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Characteristic classes of subfoliations


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CHARACTERISTIC CLASSES OF SUBFOLIATIONS

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1. Introduction.

A flag of foliations of codimensions \( q_1, q_2, \ldots, q_k \) \((q_1 \leq q_2 \leq \cdots \leq q_k)\) on a manifold \( M \) is defined by Feigin [4] as a chain of foliations \( F_1, F_2, \ldots, F_k \) on \( M \), codim \( F_i = q_i \), such that for \( i < j \) the leaves of \( F_i \) contain those of \( F_j \). In his paper, Feigin proposes two constructions for the characteristic classes of flags of foliations, in an attempt to answer the following

FEIGIN’S QUESTION. – Let \( F \) be a \( q \)-codimensional foliation on a manifold \( M \), and let \( 0 < p < q \). Do there exist foliations \( F', G \) on \( M \) of codimensions \( q, p \) respectively such that \( F' \) is integrably homotopic to \( F \) and its leaves are contained in those of \( G \) ?

In the present paper, we consider those flags of foliations with only two foliations \( F_1, F_2 \) and call the couple \((F_1, F_2)\) a \((q_1, q_2)\)-codimensional subfoliation. Then, the main aim of this work is to give a characteristic homomorphism for subfoliations through a differential-geometric construction, generalizing Bott’s construction [1] of the characteristic homomorphism of a foliation.

Previous to a detailed discussion of the contents of this paper, it must be pointed out that Cordero-Gadea [3], Moussu [12] and Suzuki [13] also give some partial answers to Feigin’s Question.

The paper is structured as follows. First, § 2 is devoted to describe the subfoliation categories; for a subfoliation \((F_1, F_2)\), its normal bundle is
defined as \( v(F_1, F_2) = vF_{21} \oplus vF_1 \), where \( vF_{21} \) is the quotient bundle \( F_1/F_2 \) and \( vF_1 \) is the usual normal bundle of \( F_1 \). So, a meaningful exact sequence of vector bundles

\[
0 \longrightarrow vF_{21} \overset{i}{\longrightarrow} vF_2 \overset{\pi}{\longrightarrow} vF_1 \longrightarrow 0
\]

appears in a canonical way. This section ends with the study of subfoliation maps and homotopic subfoliations.

\[ \S 3 \text{ is devoted to define basic connections on the normal bundle } v(F_1, F_2); \text{ the existence of such basic connections is shown, and Theorem 3.3 states the existence of triples } (V^1, V, V^2) \text{ of basic connections adapted to the subfoliation, that is, } V^1, V, V^2 \text{ are basic connections on } vF_{21}, vF_2 \text{ and } vF_1 \text{ respectively and compatible with the homomorphisms } i \text{ and } \pi \text{ in (1). The partial flatness of any basic connection on } v(F_1, F_2) \text{ with respect to } F_2 \text{ leads directly to the Bott’s Obstruction Theorem for Subfoliations (Theorem 3.9), firstly stated by Feigin [4]. At the same time, it is deduced that (1) is, in fact, an exact sequence of vector bundles all of them foliated with respect to } F_2 \text{ and the homomorphisms } i \text{ and } \pi \text{ are compatible with these structures. Nevertheless, this exact sequence does not generally admit a foliated splitting and, therefore, } vF_2 \text{ and } v(F_1, F_2) \text{ are not, in general, isomorphic as foliated bundles.} \]

In § 4, the characteristic homomorphism of a subfoliation \((F_1, F_2)\) is introduced

\[ \lambda^*_{(F_1, F_2)} : H^*(WO_1) \rightarrow H^*_{DR}(M) \]

\((WO_1, d)\) being an appropriate graded differential algebra and \( H^*(WO_1) \) the associated cohomology groups. The construction of \( \lambda^*_{(F_1, F_2)} \) is done following Bott’s technique of comparison between a basic and a metric connection on \( v(F_1, F_2) \). Of course, \( \lambda^*_{(F_1, F_2)} \) is natural with respect to subfoliation maps and homotopy invariant.

In § 5, it is first shown that \((WO_1, d)\) is a graded differential subalgebra of a convenient truncated Weil algebra \( W_1(g(N)) \) of a Lie group and Theorem 5.1 states that \( H^*(WO_1) \) and \( H^*(W_1(g(N))) \) are isomorphic, which generalizes the well known fact of foliation theory ([5], [6]). Secondly, Theorem 5.2 shows the relation between the characteristic homomorphism of a subfoliation \((F_1, F_2)\) and those of each foliation \( F_i, i = 1, 2 \).
Finally, in § 6, two applications of the results obtained in § 5 are developed; the first one is the following: Theorem 5.2 gives a necessary condition so that Feigin's Question can have an affirmative answer (Theorem 6.1), and this is used to show that any 2-codimensional foliation in the Yamato's examples [15] cannot be homotopic to $F_2$ in a (1,2)-codimensional subfoliation $(F_1, F_2)$. The second application is obtained from the techniques used to prove Theorem 5.2, and it is stated as follows: let $F$ be a $q$-codimensional foliation admitting $d$ everywhere independent transverse infinitesimal transformations $Y_1, \ldots, Y_d$ and such that $F$ and $Y_1, \ldots, Y_d$ generate a new $(q-d)$-codimensional foliation, then the characteristic homomorphism $\lambda^*_F$ of $F$ vanishes on the kernel of the canonical homomorphism $\mu^* : H^*(WO_q) \to H^*(WO_{q-d})$. This result gives a generalization of Lazarov-Shulman's results ([9], [10]).

Through this paper all manifolds are differentiable $C^\infty$-manifolds and all maps are smooth $C^\infty$-maps.

2. Subfoliation categories.

To begin with, some basic concepts associated with subfoliations are introduced.

Let $M$ be an $n$-dimensional manifold, $TM$ its tangent bundle. A $(q_1, q_2)$-codimensional subfoliation on $M$ is a couple $(F_1, F_2)$ of integrable subbundles $F_k$ of $TM$ of dimension $n - q_k$, $k = 1, 2$, and $F_2$ being at the same time a subbundle of $F_1$.

Therefore, for each $k = 1, 2$, $F_k$ defines a $q_k$-codimensional foliation on $M$, $d = q_2 - q_1 \geq 0$, and, moreover, the leaves of $F_1$ contain those of $F_2$.

Let us remark that a $q$-codimensional foliation $F$ on $M$ can be considered as a subfoliation on $M$ in three different ways:

$$(C_1) : F_1 = F_2 = F; \quad (C_2) : F_1 = TM, F_2 = F; \quad (C_3) : F_1 = F, F_2 = 0.$$ 

Let $(F_1, F_2)$ be a $(q_1, q_2)$-codimensional subfoliation on $M$, $vF_k = TM/F_k$ the normal bundle of $F_k$, and let us consider the quotient bundle $vF_{21} = F_1/F_2$. Then, the following commutative diagram of short
exact sequences of vector bundles is canonically obtained:

\[ 0 \rightarrow F_2 \rightarrow i_2 \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow 0 \]

where the i's are the canonical inclusions and the \( \pi \)'s are the canonical projections.

**Definition 2.1.** — The vector bundle

\[ v(F_1, F_2) = vF_{21} \oplus vF_1 \]

will be called the normal bundle of \((F_1, F_2)\).

Let be \( f : N \rightarrow M \) a differentiable map, \((F_1, F_2)\) a \((q_1, q_2)\)-codimensional subfoliation on \( M \); if \( f \) is transverse to \( F_2 \), the couple \( f^{-1}(F_1, F_2) = (f^{-1}(F_1), f^{-1}(F_2)) \) defines a \((q_1, q_2)\)-codimensional subfoliation on \( N \) which will be called the inverse image of \((F_1, F_2)\) and, then, \( f \) is said to be transverse to \((F_1, F_2)\). Moreover, we have
\( \nu(f^{-1}(F_1, F_2)) = f^*(\nu(F_1, F_2)) \), where \( f^* (\ ) \) denotes the pull-back of the corresponding vector bundle.

