MARIUS VAN DER PUT

The class group of a one-dimensional affinoid space


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THE CLASS GROUP
OF A ONE-DIMENSIONAL AFFINOID SPACE

by Marius van der Put

Introduction.

The field \( k \) is supposed to be complete with respect to a non-
archimedean valuation. Moreover we will assume that \( k \) is algebraically
closed. An affinoid space \( Y \) over \( k \) is the set of maximal ideals of an
affinoid algebra. The standard affinoid algebra is \( k\langle T_1, \ldots, T_n \rangle = \) the
set of all power series \( \sum a_{ij} T_1^{i_1} \cdots T_n^{i_n} \) converging on the closed polydisk
\[ \{(t_1, \ldots, t_n) \in k^n | \forall i | t_i | \leq 1 \}. \]
An affinoid algebra is a residue class ring of some \( k\langle T_1, \ldots, T_n \rangle \). An
algebraic variety over \( k \) can be studied locally by its analytic structure over
\( k \), that is by means of affinoid spaces.

We show that a one-dimensional, normal, connected affinoid space \( Y \) is
an affinoid subset of a non-singular, complete curve \( C \) over \( k \) (Thm 1.1). If
\( Y \) has a trivial class group then \( Y \) is in fact an affinoid subset of \( \mathbb{P}^1 \)
(Thm 2.1). A curve is locally a unique factorization domain (U.F.D.
for short) if and only the curve is a Mumford curve (i.e. can be
parametrized by a Schottky group). In general the class group of \( Y \)
can be expressed in terms of the Jacobi-variety of \( C \) (prop. 3.1).

Some examples show the connection between the class group of \( Y \) and
the class group of the (stable) reduction of \( Y \). For \( k \)-analytic spaces we refer
to [2], [3]. I thank A. Escassut for bringing the problem on unique
factorization on affinoid spaces to my attention. Related questions are treated
in [1].

1. Affinoid subspaces of an algebraic curve.

A curve \( C \) (non-singular and complete) over \( k \) has a natural structure as
(rigid) analytic space over \( k \). This structure is given by a collection
of subspaces \( Y \) of \( C \), called affinoid, and a sheaf \( \mathcal{O} = \mathcal{O}_C \) with respect to the Grothendieck topology of finite coverings by affinoids. For any \( Y \), \( \mathcal{O}(Y) \) is an affinoid algebra (1-dim. and normal) over \( k \) with \( \text{Sp}(\mathcal{O}(Y)) = Y \). We want to show:

1.1. — THEOREM. — Every 1-dimensional, normal, connected affinoid space \( Y = \text{sp}(A) \) is an affinoid subspace of a non-singular complete curve.

Proof. — \( Y \) is called connected and normal if the algebra \( A \) has no idempotents \( 0, 1 \) and \( A \) is integrally closed. We use the notations \( A^0 = \{ f \in A \mid ||f|| \leq 1 \} \), \( A^\infty = \{ f \in A \mid ||f|| < 1 \} \) and \( \bar{A} = A^0 / A^\infty \), where \( ||f|| = \max \{ ||f(y)|| \mid y \in Y \} \) is the spectral norm on \( Y \). The algebra \( \bar{A} \) is of finite type over \( \bar{k} = \) the residue field of \( k \) and the algebraic variety \( Y_c = \text{Max}(\bar{A}) \) is called the canonical reduction of \( Y \). There is a natural surjective map \( R : Y \rightarrow \bar{Y}_c \), also called the canonical reduction. A pure covering of an analytic space \( X \), is an allowed covering \( \mathcal{U} = (U_i) \) by affinoid spaces, such that for every \( i \neq j \) with \( U_i \cap U_j \neq \emptyset \), the set \( U_i \cap U_j \) is the inverse image of a Zariski open set \( V_{ij} \) in \( (U_i)_c \) under the map \( U_i \rightarrow (U_i)_c \). The reduction \( X' \) of \( X \) with respect to \( \mathcal{U} \) is obtained by glueing the affine algebraic varieties \( (U_i)_c \) over the open sets \( V_{ij} \). The result is an algebraic variety over \( \bar{k} \). If \( X \) is separated then the \( U_i \cap U_j \) are also affinoid, the \( V_{ij} \) are affine and equal to \( (U_i \cap U_j)_c \) and \( X' \) is separated. If \( X \) is non-singular, 1-dimensional, connected and if \( X' \) is complete then \( X \) is a non-singular complete curve over \( k \) (see [2] ch. IV 2.2).

