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SVANTE JANSON

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## BMO AND COMMUTATORS OF MARTINGALE TRANSFORMS

by Svante JANSON <sup>(1)</sup>

### 0. Introduction.

The connection between BMO and commutators of singular integrals on  $\mathbf{R}^n$  was found by Coifman, Rochberg and Weiss [1]. Their result has been further developed by Uchiyama [5] and myself [4]. This paper shows that these results hold also for the martingale transforms studied in [3].

### 1. The transform.

We state the basic definitions and properties of our transforms. More details are given in [3].

We assume that  $(\Omega, \mathfrak{F}, \mu)$  is a probability space and that  $\{\mathfrak{F}_n\}_{n=0}^\infty$  is an increasing sequence of sub- $\sigma$ -fields of  $\mathfrak{F}$  such that  $\mathfrak{F}_n$  is generated by  $d^n$  disjoint atoms of probability  $d^{-n}$ .  $d$  is here and in the sequel a fixed integer. Thus, an atom  $Q$  of  $\mathfrak{F}_n$  is the union of  $d$  atoms of  $\mathfrak{F}_{n+1}$  which will be denoted  $Q^1 \dots Q^d$ .

For  $f$  an integrable function, we define  $f_n = E(f | \mathfrak{F}_n)$ . On any atom of  $\mathfrak{F}_n$ ,  $f_n$  is constant and  $f_{n+1}$  assumes  $d$  values. Hence, still studying one atom only,  $f_{n+1} - f_n$  may be regarded as a vector in  $C^d$ , which will be called the local difference of  $f$  on the atom.

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It is easily seen that every local difference actually belongs to the  $d - 1$  dimensional space  $V = \{(x_i)_{i=1}^d ; \sum x_i = 0\}$ .

Let  $A$  be a linear operator in  $V$ .

We define, whenever possible,  $Tf$  to be the function whose local differences are obtained from those of  $f$  by the operator  $A$ . (Also  $Tf_0 = 0$ ).

We will need the fact that  $T$  is a bounded operator on  $L^p$ ,  $1 < p < \infty$ .

We will represent  $A$  by a  $d \times d$  matrix. This represents an extension of  $A$  to an operator of  $C^d$  into  $C^d$  and may be chosen in many ways, but we will use the unique choice  $(a_{ij})_{i,j}$  such that  $\sum_j a_{ij} = \sum_i a_{ij} = 0$ . Note that the identity mapping in  $V$  is represented by  $\tilde{Y} = (\delta_{ij} - 1/d)_{i,j=1}^d$ , and the corresponding transform is  $Tf = f - Ef$ .  $C$  will denote various positive constants.

### 2. The commutator.

For any integrable function  $f$  on  $\Omega$ , we define  $C_f$  to be the commutator of multiplication by  $f$  and the operator  $T$  above, i.e.  $C_f g = fTg - T(fg)$ . If  $f \in L^q$ , it is obvious that  $C_f$  is a continuous linear operator from  $L^p$  to  $L^r$ ,  $1 < r < p < \infty$  and  $1/r = 1/p + 1/q$ .

The following theorem is less trivial. Here

$$BMO = \{f ; \sup_{n,\omega} E(|f - f_n| | \mathfrak{F}_n) < \infty\}.$$

We also have  $\sup E(|f - f_n|^p | \mathfrak{F}_n) < \infty$  for  $f \in BMO$  and any  $p < \infty$ .

**THEOREM 1.** — *If  $f \in BMO$ , then  $C_f$  is a bounded linear operator in  $L^p$ ,  $1 < p < \infty$ .*

*Proof.* — This is a simple adaptation of the proof of [4], Lemma 11, but for completeness, we give the main steps. We define  $g^* = \sup_n E(|g| | \mathfrak{F}_n)$  and  $g^\# = \sup_n E(|g - g_n| | \mathfrak{F}_n)$ . Choose  $q$  and  $r$  such that  $1 < q < qr < p$ . Assume that  $\omega \in Q$ , an atom of  $\mathfrak{F}_n$ , and  $g \in L^p$ . Let  $g_1 = g \cdot X_Q$ ,  $g_2 = g - g_1$  and  $a = f_n(\omega)$ .

