proof.

Example 3.19. – The only known examples of μ constant families which are not (*b*)-regular are due to Briançon and Speder [1]. The simplest example, $F(x,y,z,t) = x^3 + txy^3 + y^4z + z^9$, has $\mu(F_t) = 56$. One can verify [14] that $\dim \Lambda^{\pi} \geqslant 1$ for this example by direct calculation. That $\dim \Lambda^{\pi} = 1$ follows since F_t is equimultiple at 0 (with multiplicity 3), so that $(b_{\mathrm{cod}\,2})$ holds, and this implies that $\dim \Lambda^{\pi} \leqslant 1$ by theorem 2.1.

In the proof of corollary 3.18 we used Teissier's theorem « μ^* constant implies (*b*)-regularity ». It is amusing to note that this is a *consequence* of corollary 3.18. For, given that $(\mu^1, \ldots, \mu^{n+1}) = \mu^*$ is constant, we can assume by induction that $(b_{\mathrm{cod}\,1})$ holds since $\mu^1, \ldots, \mu^n$ are constant. Theorem 2.1 now gives that $\dim \Lambda^{\pi} < 1$. But corollary 3.18 tells us that $\dim \Lambda^{\pi} \neq 0$, so that Λ^{π} must be empty and (b^{π}) holds. Since (*a*)-regularity follows from the constancy of μ^{n+1} as noted above, we have (*b*)-regularity.

We are now tempted to speculate on a possible direct proof of corollary 3.18, perhaps using the nice characterization of μ constant given by Lê Dũng Tràng and K. Saito [7] : the normals to level surfaces of F tend to be orthogonal to the parameter space near 0.

Example 3.20. – Michel Merle found the following example showing that theorem 3.12 does not generalize to 2-dimensional Y. Let $f : \mathbf{R}^4 \to \mathbf{R}$ be defined by $f(x,y,u,v) = x^2 + y^2 + ux + vy$. Let $Y = \{x = y = 0\}$ in

$\mathbf{R}^4$, and $X = f^{-1}(0) - Y$. Then for every hyperplane H of $\mathbf{R}^4$ containing Y, $X \cap H$ contains a curve $a(t)$ such $\lim_{t \to 0} T_{a(t)}X = H$.

Now let $X_1 = X \cap \{(x,y,u,v) | x^2 + y^2 < (u^2 + v^2)^2\}$. All curves in X_1 passing through 0 are tangent to Y at 0. We claim (*b*)-regularity holds. Firstly (*a*)-regularity holds since $\operatorname{grad} f = (2x+u, 2y+v, x, y)$ which tends to be orthogonal to Y along curves tangent to Y at 0. Secondly

$$\frac{x(\partial f/\partial x) + y(\partial f/\partial y)}{|(x,y)| \cdot |(\partial f/\partial x, \partial f/\partial y)|} = \frac{x^2 + y^2}{|(x,y)| \cdot |(2x+u, 2y+v)|} \text{ on } X_1,$$

and this ratio clearly tends to 0 on curves tangent to Y; thus (b^π) holds, giving (*b*). Kuo's ratio test fails for the vector $(1,1) \in T_0Y$ along the curve (t^2, t^2, t, t) since

$$\frac{|(x,y,u,v)| \cdot |(x,y)|}{|(x,y)| \cdot |(2x+u, 2y+v)|} = \frac{(2t^4 + 2t^2)^{\frac{1}{2}}}{(8t^4 + 8t^3 + t^2)^{\frac{1}{2}}} \to \sqrt{2} \neq 0.$$

Worse, $(b_{\text{cod}\,1})$ fails. Intersect $Y \cup X_1$ with $H = \{\lambda x + \mu y = 0\}$ giving

$$\{(x,y,u,v) | \lambda x + \mu y = 0, \ \frac{(\lambda^2 + \mu^2)}{\lambda^2} y - \frac{\mu}{\lambda} u + v = 0,$$
$$x^2 + y^2 < (u^2 + v^2)^2\},$$

i.e. Y with (part of) a 2-plane, so (*b*) fails for $(X_1 \cap H, Y)$ at 0 for dimensional reasons ($X_1 \cap H$ ought to be empty for (*b*) to hold).