Our proof consists of glueing affinoid spaces \( Y_1, \ldots, Y_s \) to \( Y \) such that the reduction of \( X = Y \cup Y_1 \cup \ldots \cup Y_s \) with respect to the pure covering \( \{ Y, Y_1, \ldots, Y_s \} \) is complete. Then clearly \( Y \) is an affinoid domain of the algebraic curve \( X \). The 1-dimensional space \( Y_c \) lies in a complete 1-dimensional \( Z \) such that \( F = Z - Y_c \) is a finite set of non-singular points. Suppose that we can find for every \( p \in F \) an affinoid space \( Y_p \) with canonical reduction \( R_p : Y_p \rightarrow (Y_p)_c \subseteq Z \) where \( (Y_p)_c \) is a neighbourhood of \( p \) and such that

\[
Y_p \supset R_p^{-1}((Y_p)_c \cap Y_c) \cong R^{-1}((Y_p)_c \cap Y_c) \subseteq Y.
\]

Then we can glue \( Y_p \) to \( Y \). The space \( X = YU \cup Y_p \) has reduction \( Z \) which is complete. So the glueing has to be done locally on \( Y \) and \( Y_c \). The component \( C \) of \( Z \) on which \( p \) lies can be projected into \( \mathbb{P}^2(\bar{k}) \) such that
The class group of a one-dimensional affinoid space is still non-singular. A good projection onto \( \mathbb{P}^1 \) maps \( p \) onto \( o \) and \( o \) is an unramified point for the projection. Replacing \( Y \) and \( Y_c \) by neighbourhoods of \( p \) we may therefore suppose:

\[
\mathcal{O}(Y) = \mathcal{O}(Y_c) = \bar{k}[t,(t,e(t))^{-1},s]/(P),
\]

where

1) \( e(t) = (t-a_1) \ldots (t-a_s) \) with \( a_1, \ldots, a_s \) different points of \( \bar{k}^* \); they are the residues of \( a_1, \ldots, a_s \in k^0 \).

2) \( P \) is a monic irreducible polynomial of degree \( n \) with coefficients in \( k[t] \).

3) \( \frac{dP}{ds} \) is invertible as element of \( \bar{k}[t,(t,e(t))^{-1},s]/(P) \).

4) the point \( \langle p \rangle \) corresponds to \( t = 0 \). Then \( \mathcal{O}(Y)^0 \) has the form \( k^0\langle T,U,S\rangle/(TE(T)U-1,Q) \) where

\[
E(T) = (T-a_1) \ldots (T-a_s) \quad \text{and} \quad Q = P.
\]

Since \( Q \) is general with respect to the variable \( S \), we can apply Weierstrass-division and assume that \( Q \) is a monic polynomial of degree \( n \) in \( S \) with coefficients in \( k^0\langle T,U\rangle/(TE(T)U-1) \). Suppose that we can find a monic polynomial \( Q^* \) of degree \( n \) in \( S \) and coefficients in \( k^0\langle T,V\rangle/(E(T)V-1) \) such that

\[
k^0\langle T,U,S\rangle/(TE(T)U-1,Q^*) \simeq \mathcal{O}(Y)^0.
\]

Then \( Y_p = Sp(k\langle T,V,S\rangle/(E(T)V-1,Q^*)) \) has the required properties. So we have to get rid of the negative powers of \( T \) in the coefficients of \( Q = S^n + a_{n-1}S^{n-1} + \cdots + a_0 \).