$C_f g = C_{f-a} g = (f - a)Tg - T(f - a)g_1 - T(f - a)g_2$ . We treat three terms separately.

$$\begin{aligned} E(|(f - a) Tg| | Q) &\leq E(|f - a|^{q'} | Q)^{1/q'} E(|Tg|^q | Q)^{1/q} \\ &\leq C((Tg^q)^*(\omega))^{1/q} \end{aligned}$$

$$\begin{aligned} E(|T(f - a) g_1| | Q) &\leq d^{-n/r} \|T(f - a) g_1\|_r \leq Cd^{-n/r} \|(f - a) g_1\|_r \\ &\leq C((g^{rq})^*(\omega))^{1/rq}, \end{aligned}$$

and  $T(f - a) g_2$  is constant on  $Q$ . Hence

$$E(|C_f g - (C_f g)_n| | Q) \leq C((Tg^q)^*(\omega))^{1/q} + C((g^{rq})^*(\omega))^{1/rq},$$

and since the right hand side is independent of  $Q$ ,

$$(C_f g)^{\#} \leq C((Tg^q)^*)^{1/q} + C((g^{rq})^*)^{1/rq} \in L^p.$$

Now  $C_f g \in L^p$  follows as in the real-variable case [2].

In order to prove the converse, we obviously have to exclude some cases, e.g. when  $T$  is the identity. The proper requirement turns out to be the following.

We define  $A$  to be degenerate if there exists  $i_0$  such that  $a_{i_0 j} = a_{j i_0} = -a_{i_0 i_0} / (d - 1)$  for every  $j \neq i_0$ , otherwise  $A$  is non-degenerate.

Equivalently  $A$  is degenerate if and only if it is a multiple of  $\tilde{I}$  plus a matrix having all entries in one row and in the corresponding column equal to zero.

*Remark.* — This property is weaker than the property required for the characterization of  $H^1$  and BMO by a different method in [3] (viz. that  $A$  has no real eigenvector). In the important special case  $a_{ij} = \alpha_{i-j}$  (where  $\alpha_{-k} = \alpha_{d-k}$ ),  $A$  is non-degenerate unless it is a multiple of the identity.

THEOREM 2. —

a) Assume that  $A$  is non-degenerate. If  $C_f$  is bounded on any  $L^p$ , then  $f \in \text{BMO}$ .

b) If  $A$  is degenerate, this fails for every  $L^p$ ,  $1 < p < \infty$ .

*Proof.* — Assume that  $C_f$  is bounded on  $L^p$ . We choose an atom  $Q$  of  $\mathfrak{F}_n$  ( $n \geq 1$ ).

Choose  $j, k \neq i$  and define  $g$  to be  $\chi_{Q^j} - \chi_{Q^k}$ . All local differences but one of  $g$  are zero and we find that  $Tg = a_{ij} - a_{ik}$

on  $Q^i$ . Since  $fg$  is zero on  $Q^i$ ,  $T(fg)$  is constant there. Thus there is a constant  $a$  such that

$$\begin{aligned} |a_{ij} - a_{ik}| E(|f - a| |Q^i) &\leq E(|(a_{ij} - a_{ik})(f - a)|^p |Q^i)^{1/p} \\ &= E(|C_f g|^p |Q^i)^{1/p} \leq |Q^i|^{-1/p} \|C_f g\|_p \\ &\leq C |Q^i|^{-1/p} \|g\|_p = C. \end{aligned}$$

Consequently  $E(|f - a| |Q^i) \leq C$  unless the  $d - 1$  values of  $a_{ij}$ ,  $j \neq i$ , all are equal. In that case their common value must be  $-a_{ii}/(d - 1)$ .

Now we note that the transpose operator  $C'_f$  is bounded on  $L^{p'}$ . However  $C'_f = -C_f^t$ , where  $C_f^t$  is the commutator of  $f$  and the operator  $T^t$  obtained as above from the transpose matrix  $A^t$ . Hence we also have  $E(|f - a| |Q^i) \leq C$  unless  $a_{ij}^t = -a_{ii}^t/(d - 1)$ . This together with the preliminary result above shows that  $E(|f - a| |Q^i) \leq C$  unless  $A$  is degenerate. Since every atom, except  $\Omega$  itself, is of the form  $Q^i$  for some  $Q$  and  $i$ , this completes the proof of part a).