**Definition 2.2.** Let \((G_1, G_2)\) and \((F_1, F_2)\) be \((q_1, q_2)\)-codimensional subfoliations on \(N\) and \(M\), respectively. A subfoliation map from \((G_1, G_2)\) to \((F_1, F_2)\) is a differentiable map \( f : N \rightarrow M \) transverse to \((F_1, F_2)\) and such that \((G_1, G_2) = f^{-1}(F_1, F_2)\).

Now, the notion of homotopy between subfoliations can be defined as follows: let \((F_1', F_2')\) and \((F_1, F_2)\) be \((q_1, q_2)\)-codimensional subfoliations on \(M\); they are said to be homotopic subfoliations if there exists a \((q_1, q_2)\)-codimensional subfoliation \((F, F)\) on \(M \times \mathbb{R}\) such that:

1) the face maps \( j_0, j_1 : M \rightarrow M \times \mathbb{R}\) both are transverse to \((F_1, F_2)\).

2) \( j_0^{-1}(F_1, F_2) = (F_1', F_2), \quad j_1^{-1}(F_1, F_2) = (F_1', F_2') \).

Of course, the normal bundles of homotopic subfoliations are isomorphic.

A subfoliation \((F_1, F_2)\) will be said with trivialized normal bundle if the vector bundles \(\nu F_1, \nu F_2\) and \(\nu(F_1, F_2)\) are all trivial vector bundles and if there have been chosen trivializations compatible with the projection map \(\pi : \nu F_2 \rightarrow \nu F_1\) and with the Whitney sum structure of \(\nu(F_1, F_2)\).

3. Basic connections and Bott's theorem for subfoliations.

We refer to Bott [1] for the well known definitions and properties of the usual theory of foliations.

Let \((F_1, F_2)\) be a \((q_1, q_2)\)-codimensional subfoliation on \(M\), \(d = q_2 - q_1\). Let us consider the exact sequence

\[
0 \rightarrow F_2 \xrightarrow{0} F_1 \xrightarrow{\pi_0} \nu F_{21} \rightarrow 0.
\]

**Définition 3.1.** A connection \(\nabla\) on \(\nu F_{21}\) is said to be basic if

\[
\nabla_X Z = \pi_0[X, \bar{Z}]
\]

for any vector field \(X \in \Gamma(F_2)\), \(\bar{Z}\) being a vector field in \(F_1\) such that \(\pi_0(\bar{Z}) = Z\).
A device similar to that of Bott in [1] permits to show the existence of basic connections in \( vF_{21} \).

Now, consider the exact sequence

\[
0 \longrightarrow vF_{21} \xrightarrow{i} vF_2 \xrightarrow{\pi} vF_1 \longrightarrow 0.
\]

Then, the following proposition is easily verified.

**Proposition 3.2.** — For any \( \mathcal{V}^1 \), \( \mathcal{V} \) and \( \mathcal{V}^2 \) basic connections on \( vF_{21} \), \( vF_2 \) and \( vF_1 \) respectively, and for any \( X \in \Gamma(vF_2) \), we have

\[
i(\mathcal{V}^1 XZ) = \mathcal{V}_X \mathcal{V}^1(Z), \quad \text{for any} \quad Z \in \Gamma(vF_{21})
\]

\[
\pi(\mathcal{V}^1 XZ) = \mathcal{V}^1_X (\mathcal{V}^2 Z), \quad \text{for any} \quad Z \in \Gamma(vF_2).
\]

In fact, we can state the following.

**Theorem 3.3.** — There exist \( \mathcal{V}^1 \), \( \mathcal{V} \) and \( \mathcal{V}^2 \) basic connections on \( vF_{21} \), \( vF_2 \) and \( vF_1 \) such that, for any vector field \( X \in \Gamma(TM) \),

\[
i(\mathcal{V}^1 XZ) = \mathcal{V}_X \mathcal{V}^1(Z), \quad \text{for any} \quad Z \in \Gamma(vF_{21})
\]

\[
\pi(\mathcal{V}^1 XZ) = \mathcal{V}^1_X (\mathcal{V}^2 Z), \quad \text{for any} \quad Z \in \Gamma(vF_2).
\]

Such a triple \( (\mathcal{V}^1, \mathcal{V}, \mathcal{V}^2) \) of basic connections will be said adapted to the subfoliation.

**Proof.** — Let us begin by considering a Riemannian metric on \( M \) which is compatible with the subfoliated structure, that is (see [14]): with respect to this metric, \( vF_2 \) (respect. \( vF_1 \)) is isomorphic to the orthogonal complement bundle to \( F_2 \) (respect. to \( F_1 \)) in \( TM \), and \( vF_{21} \) is isomorphic to the orthogonal complement bundle to \( F_2 \) in \( F_1 \); that is, by the choice of such Riemannian metric we obtain isomorphisms

\[
TM \cong F_k \oplus vF_k, \quad k = 1, 2
\]

\[
F_1 \cong vF_{21} \oplus F_2
\]

\[
vF_2 \cong vF_{21} \oplus vF_1.
\]

Now, we construct \( \mathcal{V}^1 \) and \( \mathcal{V}^2 \) basic connections on \( vF_{21} \) and \( vF_1 \), respectively, as follows: for any

\[
X \in \Gamma(TM) = \Gamma(F_2) \oplus \Gamma(vF_2) = \Gamma(F_1) \oplus \Gamma(vF_1),
\]
we write \( X = X_2 + X_2' = X_1 + X_1' \), and if \( \nabla^1 \) (respect. \( \nabla^2 \)) is an arbitrary connection on \( \nu F_{21} \) (respect. on \( \nu F_1 \)), we define

\[
\nabla^1 X Z = \pi_0 [X_2, \tilde{Z}] + \nabla^1_{X_2} Z, \quad \text{for any} \quad Z \in \Gamma(\nu F_{21})
\]

\[
\nabla^2 Z_1 = \pi_1 [X_1, \tilde{Z}_1] + \nabla^2_{X_1} Z_1, \quad \text{for any} \quad Z_1 \in \Gamma(\nu F_1)
\]

where \( \tilde{Z} \in \Gamma(F_1) \) and \( \tilde{Z}_1 \in \Gamma(TM) \) are such that \( \pi_0(\tilde{Z}) = Z \), \( \pi_1(\tilde{Z}_1) = Z_1 \).

The basic connection \( \nabla \) on \( \nu F_2 \) is constructed as follows: for any \( Z_2 \in \Gamma(\nu F_2) = \Gamma(\nu F_{21}) \oplus \Gamma(\nu F_1) \), we write \( Z_2 = Z'_2 + Z''_2 \), and define a connection \( \nabla \) on \( \nu F_2 \) by

\[
\nabla X Z_2 = \nabla^1 X Z'_2 + \nabla^2 X Z''_2
\]

for any \( X \in \Gamma(TM) \); then, \( \nabla \) is given by

\[
\hat{\nabla} X Z_2 = \pi_2 [X_2, \tilde{Z}_2] + \nabla_{X_2} Z_2
\]

where \( \tilde{Z}_2 \in \Gamma(TM) \) verifies \( \pi_2(\tilde{Z}_2) = Z_2 \).

Now, for any \( X \in \Gamma(TM) \) and \( Z \in \Gamma(\nu F_{21}) \), we have

\[
i(\nabla^1 X Z) = i(\pi_0 [X_2, \tilde{Z}] + \nabla^1_{X_2} Z) = \pi_2 [X_2, \tilde{Z}] + i(\nabla^1_{X_2} Z)
\]

\( \tilde{Z} \in \Gamma(F_1) \) being such that \( \pi_0(\tilde{Z}) = Z \). On the other hand,

\[
\nabla_X i(Z) = \pi_2 [X_2, \tilde{Z}] + \nabla_{X_2} i(Z)
\]

but \( i(Z) \in \text{Ker} \pi \), then \( \hat{\nabla}_{X_2} i(Z) = \nabla^1_{X_2} i(Z) \), and, since \( X_2' \in \Gamma(\nu F_2) \), the result follows immediately.