Thus we cannot generalize theorem 3.16 to Y of dimension 2 or more.

Question 3.21. — Is there an example of a semialgebraic (*b*)-regular (*r*)-fault (X,Y) for which $(b_{\text{cod}\,1})$ fails, as in example 3.20, and in addition $X \cup Y$ is locally closed at 0 ?

Note that in previous examples of semialgebraic (*b*)-regular (*r*)-faults (X,Y) given in [13], and [4] (where they are real algebraic), $X \cup Y$ is locally closed, and in fact is a C^1 submanifold so that generic wings attached to Y miss X and $(b_{\text{cod}\,1})$ is vacuously satisfied.

4. A sufficient condition for regularity to imply regularity at level k.

In this section we give a condition sufficient to imply that generic wings are transverse to all limits of tangent planes, not merely those given by sequences contained in the wing, or « tangent » to it.

THEOREM 4.1. — *Let* $\Gamma \subset G^n_{p+q}$ *be a subset of Hausdorff dimension less than* $p-k+1$. *Let* $Y = \mathbf{R}^q \times 0^{n-q} \subset \mathbf{R}^n$. *Then*

$$\Omega = \{H \in G^n_{n-k} \mid Y \subset H \text{ and } H \pitchfork T, \forall T \in \Gamma\}$$

is the complement of a set of measure zero in $\{H \in G^n_{n-k} \mid Y \subset H\}$, *and is thus dense.*

Proof. — Write $Y^\perp = 0^q \times \mathbf{R}^{n-q}$ and let

$$\Gamma_i = \{T \in \Gamma \mid \dim_{\mathbf{R}}(T \cap Y^\perp) = p+i\}.$$

Then $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_r$, where $r = \inf(q, n-p-q)$, and

$$\Omega = \bigcap_{i=0}^{r} \{H \in G^n_{n-k} \mid Y \subset H \quad \text{and} \quad H \pitchfork T,\ \forall T \in \Gamma_i\}.$$

Note that $\dim \Gamma_i \leqslant \dim \Gamma < p-k+1$ so that we can assume $\Gamma = \Gamma_i$.

If $H \cap Y^\perp$ and $T \cap Y^\perp$ are transverse in $Y^\perp$ it follows that H and T are transverse in $\mathbf{R}^n$, and thus it will suffice to prove the theorem in the case of $q = 0$. Also because

$$\dim \Gamma_i < p-k+1 < (p+i)-k+1 \quad (i \geqslant 1),$$

proving the result when $\Gamma = \Gamma_0$ implies the result for $\Gamma = \Gamma_i\,(i \geqslant 1)$.

We are left with the following lemma to prove.

LEMMA 4.2. — *Let* $\Gamma_0 \subset G^n_p$ *have Hausdorff dimension less than* $p-k+1$. *Then* $\Omega_0 = \{H \in G^n_{n-k} \mid H \pitchfork T, \forall T \in \Gamma_0\}$ *is the complement of a set of measure zero in* G^n_{n-k}.

Proof of lemma 4.2. — Set

$$A_j = \{(H,T) \in G^n_{n-k} \times G^n_p \mid \dim_{\mathbf{R}}(H \cap T) = p-k+j \quad \text{and} \quad T \in \Gamma_0\} \qquad (1 \leqslant j \leqslant n-p).$$

Then if $\pi_1 : G^n_{n-k} \times G^n_p \to G^n_{n-k}$ denotes projection onto the first factor, $\Omega_0 = G^n_{n-k} - \left(\pi_1\left(\bigcap_{j=1}^{n-p} A_j\right)\right)$.