1.2. - Lemma. - If \( Q^* = S^n + a_{n-1}S^{n-1} + \cdots + a_0 \) has coefficients in \( A = k^0\langle T,U\rangle/(TE(T)U-1) \) and \( Q^* = Q = P \), then

a) \( Q^* \) is irreducible

b) \( Q^* \) has a zero in \( \mathcal{O}(Y)^0 \)

c) \( k\langle T,U,S\rangle/(TE(T)U-1,Q^*) \simeq \mathcal{O}(Y) \).

Proof: - a) Let \( Q^* \) be reducible over the quotient field of \( A \). Since \( A \) is normal, \( Q^* \) is a product of monic polynomials with coefficients in \( A \). This contradicts the irreducibility of \( Q^* = P \).
b) First we show that \( \left\{ Q^*, \frac{dQ^*}{dS} \right\} \) generates the unit ideal in \( A[S] \). Let 

\( m \) be a maximal ideal containing \( Q^* \) and \( \frac{dQ^*}{dS} \). If \( m \cap k^0 \neq 0 \) then \( m \)

induces a maximal ideal of \( \bar{k}[t, (te(t))^{-1}][S] = \bar{A}[S] \) containing \( P \) and \( \frac{dP}{dS} \). This contradicts our assumptions on \( P \). So \( m \) corresponds to a maximal ideal \( m_1 \), of \( k\langle T, U \rangle/(TE(T)U - 1)[S] \), containing \( Q^* \) and \( \frac{dQ^*}{dS} \).

If \( m_1 \cap k\langle T, U \rangle/(TE(T)U - 1) \neq 0 \) then \( m_1 \), is the kernel of a homomorphism in \( k \) given by \( T \mapsto \lambda_1 \in k, S \mapsto \lambda_2 \in k \) with

\[ |\lambda_1| \leq 1, \quad |\lambda_1 E(\lambda_1)| = 1, \quad |\lambda_2| \leq 1 \]

since \( Q^*(\lambda_2) = 0 \). From \( \left( P, \frac{dP}{dS} \right) = \bar{k}[t, (te(t))^{-1}, S] \) it follows that

\[ Z_1(S)Q^* + Z_2(S)\frac{dQ^*}{dS} = 1 + \sum_{i>0} a_i S^i \]

for certain \( Z_1, Z_2 \in A[S] \) and \( a_i \in A \) with \( ||a_i|| < 1 \). The substitution \( T \mapsto \lambda_1 ; S \mapsto \lambda_2 \) makes \( 0 = 1 + \sum_{i>0} a_i(\lambda_1)\lambda_2^i \), which is impossible. So \( m \) and \( m_1 \) correspond to an ideal of \( L[S] \) with \( L \) the quotient field of \( A \).

Since \( Q^* \) is irreducible, this means that \( \frac{dQ^*}{dS} = 0 \). This is obviously in contradiction with \( \left( P, \frac{dP}{dS} \right) = \bar{k}[t, (te(t))^{-1}] \).

We conclude the existence of \( Z_1, Z_2 \in A[S] \) with

\[ 1 = Z_1(S)Q^* + Z_2(S)\frac{dQ^*}{dS} . \]

By Newton’s method we will show that \( Q^* \) has a zero in \( \mathcal{O}(Y)^0 \). Let \( \eta \in \mathcal{O}(Y)^0 \) satisfy \( ||Q^*(\eta)|| < 1 \) (e.g. \( \eta \) is the residue of \( S \) mod \( Q \) in \( \mathcal{O}(Y)^0 \)). Then \( 1 - Z_1(\eta)Q^*(\eta) = Z_2(\eta)\frac{dQ^*}{dS}(\eta) \) and since
The principal ideal domain $\mathcal{O}(Y)$ of $\mathcal{O}(Y)$ show the existence of a root of $Q^*$ in $\mathcal{O}(Y)$.