Analogously, for any \( X \in \Gamma(TM) \) and \( Z_2 \in \Gamma(\nu F_2) \) we have

\[
\pi(\nabla X Z_2) = \pi(\pi_2 [X_2, \tilde{Z}_2] + \nabla_{X_2} Z_2) = \pi_1 [X_2, \tilde{Z}_2] + \pi(\nabla_{X_2} Z_2)
\]

\( \tilde{Z}_2 \in \Gamma(TM) \) being such that \( \pi_2(\tilde{Z}_2) = Z_2 \). Since \( Z_2 = Z'_2 + Z''_2 \) and \( X'_2 \in \Gamma(\nu F_2) \), and then \( X'_2 = (X'_2)' + (X'_2)'' \in \Gamma(\nu F_{21}) \oplus \Gamma(\nu F_1) \), we have

\[
\pi(\nabla_{X_2} Z_2) = \nabla^2_{X_2} Z''_2 = \nabla^2_{X_2} \pi(Z_2) = \pi_1 [(X'_2)', \tilde{Z}_2] + \nabla^2_{X_2} \pi(Z_2)
\]

and, since \( X_2 + (X'_2)' = X_1 \) and \( (X'_2)'' = X_1' \), the second part of the theorem is proved.

Q.E.D.
For the later use, we shall explain the relation between the local connection forms of an adapted triple \((\mathfrak{V}^1, \mathfrak{V}, \mathfrak{V}^2)\) of basic connections.

Let \(U \subset M\) be an open set of local triviality for \(\mathfrak{V}_{F_1}, \mathfrak{V}_{F_2}\) and \(\mathfrak{V}_{F}\). A local basis \(\{Z_i, i = 1, 2, \ldots, q_2\}\) of sections for \(\mathfrak{V}_{F_2}\) will be said adapted if \(\{\pi(Z_u), u = d + 1, \ldots, q_2\}\) is a local basis of sections of \(\mathfrak{V}_{F_1}\) and \(\{Z_a, a = 1, 2, \ldots, d\}\) is a local basis of sections of \(\mathfrak{V}_{F_{21}}\).

Now, given an adapted local basis \(\{Z_i\}\) of sections, let \(\{\tilde{Z}_b, b = 1, 2, \ldots, q_2\}\) be local vector fields on \(M\) such that \(\pi(Z_i) = Z_i, 1 \leq i \leq q_2\), and let \(\{\omega_b, b = 1, 2, \ldots, q_2\}\) be the local 1-forms on \(M\) dual to the local vector fields \(\{\tilde{Z}_b\}\). Then, the 1-forms \(\{\omega_b, 1 \leq b \leq q_2\}\) are annihilated by \(F_2\) and the 1-forms \(\{\omega_{d + 1} \leq u \leq q_2\}\) are annihilated by \(F_1\). Therefore, by the integrability conditions, there exist local 1-forms \(\tau_{ba}, \tau_{ua}, \tau_{uv}, 1 \leq a, b \leq d, d + 1 \leq u, v \leq q_2\), on \(M\) such that

\[
d\omega_a = \sum_{b=1}^{d} \omega_b \wedge \tau_{ba} + \sum_{u=d+1}^{q_2} \omega_u \wedge \tau_{ua}
\]

\[
d\omega_u = \sum_{v=d+1}^{q_2} \omega_v \wedge \tau_{uv}.
\]

Let \((\mathfrak{V}^1, \mathfrak{V}, \mathfrak{V}^2)\) be an adapted triple of basic connections, and suppose that \((\theta_{ab}^1), (\theta_{d+1}), (\theta_{uv}^2)\) are respectively their local connection forms with respect to an adapted local basis of sections over an open set \(U \subset M\). Then, for any local vector field \(X\) on \(M\)

\[
\nabla^1_X Z_a = \sum_{b=1}^{d} \theta_{ab}^1(X) Z_b
\]

\[
\nabla_X Z_i = \sum_{j=1}^{q_2} \theta_{ij}(X) Z_j
\]

\[
\nabla^2_X \pi(Z_u) = \sum_{v=d+1}^{q_2} \theta_{uv}^2(X) \pi(Z_v).
\]

Then, a direct computation leads to

\[
\theta_{ab}^1 = \theta_{ab}, \quad 1 \leq a, b \leq d
\]

\[
\theta_{aa} = 0, \quad 1 \leq a \leq d, \quad d + 1 \leq u \leq q_2
\]

\[
\theta_{uv} = \theta_{uv}^2, \quad d + 1 \leq u, v \leq q_2.
\]
and we can state the following:

**Lemma 3.4.** — With respect to an adapted local basis of sections, the local connection forms of an adapted triple \((\nabla^1, \nabla, \nabla^2)\) of basic connections are, respectively

\[
\begin{align*}
\nabla^1 &: (\theta_{ab}), \\
\nabla &: \begin{bmatrix} \theta_{ab} & 0 \\ \theta_{ab} & \theta_{uv} \end{bmatrix}, \\
\nabla^2 &: (\theta_{uv}),
\end{align*}
\]

\(1 \leq a, b \leq d, \quad d + 1 \leq u, v \leq q_2.\)

Moreover, by a similar device to that used in the foliation theory, we get

**Lemma 3.5.** — Let \(\nabla^1, \nabla\) and \(\nabla^2\) be any basic connections on \(vF_{21}\), \(vF_2\) and \(vF_1\), respectively. Let us consider an adapted local basis of sections over an open set \(U \subset M\), and suppose that \((\theta^1_{ab}), (\theta_{ij})\) and \((\theta^2_{uv})\) are the respective local connection forms. Then, we have:

1. \(\theta^1_{ab}(X) = \tau_{ab}(X)\), for any local vector field \(X \in \Gamma(F_2)\), \(1 \leq a, b \leq d\).
2. \(\theta_{ab}(X) = \tau_{ab}(X), \theta_{uv}(X) = 0, \theta_{uv}(X) = \tau_{uv}(X)\) and \(\theta_{uv}(X) = \tau_{uv}(X)\) for any local vector field \(X \in \Gamma(F_2)\), \(1 \leq a, b \leq d, d + 1 \leq u, v \leq q_2.\)
3. \(\theta^2_{uv}(X) = \tau_{uv}(X),\) for any local vector field \(X \in \Gamma(F_1)\), \(d + 1 \leq u, v \leq q_2.\)

Now, let us remark once more what happens for the particular case of a foliation considered as a subfoliation:

\((C_1)\) : \(vF_{21}\) is the zero vector bundle and the only possible connection is the zero one, and that is trivially a basic connection.

\((C_2)\) : \(vF_{21} = vF\), the normal bundle of \(F\), and Definition 3.1 is the usual definition of basic connections.

\((C_3)\) : \(vF_{21} = F\) and, since \(F_2 = 0\), any connection on \(vF_{21}\) is basic.

From Definition 3.1, we get through a straightforward computation

**Lemma 3.6.** — Let \(\nabla^1\) be a basic connection on \(vF_{21}\), \(K^1\) the curvature of \(\nabla^1\). Then \(K^1(X,Y) \equiv 0\) for any vector fields \(X, Y\) in \(F_2\).

Now, we shall adopt the following

**Definition 3.7.** — A connection \(\nabla = \nabla^1 \oplus \nabla^2\) on \(v(F_1, F_2) = vF_{21} \oplus vF_1\) is said to be basic if and only if \(\nabla^1\) is a basic connection on \(vF_{21}\) and \(\nabla^2\) is a basic connection on \(vF_1\).
It is a well known fact that the curvature $K^2$ of a basic connection $\nabla^2$ on $vF_1$ verifies $K^2(X,Y) = 0$ for any $X$, $Y$ vector fields in $F_1$. Therefore, the following is immediate:

**Corollary 3.8.** — Let $\nabla$ be a basic connection on $v(F_1,F_2)$, $K$ the curvature of $\nabla$. Then $K(X,Y) = 0$ for any $X$, $Y$ vector fields in $F_2$.