It will suffice to show that $\dim_H A_j < k(n-k) = \dim G^n_{n-k}$ for each j, $1 \leqslant j \leqslant n-p$, where $\dim_H$ denotes Hausdorff dimension. For then

$\dim_H \pi_1\left(\bigcup_{j=1}^{n-p} A_j\right) < k(n-k)$ so that $\pi_1\left(\bigcup_{j=1}^{n-p} A_j\right)$ has zero measure in G^n_{n-k}.

Now A_j fibres over Γ_0 by projection $(H,T) \longmapsto T$, with fibre

$$A_{j,T} = \{H \in G^n_{n-k} | \dim_{\mathbf{R}}(H \cap T) = p - k + j\}.$$

Hence $\dim_H A_j = \dim A_{j,T} + \dim_H \Gamma_0$. Also $A_{j,T}$ fibres over G^p_{p-k+j} by intersection $H \longmapsto H \cap T$ with fibre isomorphic with $G^{n-p+k-j}_{n-p-j}$. Thus

$$\begin{aligned}\dim A_{j,T} &= \dim G^p_{p-k+j} + \dim G^{n-p+k-j}_{n-p-j} \\ &= (k-j)(p-k+j) + k(n-p-j) \\ &= k(n-k) - j(p-k+j).\end{aligned}$$

Since $\dim_H \Gamma_0 < p - k + 1$ we obtain

$$\dim_H A_j < k(n-k) - (j-1)(p-k+j),$$

for $1 \leqslant j \leqslant n-p$, and hence $\dim_H A_j < k(n-k)$ as required. This completes the proof of lemma 4.2 and hence of theorem 4.1.

Recall the notation of Whitney [19] : for X a C^1 submanifold of $\mathbf{R}^n$, $0 \in \bar{X}$, the space of limits of tangent spaces to X at 0 is

$$\tau(X,0) = \{T \in G^n_{\dim X} | \exists \{x_i\} \in X,\ x_i \to 0,\ T_{x_i}X \to T\}.$$

Corollary 4.3. – *Let* X, Y *be* C^1 *submanifolds of* $\mathbf{R}^n$, *with* $0 \in Y \cap (\bar{X} - X)$. *Let* $\tau(X,0)$ *have Hausdorff dimension* $\leqslant \dim X - \dim Y - k$. *Then in the space of planes of codimension* k *containing* T_0Y *those planes transverse to every element of* $\tau(X,0)$ *form an open dense subset.*

Proof. – Apply theorem 4.1 to obtain a dense subset. Since $\tau(X,0)$ is closed the required openness follows easily.

Corollary 4.4. – *Let* X, Y *be* C^1 *submanifolds of* $\mathbf{R}^n$ *with* $0 \in Y \subset (\bar{X} - X)$. *If* $\dim \tau(X,0) \leqslant \dim X - \dim Y - k$ *and* (X,Y) *is* (b)-*regular at* 0, *then* (X,Y) *is* $(b_{\mathrm{cod}\,k})$-*regular.*

Proof. – Apply corollary 1.2 and corollary 4.3.

Next we give examples showing that the restriction on the dimension of

$\tau(X,0)$ in corollary 4.4 cannot be dropped in general, or even when X is semialgebraic. Teissier has recently shown [11] that the restriction on $\dim \tau(X,0)$ is unnecessary in the complex analytic case, by proving that (*b*)-regularity implies $\dim \tau(X,0) \leqslant \dim X - \dim Y - 1$.

Example 4.5. – (Goresky Whirlpool). – A picture like the one below was drawn by Mark Goresky during a seminar at I.H.E.S. in June 1979, as an example (in his terms) of a Whitney object without a Whitney triangulation.