For the converse, let us assume that  $A$  is degenerate;  $A = \lambda \tilde{I} + A'$  where  $a'_{i_0 j} = a'_{j i_0} = 0$ ,  $j = 1 \dots d$ . Hence  $Tg = \lambda(g - Eg) + T'g$ . We choose a positive integer  $N$ .  $\Omega, \Omega^{i_0}, (\Omega^{i_0})^{i_0} \dots$  is a sequence of atoms of  $\mathfrak{F}_0, \mathfrak{F}_1, \dots$  respectively. Let  $Q \in \mathfrak{F}_N$  be the  $(N + 1)$ :th of these, and define  $f = X_Q$ . Then it is easy to see, for any  $g$ , that  $T(f(g - g_N)) = fT(g - g_N)$ ,  $T'(fg_N) = 0$  and  $fT'g_N = 0$ . Consequently

$$\begin{aligned} C_f g &= \lambda f(g - Eg) + fT'g - \lambda(fg - E(fg)) - T'(fg) \\ &= \lambda(-fEg + E(fg)) \end{aligned}$$

and  $\|C_f g\|_p \leq |\lambda| (\|f\|_p + \|f\|_{p'}) \|g\|_p$ . Hence

$$\|C_f\| \leq |\lambda| (\|f\|_p + \|f\|_{p'}).$$

If the result of part a) were to hold, we would by the closed graph theorem have  $\|f\|_{\text{BMO}} \leq C \|C_f\| \leq C(\|f\|_p + \|f\|_{p'})$ , but we see that this is impossible by letting  $N \rightarrow \infty$ .

3. Various extensions.

On  $\mathbb{R}^n$ , Uchiyama [5] showed that the commutator is compact if and only if the function belongs to the subspace CMO of BMO. This holds in our case too.

DEFINITION. —  $CMO = \{f \in BMO ; f_n \rightarrow f \text{ in BMO as } n \rightarrow \infty\} = \{f ; \sup E(|f - f_n| | \mathfrak{F}_n) \rightarrow 0, n \rightarrow \infty\}$ . CMO is the closure in BMO of the functions that are  $\mathfrak{F}_n$ -measurable for some  $n$ .

THEOREM 3. — Assume that  $A$  is non-degenerate. Then  $C_f$  is a compact operator of  $L^p$  into itself if and only if  $f \in CMO$  ( $1 < p < \infty$ ).

Proof. — If  $f \in CMO$ ,  $\|C_{f_n} - C_f\| = \|C_{f_n - f}\| \leq C \|f_n - f\|_{BMO}$  by Theorem 1. Hence  $C_{f_n} \rightarrow C_f$ , and since the range of  $C_{f_n}$  is finite-dimensional,  $C_f$  is compact.

Conversely, if  $f \notin CMO$ , there exists an infinite sequence  $Q_n$  of atoms such that  $E(|f - E(f|Q_n)| | Q_n) \geq C$  for some positive  $C$ . Thus, by the proof of Theorem 2, there exist functions  $g_n$  such that  $g_n$  and  $C_f g_n$  are supported on  $Q_n$ ,  $\|g_n\| \leq 1$  and  $\|C_f g_n\| \geq C$ . There is no convergent subsequence of  $\{C_f g_n\}$ , and hence  $C_f$  is not compact.

Quantitative estimates of the rate of convergence of  $f_n$  to  $f$  correspond to the commutator mapping one space into another.

THEOREM 4. — Assume that  $A$  is non-degenerate. Let  $1 < p < \infty$  and let  $\varphi$  be a positive increasing convex function on  $\mathbb{R}^+$  such that  $\varphi(0) = 0$ ,  $\varphi(2t) \leq C\varphi(t)$  and  $t^{-1/p} \varphi^{-1}(t)$  is decreasing. Then  $E(|f - f_n| | \mathfrak{F}_n) \leq Cd^{-n/p} \varphi^{-1}(d^n)$  if and only if  $C_f$  maps  $L^p$  into the Orlicz space  $L\varphi$ .

The proof is similar to the one for  $\mathbb{R}^n$  given in [4].

We may also study more general operators. Let us assume that we begin with one linear operator  $A_Q$  in  $V$  for every atom  $Q$ . Define the operator  $T$  as before, now applying  $A_Q$  to the local difference on  $Q$ .

It is clear, by the same proof as above, that Theorem 1 (and one direction of Theorems 3 and 4) holds if the operators  $A_Q$  are uniformly bounded. Also, if the operators are uniformly non-degenerate, all theorems above hold.

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Svante JANSON,  
Uppsala University  
Department of Mathematics  
Thunbergsvägen 3  
S 752 38 Uppsala (Suède).