**Remarks.** — 1) Lemma 3.6 implies that $vF_{21}$ has a foliated vector bundle structure with respect to $F_2$ defined by considering the horizontal lift of $F_2$ with respect to a basic connection $\nabla^1$ on $vF_{21}$. Lemma 3.5 points out that the foliated structure does not depend on the choice of $\nabla^1$.

2) Corollary 3.8 implies that $v(F_1,F_2)$ has also a well-defined foliated bundle structure with respect to $F_2$ adapted to the Whitney sum structure. Nevertheless, although both $vF_2$ and $v(F_1,F_2)$ have foliated structures with respect to $F_2$ and are isomorphic as vector bundles, they are not, in general, isomorphic as foliated vector bundles.

3) The result in Proposition 3.2 implies that homomorphisms $\iota$ and $\pi$ in the exact sequence of vector bundles

$$0 \longrightarrow vF_{21} \longrightarrow vF_2 \longrightarrow vF_1 \longrightarrow 0$$

are, in fact, foliated homomorphisms (i.e. compatible with the respective foliated structures with respect to $F_2$).

Now, let $U \subset M$ be a simultaneously trivializing neighborhood for $vF_1$, $vF_2$ and $vF_{21}$; over $U$, $F_1$ can be described as the set of tangent vectors on which certain local 1-forms $\omega_{d+1}, \ldots, \omega_{q_2}$ vanish, these 1-forms being linearly independent at each point of $U$. Analogously, $F_2$ can be described over $U$, and, since $F_2 \subset F_1$ we can suppose that the family of local 1-forms which annihilate $F_2$ is obtained by adding local 1-forms $\omega_1, \ldots, \omega_d$ to the family above, being also linearly independent at each point of $U$. Let be

$$I_{U}^1 = \text{ideal in } \Lambda^*(U) \text{ generated by } \omega_{d+1}, \ldots, \omega_{q_2}$$
$$I_{U}^2 = \text{ideal in } \Lambda^*(U) \text{ generated by } \omega_1, \ldots, \omega_d, \omega_{d+1}, \ldots, \omega_{q_2}.$$  

Obviously,

$$I_{U}^1 \subset I_{U}^2, \quad (I_{U}^1)^{q_1+1} = 0, \quad (I_{U}^2)^{q_2+1} = 0.$$
Now, if $V = V^1 \oplus V^2$ is a basic connection on $\nu(F_1, F_2)$, then the curvature matrix $K_U$ of $V$ over $U$ will be

$$K_U = \begin{bmatrix} (K^1_{U})_{ab} & 0 \\ 0 & (K^2_{U})_{uv} \end{bmatrix}$$

with respect to a local basis of sections of $\nu(F_1, F_2)$, dual to the local 1-forms $\omega_i$, $1 \leq i \leq q_2$. Here, $((K^1_{U})_{ab})$ (respect. $((K^2_{U})_{uv})$) denotes the curvature matrix of $\nu^1$ (respect. of $\nu^2$) over $U$. Taking into account earlier results, we get

$$(K^1_{U})_{ab} \in I^2_U, \quad 1 \leq a, b \leq d$$

$$(K^2_{U})_{uv} \in I^1_U, \quad d + 1 \leq u, v \leq q_2.$$

Hence, the following Bott's Obstruction Theorem for Subfoliations is obtained

**Theorem 3.9.** — Let $P_1$ and $P_2$ be homogeneous polynomials in the real Pontryagin classes of $\nu F_1^i$ and $\nu F_1$ respectively, of degree $l_k$, $k = 1, 2$. If at least one of the inequalities $l_2 > 2q_1$, $l_1 + l_2 > 2q_2$ is satisfied, then $P_1P_2 = 0$.

4. The characteristic homomorphism for subfoliations.

In this section we define a characteristic homomorphism for subfoliations, which generalizes the usual characteristic homomorphism for foliations. For this purpose, the technique used by Bott in [1] shall be adopted here.

Let $\mathfrak{g}l_n$ denote the Lie algebra of $Gl(n, \mathbb{R})$,

$I(\mathfrak{g}l_n) = \mathbb{R}[c_1, \ldots, c_n]$ the ring of symmetric invariant polynomials on $\mathfrak{g}l_n$, $c_1, \ldots, c_n$ being the Chern polynomials given by

$$\det \left( I + \frac{t}{2\pi} A \right) = \sum_{j=0}^{n} c_j(A)t^j$$

where $c_j(A) = \left( \frac{1}{2\pi} \right)^j \text{trace} A^j$, for any $A \in \mathfrak{g}l_n$.

In addition, if $E \to M$ is a vector bundle of dimension $n$ over $M$ and
\( \nabla \) is a connection on \( E \) of curvature \( K \), denote
\[
\lambda(\nabla) : I(gl_n) \rightarrow \Lambda^*(M)
\]
the ring homomorphism defined by
\[
\lambda(\nabla)(c_j) = c_j(K).
\]

Moreover, if \( \nabla^0, \nabla^1, \ldots, \nabla^m \) are connections on \( E \), define
\[
\lambda(\nabla^0, \nabla^1, \ldots, \nabla^m)(c_j) = \pi_*[c_j(K(0, \ldots, \cdot, \cdot, \cdot, m))]
\]
where \( \Delta^m \) is the standard \( m \)-simplex, \( K(0, \ldots, \cdot, \cdot, \cdot, m) \) is the curvature of the connection
\[
\nabla^0, \ldots, \nabla^m = (1-t_1-\cdots-t_m)\nabla^0 + t_1\nabla^1 + \cdots + t_m\nabla^m
\]
on the vector bundle \( E \times \mathbb{R}^m \rightarrow M \times \mathbb{R}^m \) and
\[
\pi_* : \Lambda^p(M \times \Delta^m) \rightarrow \Lambda^{p-m}(M)
\]
denotes the integration along \( \Delta^m \).

The following useful properties are verified:

1. \( df(\lambda(\nabla^0, \nabla^1, \ldots, \nabla^m)(c_j)) = \sum_{i=0}^m (-1)^i \lambda(\nabla^0, \ldots, \hat{\nabla}^i, \ldots, \nabla^m)(c_j). \)

2. If \( f : N \rightarrow M \) is a differentiable map, then
\[
f^*(\lambda(\nabla^0, \nabla^1, \ldots, \nabla^m)(c_j)) = \lambda(f^*(\nabla^0), f^*(\nabla^1), \ldots, f^*(\nabla^m))(c_j).
\]

Now, in order to construct an appropriate cochain complex \( (W_0, \partial) \), let us consider \( q_1, q_2 \in \mathbb{N} \), with \( q_2 \geq q_1 \) and \( d = q_2 - q_1 \); denote by
\[
I(gl_d) = R[c_1', \ldots, c_d'], \quad I(gl_{q_1}) = R[c_1'', \ldots, c_{q_1}'']
\]
the rings of symmetric invariant polynomials, \( c_i' \) and \( c_i'' \) being the corresponding Chern polynomials. Let \( I(gl_d) \otimes I(gl_{q_1}) \) be the tensor product and denote by \( I \) the homogeneous ideal generated by binomials \( \varphi' \otimes \varphi'' \in I'(gl_d) \otimes I'(gl_{q_1}) \) whose dimensions verify at least one of the inequalities \( s'' > q_1, s' + s'' > q_2 \).