C) The quotient field of $A[S]/Q^*$ is contained in that of $A[S]/Q$, because of (b). Both fields are extensions of degree $n$ of the quotient field of $A$. So they are equal. The rings $k\langle T,U,S\rangle/(TE(T)U-1,Q^*)$ and $\mathcal{O}(Y)$ are both the integral closure of $k\langle T,U\rangle/(TE(T)U-1)$ in that field. So they are equal.

End of the proof of 1.1. — We choose $Q^*$ with coefficients in $k^0\langle T,V\rangle/(VE(T)-1)$ and $Q^* = P$.

1.3. — Corollary. — Let $Y$ be as in (1.1); then $Y$ is affinoid in a curve $X$ (complete non-singular) such that $X - Y_e$ is a finite set of non-singular points.

2. Unique factorization.

We want to show the following:

2.1. — Theorem. — Let $Y = \text{Sp} \ A$ be a 1-dimensional connected affinoid space. Then $A$ has unique factorization if and only if $Y$ is an affinoid subspace of $\mathbb{P}^1(k)$.

Remarks. — 1) Since $A$ has dimension 1 the condition « $A$ has unique factorization » is equivalent to « $A$ is a principal ideal domain ».

2) It seems that this theorem has also been proved by M. Raynaud.

A connected affinoid subspace $Y$ of $\mathbb{P}^1(k)$ has clearly a U.F.D. as affinoid algebra. Before we start the proof of 2.1, we like to state its algebraic analogue. It is:

2.2. — Proposition. — Let $A$ be a finitely generated algebra over an algebraically closed field $k$. Suppose that $A$ is 1-dimensional and a U.F.D. Then $A$ is isomorphic to the coordinate ring of a Zariski-open subset of $\mathbb{P}^1(k)$.

Proof. — $A$ is the coordinate ring of a Zariski-open subset $X$ of some non-singular complete curve $C$; put $X = C - \{p_1, \ldots, p_s\}$. Let $D$ be a
A divisor of degree 0 on \( C \); since \( A \) is a U.F.D. there is a rational function \( f \) on \( C \) with \( D = (f) \) on \( X \). This means that the map \( \left\{ \sum_{i=1}^{s} n_i p_i n_i \in \mathbb{Z} \right\} \rightarrow J(C) = \text{the Jacobi-variety of } C \), is surjective. If \( C \) is not a rational curve then its Jacobi variety (or better its points in \( k \)) is not a finitely generated group. Hence \( C \simeq \mathbb{P}^1(k) \).

We prove the theorem in some steps.

2.3. - Lemma. - Suppose that \( \mathcal{O}(Y) \) is a U.F.D. and that \( Y \) is irreducible, then \( H^1(Y, \mathcal{O}^*) = 0 \).

Proof. - \( \bar{Y} \) denotes the canonical reduction of \( Y \). An element \( \xi \in H^1(\bar{Y}, \mathcal{O}^*) \) corresponds to a projective, rank 1, \( \mathcal{O}(\bar{Y}) \)-module \( N \); let \( F \) be a free \( \mathcal{O}(\bar{Y}) \)-module, \( \sigma : F \rightarrow F \) an idempotent endomorphism with \( \text{im } \sigma = N \). Then \( F, \sigma \) lift to similar things over \( \mathcal{O}(Y)^0 \) since \( \mathcal{O}(Y)^0 \) is complete and \( \mathcal{O}(\bar{Y}) = \mathcal{O}(Y)^0 \otimes \overline{k} \). So we find a projective, rank 1, \( \mathcal{O}(Y)^0 \)-module \( M \) with \( M \otimes \overline{k} = N \).

Further \( M \otimes \mathcal{O}(Y) \simeq \mathcal{O}(Y) \) since \( \mathcal{O}(Y) \) is a U.F.D. There exists a Zariski-open covering of \( \bar{Y} \) such that \( N \) is free on the sets of this covering. That implies the existence of \( f_1, \ldots, f_s \in \mathcal{O}(Y)^0 \) such that

- \( a) \) each \( ||f_i|| = 1 \) and \( (f_1, \ldots, f_s)\mathcal{O}(Y)^0 = \mathcal{O}(Y)^0 \).
- \( b) \) \( M \otimes \mathcal{O}(X)^0 \langle S \rangle / (S f_i - 1) \) is a free \( \mathcal{O}(X)^0 \langle S \rangle / (S f_i - 1) \)-module.