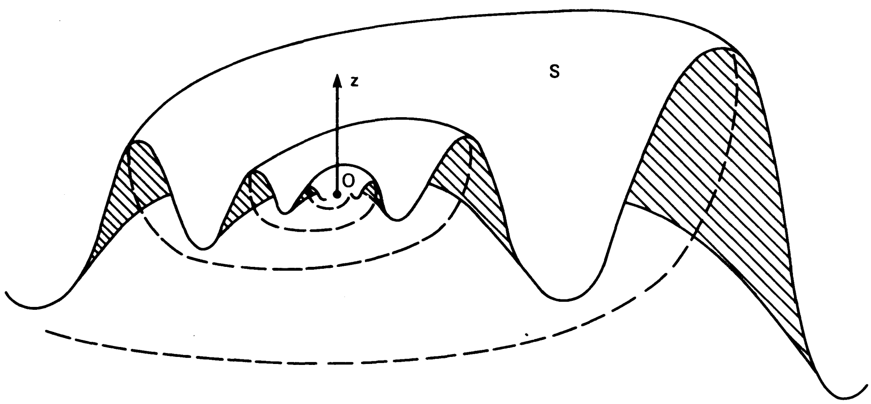

Figure : A portion of the Goresky Whirlpool.

The figure is of a surface S in $\mathbf{R}^3$ (based on a rapid spiral around 0 in the plane and a curve of the type $z = r \sin(1/r)$ in each vertical slice) which is smooth everywhere except at the origin, and so that the pair $(S - \{0\}, \{0\})$ is (*b*)-regular, but not $(b_{\text{cod}\,1})$-regular. Moreover the space of limits of tangents at 0 has dimension 2. This shows that in general (*b*)-regularity for a pair of strata (X,Y) does not imply that $\dim \tau(X,0) \leqslant \dim X - \dim Y - 1$

Here is an explicit representation of such a surface. Write

$$c(n,\theta) = (1/2)(e^{-(n\pi+\theta)^2} + e^{-((n+1)\pi+\theta)^2})$$

and

$$w(n,\theta) = (1/2)(e^{-(n\pi+\theta)^2} - e^{((n+1)\pi+\theta)^2}).$$

Then in cylindrical coordinates (r,θ,z),

$$\bigcup_{n\in\mathbf{N}} \{w(n,\theta)z = (w(n,\theta) - ((r-c(n,\theta))^2/w(n,\theta)))^2, \quad e^{-(n\pi+\theta)^2} \geqslant r \geqslant e^{-((n+1)\pi+\theta)^2}\}$$

defines a surface of class C^1 containing 0 in its closure, which has the required properties. Note that S intersects the horizontal plane $\{z=0\}$ in the two rapid spirals $\{(r,\theta)|r = e^{-t^2}, \theta = t \pmod{2\pi}\}$ and $\{(r,\theta)|r = e^{-t^2}, \theta = \pi + t \pmod{2\pi}\}$. Each rapid spiral has the property that the angle between the radial vector Ox defined by a point x on the spiral and the tangent to the spiral at x tends to zero as x approaches 0. Intersecting S with the cones $\{z/r = \text{constant}\}$ again gives 2 rapid spirals with the same property. It follows that (b^π)-regularity holds, where π is the canonical retraction along lines through 0, and hence (b) holds since (a) is trivially satisfied.

Fixing θ and letting r tend to zero we find a 1-dimensional set of unit vectors which are limits of unit normals to S, all contained in the plane $\theta = \text{constant}$. Varying θ we obtain $\dim \tau(S-\{0\},0) = 2$. To see that $(b_{\text{cod}\,1})$ fails take a sequence of points defined by $\theta = \text{constant}$ and

$$r = c(n,\theta) + (w(n,\theta)/\sqrt{3})$$

(compare [13], [14]).

Note 4.6. – The previous example is not subanalytic (at 0). Example 3.20 is semialgebraic and may also be used. We have already seen that (b) holds and $(b_{\text{cod}\,1})$ fails for the pair (X_1,Y) of example 3.20. One finds easily that

$$\dim\ \tau(X_1,0) = 1 = \dim\ X_1 - \dim\ Y.$$

Thus (b)-regularity for a pair (X,Y) does not imply that $\dim \tau(X,0) \leqslant \dim X - \dim Y - 1$ for semi-algebraic strata. However this example is unsatisfactory in that $X_1 \cup Y$ is not closed. See question 3.2.1.

After B. Teissier's recent result [11] in the complex analytic case there remains the question of what happens in the real algebraic/analytic case.

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