On the other hand, consider the exterior algebras over \( \mathbb{R} \)
\[
\Lambda(h_1', h_2', \ldots, h_l'), \quad \Lambda(h_1'', h_2'', \ldots, h_l'')
\]
generated by the elements \( h'_i, h''_i \) respectively and where

\[
I' = 2 \left[ \frac{d+1}{2} \right] - 1, \quad I'' = 2 \left[ \frac{q_1+1}{2} \right] - 1.
\]

Now we build a graded differential algebra \( \text{WO}_1 \) as follows:

\[
\text{WO}_1 = \Lambda(h'_1, h'_3, \ldots, h'_i) \otimes \Lambda(h''_1, h''_3, \ldots, h''_i) \otimes \frac{I(\text{gl}_d) \otimes I(\text{gl}_q)}{I}
\]

where

- \( \text{degree} (h'_i) = \text{degree} (h''_i) = 2i - 1 \)
- \( \text{degree} (c'_i) = \text{degree} (c''_i) = 2i \),

and the unique antiderivation of degree 1, \( d : \text{WO}_1 \to \text{WO}_1^{i+1} \), is defined by

\[
d(h'_i) = c'_i, \quad d(h''_i) = c''_i, \quad d(c'_i) = d(c''_i) = 0.
\]

We shall denote \( H^*(\text{WO}_1) \) the cohomology of the cochain complex \( (\text{WO}_1, d) \).

Let \( (F_1, F_2) \) be a \((q_1, q_2)\)-codimensional subfoliation on the manifold \( M \), \( \nu(F_1, F_2) = \nu F_{21} \otimes \nu F_1 \) its normal bundle. Let \( \nabla^0 = 1\nabla^0 \oplus 2\nabla^0 \), \( \nabla^1 = 1\nabla^1 \oplus 2\nabla^1 \) be connections on \( \nu(F_1, F_2) \), where \( 1\nabla^0 \) (respect. \( 2\nabla^0 \)) is a Riemannian connection on \( \nu F_{21} \) (respect. on \( \nu F_1 \)) and \( \nabla^1 \) is a basic connection. Then, a graded algebra homomorphism

\[
\lambda_{(F_1, F_2)} : \text{WO}_1 \to \Lambda^*(M)
\]

is defined by

\[
\lambda_{(F_1, F_2)}(c'_i) = \lambda(1\nabla^1)(c'_i), \quad \lambda_{(F_1, F_2)}(c''_i) = \lambda(2\nabla^1)(c''_i)
\]
\[
\lambda_{(F_1, F_2)}(h'_i) = \lambda(1\nabla^0, 1\nabla^1)(c'_i), \quad \lambda_{(F_1, F_2)}(h''_i) = \lambda(2\nabla^0, 2\nabla^1)(c''_i).
\]

The Obstruction Theorem 3.9 implies that \( \lambda_{(F_1, F_2)} \) is well defined and, in fact, it is a cochain complex homomorphism; therefore, it induces a homomorphism of graded algebras on cohomology:

\[
\lambda^*_{(F_1, F_2)} : H^*(\text{WO}_1) \to H^*_{DR}(M).
\]

While the cochain homomorphism \( \lambda_{(F_1, F_2)} \) depends on the choice of \( \nabla^0 \) and \( \nabla^1 \), the induced homomorphism \( \lambda^*_{(F_1, F_2)} \) does not, as one can easily show by means of standard techniques.
**Definition 4.1.** — $\lambda^*_{{(F_1, F_2)}}$ is called the characteristic homomorphism of $(F_1, F_2)$ and the classes in the image of $\lambda^*_{{(F_1, F_2)}}$ are said the secondary subfoliation classes of $(F_1, F_2)$.

The characteristic homomorphism $\lambda^*_{{(F_1, F_2)}}$ has a certain naturality property which can be expressed as follows:

**Proposition 4.2.** — Let $(G_1, G_2)$ and $(F_1, F_2)$ be subfoliations on $N$ and $M$ respectively, $f : N \to M$ a subfoliation map from $(G_1, G_2)$ to $(F_1, F_2)$. Then, the diagram

$$
\begin{array}{ccc}
H^*(W_0) & \xrightarrow{\lambda^*_{{(F_1, F_2)}}} & H^*_{\text{DR}}(M) \\
\downarrow{\lambda^*_{{(G_1, G_2)}}} & & \downarrow{f^*} \\
H^*_{\text{DR}}(N) & &
\end{array}
$$

commutes.

**Proof.** — This result is an immediate consequence of the following fact: if $\nabla^0$ and $\nabla^1$ are, respectively, a Riemannian and a basic connection on $\nu(F_1, F_2)$, then $f^*(\nabla^0)$ and $f^*(\nabla^1)$ are connections of the same type on $\nu(G_1, G_2) = f^*(\nu(F_1, F_2))$; therefore, from the definition of $\lambda_{{(F_1, F_2)}}$, the commutativity of the above diagram follows immediately at the cochain complex level. Q.E.D.

Then, from the naturality and through purely homotopy theoretic reasons, we obtain

**Corollary 4.3.** — $\lambda^*_{{(F_1, F_2)}}$ only depends on the homotopy class of $(F_1, F_2)$.

To construct the secondary subfoliation classes of a subfoliation $(F_1, F_2)$ with a trivialized normal bundle, one proceeds exactly as before except that Riemannian connections $\nabla^0 = 1 \nabla^0 \oplus 2 \nabla^0$ are replaced by flat connections (Whitney sum of flat connections) and the cochain complex $(W_0, d)$ is replaced by the cochain complex $(W, d)$, where

$$W_1 = \Lambda(h_1', h_2', \ldots, h_d') \otimes \Lambda(h_1'', h_2'', \ldots, h_{q_1}'') \otimes \frac{I(gl_d) \otimes I(gl_{q_1})}{I}$$

and with gradation and differential $d$ defined in the same form.
5. Relation between the characteristic homomorphisms
of \((F_1, F_2)\) and \(F_k, k = 1, 2\).

In order to justify our later constructions, let us recall some well known
facts of the theory of characteristic classes of foliations and their relation
with the cohomology of truncated relative Weil algebras.

Let \(W(gl_n)\) denote the Weil algebra of \(gl_n\); \(W(gl_n)\) is
a differential graded algebra \(W(gl_n) = \bigoplus_{r \geq 0} W^r(gl_n)\), where
\[ W^r(gl_n) = \bigoplus_{i+j=r} \{\Lambda^i(gl_n) \otimes S^j(gl_n)\}. \]

In fact, \(W(gl_n)\) is a \(gl_n\)-algebra and, if \(I(gl_n)\) denotes the subalgebra of
\(gl_n\)-basic elements of \(W(gl_n)\), then \(I(gl_n)\) admits the Chern polynomials
\(c_1, c_2, \ldots, c_n\) as a system of generators, which are cocycles of degree 2i.
Moreover, since the complex \((W(gl_n); \partial)\) is acyclic, there exist
\(h_i \in W^{2i-1}(gl_n), \quad 1 \leq i \leq n,\) such that \(\partial h_i = c_i\). Also, if \(W(gl_n, O_n)\)
denotes the differential subalgebra of the \(O_n\)-basic elements of \(W(gl_n)\)
\((O_n = O(n, \mathbb{R})\) being the orthogonal group), the generators \(c_i\) of \(I(gl_n)\) can
be chosen in such way that the \(h_i\) be \(O_n\)-basic for each odd \(i\).

Now, let \(J_n\) be the homogeneous ideal of \(W(gl_n)\) generated by the
elements of \(S(gl_n)\) of degree greater than 2n; denote \(W_n(gl_n) = W(gl_n)/J_n\)
the quotient algebra and \(W_n(gl_n, O_n)\) the subalgebra of \(O_n\)-basic elements
of \(W_n(gl_n)\). Then, one has the following well-known theorem (see [6]
or [5]):

**Theorem.** — \(W_n(gl_n)\) (respect. \(W_n(gl_n, O_n)\)) has the same cohomology that
its subalgebra
\[
W_n = \Lambda(h_1, h_2, \ldots, h_n) \otimes \frac{I(gl_n)}{J_n}
\]
(respect. \(WO_n = \Lambda(h_1, h_3, \ldots) \otimes \frac{I(gl_n)}{J_n}\))

where the \(h_i\) are, for odd \(i\), the \(O_n\)-basic elements such that \(\partial h_i = c_i\).

This theorem plays a fundamental role in the theory of characteristic
classes of foliations because the cohomology ring \(H^*(WO_n)\) (respect.
\( H^*(W_n) \) is the domain of the characteristic homomorphism of \( n \) codimensional foliations (respect. with trivialized normal bundle).

Similarly, the domain of the characteristic homomorphism for subfoliations, as defined in the earlier section, can be seen having as domain the cohomology ring of a convenable relative truncated Weil algebra. To explain that, let us consider the Lie algebra \( g(N) = gl_{n_1} \times gl_{n_2} \), its Weil algebra \( W(g(N)) \) and the homogeneous ideal \( I \) of \( W(g(N)) \) generated by the subspaces \( S^1(gl_{n_1}) \otimes S^2(gl_{n_2}) \), where the integers \( i_1, i_2 \) satisfy at least one of the inequalities \( i_2 > n_2, i_1 + i_2 > n_1 + n_2 \). Clearly, \( I \) is a graded subcomplex of \( W(g(N)) \), so the quotient \( W_1(g(N)) = W(g(N))/I \) is a multiplicative graded complex; moreover, if \( O_N = O_{n_1} \times O_{n_2} \) is the product Lie group, we denote \( W_1(g(N), O_N) \) the graded subcomplex of \( W_1(g(N)) \) of \( O_N \)-basic elements.

Now, through the canonical isomorphism

\[
W(g(N)) \cong W(gl_{n_1}) \otimes W(gl_{n_2})
\]

let us consider the graded differential subalgebras \( W_1, W_0 \) of \( W_1(g(N)) \) and \( W_1(g(N), O_N) \) respectively, given by

\[
W_1 = \Lambda(h'_1, h'_2, \ldots, h'_{n_1}) \otimes \Lambda(h''_1, h''_2, \ldots, h''_{n_2}) \otimes \frac{I(gl_{n_1}) \otimes I(gl_{n_2})}{I},
\]

\[
W_0 = \Lambda(h'_1, h'_3, \ldots, ) \otimes \Lambda(h''_1, h''_3, \ldots, ) \otimes \frac{I(gl_{n_1}) \otimes I(gl_{n_2})}{I},
\]

the \( h'_i \) (respect. \( h''_i \)) being such that \( dh'_i = c'_i \) (respect. \( dh''_i = c''_i \)) and \( c'_i, c''_2, \ldots, c''_{n_1} \) (respect. \( c'_i, c''_2, \ldots, c''_{n_2} \)) the Chern polynomials which generate \( I(gl_{n_i}) \) (respect. \( I(gl_{n_2}) \)), and for odd \( i \), \( h'_i \) (respect. \( h''_i \)) is supposed to be \( O_{n_1} \)-basic (respect. \( O_{n_2} \)-basic).

Then, we have

**Theorem 5.1.** — The canonical injections

\[
W_1 \longrightarrow W_1(g(N))
\]

\[
W_0 \longrightarrow W_1(g(N), O_N)
\]

induce isomorphisms on cohomology.

**Proof.** — The result follows from a device similar to that used in the ordinary case.
We start by defining an even filtration of $W_1$ and $W_{01}$ as follows:

$$F^{2p}(W_i) = \Lambda(h'_1, h'_2, \ldots) \otimes \Lambda(h''_1, h''_2, \ldots) \otimes \left\{ \bigoplus_{i_1 + i_2 \geq p} I^{i_1}(gl_{n_1}) \otimes I^{i_2}(gl_{n_2}) \right\}$$

$$F^{2p}(W_{0i}) = \Lambda(h'_1, h'_3, \ldots) \otimes \Lambda(h''_1, h''_3, \ldots) \otimes \left\{ \bigoplus_{i_1 + i_2 \geq p} I^{i_1}(gl_{n_1}) \otimes I^{i_2}(gl_{n_2}) \right\}$$

and the second term $E_2$ of the spectral sequence associated to $W_1$ (respect. $W_{0i}$) can be canonically identified with $W_1$ (respect. with $W_{0i}$).

On the other hand, we define an even filtration of $W_i(g(N))$ by

$$F^{2p}(W_i(g(N))) = \Lambda^*(g(N)) \otimes \left\{ \bigoplus_{i_1 + i_2 \geq p} (S^{i_1}(gl_{n_1}) \otimes S^{i_2}(gl_{n_2})) \right\}.$$ 

Since this filtration is by $O_N$-invariant ideals, it induces a new and similar filtration of $W_i(g(N), O_N)$. In both cases, the associated spectral sequence is of the Hochschild-Serre type [7], hence its second term is given by

$$E_2^{p,q} = H^q(g(N), R) \otimes \left\{ \bigoplus_{i_1, i_2} ([S^{i_1}(gl_{n_1})]^{g_{i_1}} \otimes [S^{i_2}(gl_{n_2})]^{g_{i_2}}) \right\}$$

for that associated to $W_i(g(N))$, and by

$$E_2^{p,q} = H^q(g(N, O_N)) \otimes \left\{ \bigoplus_{i_1, i_2} ([S^{i_1}(gl_{n_1})]^{g_{i_1}} \otimes [S^{i_2}(gl_{n_2})]^{g_{i_2}}) \right\}$$

for that associated to $W_i(g(N, O_N))$; here, in both cases, the summation on the right extends over all couples $i_1, i_2$ such that $i_1 + i_2 = p$, $i_2 \leq n_2$ and $i_1 + i_2 \leq n_1 + n_2$, and the symbol $[\ldots]$ denotes the set of $g_{i_1}$-invariants.

Now, the canonical projection of $\Lambda(h'_1, h'_2, \ldots) \otimes \Lambda(h''_1, h''_2, \ldots)$ (respect. $\Lambda(h'_1, h'_3, \ldots) \otimes \Lambda(h''_1, h''_3, \ldots)$) over $\Lambda^*(g(N))$ (respect. over $[\Lambda^*(g(N))]_{O_N}$) induces an isomorphism on cohomology and the result follows by applying a comparison theorem.

Q.E.D.

Thus, Theorem 5.1 shows that the graded differential algebras $W_1$ and $W_{01}$, which were introduced in the earlier section, play in the context of
subfoliation theory the same role as that of $W_n$ and $WO_n$ in the context of the foliation theory.

Now, let us consider the canonical projection homomorphism of Lie algebras

$$g(N) = gl_{n_1} \times gl_{n_2} \longrightarrow gl_{n_2}.$$ 

This homomorphism induces an homomorphism of graded differential algebras

$$\mu_1 : W(gl_{n_2}) \longrightarrow W(g(N))$$

which is compatible with the truncation by $I_{n_2}$ and $I$, and since $O_N = O_{n_1} \times O_{n_2}$ applies onto $O_{n_2}$, we obtain a graded differential algebra homomorphism

$$\mu_1 : W_{n_2}(gl_{n_2}O_{n_2}) \longrightarrow W_1(g(N),O_N)$$

and this homomorphism $\mu_1$ acts on the generators as follows:

$$\mu_1(h_i) = h_i'', \quad \mu_1(c_i) = c_i''.$$ 

Analogously, the canonical injective homomorphism of Lie algebras

$$gl(N) = gl_{n_1} \times gl_{n_2} \longrightarrow gl_{n_1 + n_2}$$

induces a graded differential algebra homomorphism

$$\mu_2 : W_{n_1 + n_2}(gl_{n_1 + n_2}O_{n_1 + n_2}) \longrightarrow W_1(g(N),O_N)$$

and this homomorphism $\mu_2$ acts on the generators as follows:

$$\mu_2(h_i) = h_i'' + h_i''', \quad \mu_2(c_i) = c_i' + c_i'''.$$ 

Consequently, and by restricting to the corresponding subalgebras, we obtain two graded differential algebra homomorphisms

$$\mu_1 : WO_{n_2} \longrightarrow WO_1, \quad \mu_2 : WO_{n_1 + n_2} \longrightarrow WO_1.$$ 

Hereafter, we shall denote $\mu_k^*, \ k = 1, 2$, the induced homomorphism at cohomology level.

Let $(F_1, F_2)$ be a $(q_1, q_2)$-codimensional subfoliation on a manifold $M$
and put \( n_1 = d = q_2 - q_1 \), \( n_2 = q_1 \). Also, denote by

\[
\lambda^*_{F_k} : H^*(WO_{q_k}) \rightarrow H^*_{DR}(M)
\]

the characteristic homomorphism of \( F_k \), \( k = 1,2 \), as defined by Bott [1]. Then, we have

**Theorem 5.2.** — For each \( k = 1,2 \), the diagram

\[
\begin{array}{ccc}
H^*(WO_{q_k}) & \xrightarrow{\mu^*} & H^*(WO_{q_1}) \\
\downarrow{\lambda^*_{F_k}} & & \downarrow{\lambda^*_{(F_1,F_2)}} \\
H^*_{DR}(M) & \rightarrow & H^*_{DR}(M)
\end{array}
\]

is commutative.

**Proof.** — We shall proceed separately for each value of \( k \).

1. \( k = 1 \).

Let \( V^0 = \nabla^0 \oplus \nabla^0 \) (respect. \( V^1 = \nabla^1 \oplus \nabla^1 \)) be a Riemannian connection (respect. a basic connection) on \( v(F_1,F_2) = vF_{21} \oplus vF_1 \). Then, \( \nabla^0 \) (respect. \( \nabla^1 \)) is a Riemannian connection (respect. a basic connection) on \( vF_1 \). Therefore, from the definitions of \( \lambda_{(F_1,F_2)} \), \( \lambda_{F_1} \) and \( \mu_1 \), we have \( \lambda_{F_1} = \lambda_{(F_1,F_2)} \circ \mu_1 \), the commutativity of the diagram at the cochain level.

2. \( k = 2 \).

Let us consider \( vF_{21} \), \( vF_2 \) and \( vF_1 \) as vector subbundles of \( TM \) in such way that \( vF_2 \cong F_{21} \oplus vF_1 = v(F_1,F_2) \); that can be done, for example, by using a Riemannian metric to split the exact sequence

\[
0 \rightarrow vF_{21} \rightarrow vF_2 \rightarrow vF_1 \rightarrow 0.
\]

Then, let \( \{Z_{ia}, i=1,2,\ldots,q_2\} \) be a local basis of sections for \( vF_2 \) over an open set \( U \subset M \), adapted to such splitting; that is, being \( \{Z_{a}, 1 \leq a \leq d\} \) (respect. \( \{Z_{a}, d+1 \leq u \leq q_2\} \)) a local basis of sections for \( vF_{21} \) (respect. for \( vF_1 \)); then the change of such local trivializations will be given by a matrix of the form

\[
\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}
\]
where $A$ (respect. $B$) is the matrix of change of local trivialization for $vF_{21}$ (respect. for $vF_1$).

Now, let us consider an adapted triple $(\nabla^1, \nabla, \nabla^2)$ of basic connections; with respect to the local basis of sections $\{Z^i, 1 \leq i \leq q_2\}$ above, the matrices of local 1-forms which define these connections are, respectively,

$$\theta^1 = [\theta^1_{ab}], \quad \theta = \begin{bmatrix} \theta^1_{ab} & 0 \\ \theta^2_{ab} & \theta^2_{uv} \end{bmatrix}, \quad \theta^2 = [\theta^2_{uv}]$$

where $(\theta_{ab})$ is non zero in general.

On the other hand, we can consider on $vF_2$ the connection sum $\nabla = \nabla^1 \oplus \nabla^2$, which is locally defined by the matrix of local 1-forms

$$\bar{\theta} = \begin{bmatrix} \theta^1_{ab} & 0 \\ 0 & \theta^2_{uv} \end{bmatrix}.$$

Let us remark that $\nabla$ is not a basic connection on $vF_2$, in general, but it is differentiably $J(>q_2)$-homotopic to the basic connection $\nabla$, in the sense of Lehmann [11], as we shall state in Lemma 5.3 below; therefore, $\nabla$ can be used in order to compute $\lambda^*_{F_2}$.

Keeping all that in mind, the commutativity of the diagram follows directly from the definitions of $\lambda^*_{(F_1,F_2)}$, $\mu^*_2$ and $\lambda^*_{F_2}$, through a straightforward calculation, taking $\nabla = \nabla^1 \oplus \nabla^2$ in the place of a basic connection on $vF_2$, the same $\nabla$ as a basic connection on $v(F_1,F_2)$ and $\nabla^0 = 0\nabla^1 \oplus 0\nabla^2$ as Riemannian connection in both fiber bundles, $0\nabla^1$ (respect. $0\nabla^2$) being a Riemannian connection on $vF_{21}$ (respect. $vF_1$).

Q.E.D.

**Lemma 5.3.** — Let $(\nabla^1, \nabla, \nabla^2)$ be an adapted triple of basic connections, $\nabla = \nabla^1 \oplus \nabla^2$ the connection on $vF_2$ defined as in Theorem 5.2 above. Then, $\nabla$ and $\nabla$ are differentiably $J(>q_2)$-homotopic connections on $vF_2$ (in the sense of Lehmann [11]).

**Proof.** — First of all, let us remark that both $\nabla$ and $\nabla$ are $J(>q_2)$-connections, because their curvature tensors vanish over $F_2$.

Now, we define the connection $\nabla' = t\nabla + (1-t)\nabla$ on the vector bundle $vF_2 \times I \rightarrow M \times I$, $I = [0,1]$. With respect to a local basis of sections $\{Z^i, 1 \leq i \leq q_2\}$ for this bundle, as above, $\nabla'$ is locally defined by
the matrix of local 1-forms

\[
\begin{bmatrix}
\theta^1_{ab} & 0 \\
t\theta_{ab} & \theta^2_{ab}
\end{bmatrix}.
\]

Therefore, \( V' \) is also a \( J(>q_2) \)-connection, because if \( K' \) is the curvature form of \( V' \) and \( f \in \Gamma'(gl_{q_2}) \), \( f(K') = 0 \) for \( r > q_2 \); in fact, locally \( f(K') \) is given by

\[
f(K') = dt \wedge rf(\theta_0, K_r) + f(K_r)
\]

where \( K_r \) denotes the curvature form of the connection for each fixed \( t \). Since \( f(\theta - \theta_0, K_r) = 0 \) for every \( r \), because

\[
\theta - \theta_0 = \begin{bmatrix} 0 & 0 \\ t\theta_{ab} & 0 \end{bmatrix}
\]

and \( f(K_r) = 0 \) for \( r > q_2 \), the result follows immediately.

Q.E.D.

Analogously, we can state:

**Theorem 5.4.** — Let \((F_1, F_2)\) be a \((q_1, q_2)\)-codimensional subfoliation on \( M \) with a trivialized normal bundle. Then, for each \( k = 1, 2 \) the diagram

\[
\begin{array}{ccc}
H^*(W_{q_2}) & \xrightarrow{\mu^*_k} & H^*(W) \\
\downarrow{\lambda^*_k} & & \downarrow{\lambda^*_k} \\
\mathcal{H}^*_{DR}(M) & & \mathcal{H}^*_{DR}(M)
\end{array}
\]

commutes.

6. Applications.

In this section, we shall explain two applications of the results and techniques developed in the earlier section.

1. — First, let us remark that Theorems 5.2 and 5.4 allow to approach an answer to Feigin's Question. Obviously, from those Theorems we deduce the following
THEOREM 6.1. — A necessary condition so that Feigin's Question can have an affirmative answer is the vanishing of those exotic classes of $F$ which are obtained from the elements in $\text{Ker} \, \mu^+_2$.

In fact, the spirit of the final note in Moussu's article [12] is that of this theorem. On the other hand, Feigin [4] constructs a 2-codimensional foliation with a trivialized normal bundle for which the answer to his Question is negative. Now, by using the results of Yamato [15] and Theorem 5.2, we shall give another new example for which the answer is also negative.

For this purpose, let us remark that the groups $H^r(WO_i)$ have, for $q_1 = 1$, $q_2 = 2$ and $d = 1$, the following dimensions:

$$\dim H^r(WO_i) = \begin{cases} 2 & \text{for } r = 5, 6 \\ 1 & \text{for } r = 0, 3 \\ 0 & \text{for the remaining } r \end{cases}$$

and, in fact, for $r = 5$, the cohomology classes of $h'_1 \otimes (c_1)^2$ and $h'_1 \otimes c'_1 \otimes c''_1$ (or its cohomologous $h''_1 \otimes (c'_1)^2$) are generators of $H^5(WO_1)$.

Now, let us recall Yamato's Theorem:

THEOREM [15]. — For any integer $q \geq 1$, there exists a $q$-codimensional foliation $F$ on a closed $(2q+1)$-manifold $M$ such that all the exotic characteristic classes of $F$ which correspond to the canonical generators $[h \otimes \varphi]$ of $H^{2q+1}(WO_q)$ are non zero, where $\varphi \in \mathbb{R}[c_1, \ldots, c_q]$ is a monomial of degree $2(q-j+1)$.

Hence, for $q = 2$, the canonical generators of $H^5(WO_2)$ are the cohomology classes of $h_1 \otimes c^2_1$ and $h_1 \otimes c_2$. Hence, Yamato's theorem implies that $\lambda^*_F([h_1 \otimes c_2])$ is not zero, while the class $[h_1 \otimes c_2]$ belongs to $\text{Ker} \, \mu^+_2$; therefore, on such a manifold $M$ does not exist any $(1,2)$-codimensional subfoliation $(F_1, F_2)$ such that $F_2$ be homotopic to the Yamato's 2-codimensional foliation on $M$.

2. — Let $F_2$ be a $q_2$-codimensional foliation on $M$ and suppose there exist vector fields $Y_1, \ldots, Y_d$ on $M$ such that:

(i) $Y_1, \ldots, Y_d$ are everywhere independent,

(ii) $Y_1, \ldots, Y_d$ are infinitesimal transformations of $F_2$ and everywhere transverse to $F_2$,

(iii) $F_2$ and $Y_1, \ldots, Y_d$ define a new $q_1$-codimensional foliation $F_1$
on $M$, $q_1 = q_2 - d$, and therefore $F_2 \subset F_1$; for this subfoliation $(F_1, F_2)$, the vector bundle $vF_{21}$ is trivial, being its trivialization defined by $Y_1, \ldots, Y_d$.

The techniques used in the proof of Theorem 5.2 and Lemma 5.3, lead us to the following:

**Theorem 6.2.** — *Under the hypothesis above, the diagram*

\[
\begin{array}{ccc}
H^\bullet(\mathcal{W}O_{q_2}) & \xrightarrow{\lambda^*_F} & H^\bullet_{DR}(M) \\
\mu^* & & \lambda^*_F \\
& H^\bullet(\mathcal{W}O_{q_1}) &
\end{array}
\]

*commutes.*

Here, $\mu^*$ is the homomorphism induced from the canonical injection $gl_{q_1} \longrightarrow gl_d \times gl_{q_1} \longrightarrow gl_{q_2}$.

**Corollary 6.3.** — *The existence of $d$ everywhere independent and transverse infinitesimal transformations of a $q$-codimensional foliation $F$, satisfying (iii) above, implies the vanishing of $\lambda^*_F$ on the Kernel of $\mu^*: H^\bullet(\mathcal{W}O_q) \longrightarrow H^\bullet(\mathcal{W}O_{q-d})$.*

**Proof of Theorem 6.2.** — With the help of a convenable Riemannian metric on $M$, we may consider the normal vector bundles $vF_1$, $vF_2$ and $v(F_1, F_2)$ as vector subbundles of $TM$ in such way that $vF_2 = vF_{21} \oplus vF_1$, and $vF_{21}$ being still trivialized by $Y_1, \ldots, Y_d$. In fact, with this identification, the vector fields $Y_1, \ldots, Y_d$ on $M$ are considered as foliated sections for $vF_2$ and, consequently, the flat connection defined on $vF_{21}$ by this trivialization is a basic connection in the sense of Definition 3.1.

Now, if $^0\nabla^2$ and $^1\nabla^2$ are, respectively, a metric and a basic connection on $vF_1$, by using the flat connection on $vF_{21}$, we get, as in the proof of Theorem 5.2 and in order to compute $\lambda^*_F$, that the curvature forms of the corresponding connections on $vF_2$ are expressed, with respect to an adapted basis of sections, by

\[
^0K = \begin{bmatrix} 0 & 0 \\ 0 & ^0K^2 \end{bmatrix}, \quad ^1K = \begin{bmatrix} 0 & 0 \\ 0 & ^1K^2 \end{bmatrix}
\]
where $^0K^2$ (respect. $^1K^2$) is the curvature form of $^0\nabla^2$ (respect. of $^1\nabla^2$).

The result follows now immediately.

Q.E.D.

The following are two examples where Theorem 6.2 and Corollary 6.3 are applied.

Example 1. — Let $G$ be a $d$-dimensional Lie group acting locally and freely transverse to a foliation $F$ on $M$ and mapping leaves of $F$ into leaves of $F$. Then, the Lie algebra of $G$ gives rise to $d$ infinitesimal transformations of $F$, $Y_1, \ldots, Y_d$, satisfying the hypothesis above. Moreover,

$$[Y_a, Y_b] = \sum_{c=1}^{d} C_{ab}^c Y_c, \quad 1 \leq a, b \leq d$$

where $C_{ab}^c \in \mathbb{R}$.

Let us remark that this particular case has been firstly considered by Lazarov-Shulman [9], their results being weaker than that of Corollary 6.3. In fact, Lazarov-Shulman announce the result of Corollary 6.3 for the most particular case where $C_{ab}^c = 0, \ 1 \leq a, b, c \leq d$ ([10]).

Example 2. — Let $\pi : P \to M$ be a foliated principal bundle, $F$ being the $q$-codimensional foliation on $M$ and $\tilde{F}$ the foliation on $P$. Let us consider the canonical subfoliation $(\pi^{-1}F, \tilde{F})$ on $P$; then, the following diagram commutes

$$
\begin{array}{ccc}
H^*(WO,\phi) & \xrightarrow{\lambda^*_F} & H^*_DR(P) \\
\mu^* \downarrow \ & & \downarrow \pi^* \\
H^*(WO,\phi) & \xrightarrow{\lambda^*_F} & H^*_DR(M)
\end{array}
$$

The commutativity of $\heartsuit$ is consequence of Theorem 6.2 and that of $\spadesuit$ is given by the naturality of the exotic homomorphism of a foliation. That means

$$\lambda^*_F = \pi^* \circ \lambda^*_F \circ \mu^*.$$

If, moreover, $vF$ is a trivialized vector bundle, we have a similar
commutative diagram

\[
\begin{array}{ccc}
H^*(W_q) & \xrightarrow{\lambda_F^*} & H^*_{DR}(P) \\
\mu^* & & \pi^* \\
\downarrow & & \\
H^*(W_q) & \xrightarrow{\lambda_F^*} & H^*_{DR}(M)
\end{array}
\]

Now, let us remark that, in both cases, if \( \pi^* \) is injective, the vanishing of an exotic class of \( \mathcal{F} \) implies the vanishing of the corresponding one of \( \mathcal{F} \). A situation where this is applied is the following: suppose \( \nu F \) trivialized and \( P \) being the principal bundle of transverse references of \( \mathcal{F} \); then \( \pi^* \) is injective because \( P \) is topologically a product bundle.

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