We identify \( M \) with \( M \otimes \mathcal{O}(Y)^0 \subset \mathcal{O}(Y) \) and we may suppose that \( M \subset \mathcal{O}(Y)^0 \); \( \max \{||m|| \mid m \in M\} = 1 \) and \( M \supset \lambda \mathcal{O}(Y)^0 \) for certain \( \lambda \in k^0 \), \( \lambda \neq 0 \). Then

\[
M \otimes \mathcal{O}(Y)^0 \langle S \rangle / (S f_i - 1) \subset \mathcal{O}(Y)^0 \langle S \rangle / (S f_i - 1)
\]

is generated by one element \( h \). This element has norm 1 and it has no zeros is \( \{y \in Y \mid |f_i(y)| = 1\} = Y_i \). So \( h \) is invertible in \( \mathcal{O}(Y)_i \). Its inverse \( h^{-1} \) has also norm 1 since \( Y_i \) is irreducible and the norm on \( \mathcal{O}(Y_i) \) is, as a consequence, multiplicative. Hence \( \mathcal{O}(Y_i)^0 = \mathcal{O}(Y)^0 \). It follows that some power of \( f_i \) lies in \( M \). Since \( (f_1, \ldots, f_s) = \mathcal{O}(Y)^0 \) we find that \( M = \mathcal{O}(Y)^0 \). So \( N \) is free and \( \xi = 0 \).

2.4. - Lemma. - Let \( L \) be affine, 1-dimensional and irreducible over \( k \). If \( H^1(L, \mathcal{O}^*_L) = 0 \) then \( L \) is rational and non-singular.
Proof. Let \( \pi : L_1 \rightarrow L \) be the normalization of \( L \). We have an exact sequence of sheaves on \( L : 0 \rightarrow \mathcal{O}_L^* \rightarrow \pi_*\mathcal{O}_{L_1}^* \rightarrow F \rightarrow 0 \) where \( F \) is the skyscraper sheaf with stalks, \( F_p = \mathcal{O}_{L,p}^*/\mathcal{O}_{L_1,p}^* \) and \( \mathcal{O}_{L_1,p}^* \) is the integral closure of \( \mathcal{O}_{L,p}^* \).

One finds an exact sequence
\[
0 \rightarrow \mathcal{O}(L)^* \rightarrow \mathcal{O}(L_1)^* \rightarrow H^0(F) \rightarrow H^1(L,\mathcal{O}_L^*) \rightarrow H^1(L_1,\mathcal{O}_{L_1}^*) \rightarrow 0.
\]
So clearly (by 2.2) \( L_1 = \mathbb{P}^1(k) - \{p_1, \ldots, p_s\} \) and the group \( \mathcal{O}(L_1)^* \) is isomorphic to \( k^* \oplus N \) where \( N \) is a subgroup of \( Z^{s-1} \).

So we find that \( H^0(F) \) is a finitely generated \( Z \)-module.

If \( L \) has a singular point \( p \) then \( H^0(F) \) has \( \mathcal{O}_{L,p}^*/\mathcal{O}_{L_1,p}^* \) as component. The last group has \( k \) or \( k^* \) as quotient group. It is not finitely generated. So we conclude that \( L \) is non-singular, and hence a Zariski-open subset of \( \mathbb{P}^1(k) \).

2.5. Continuation of the proof of 2.1.

We have to consider the case where \( Y \), the canonical reduction of \( Y \), has more than one component. Let \( L \) be a component and \( L_1 = L - \{\text{the intersection of } L \text{ with the other components}\} ; Y_1 = \mathbb{R}^{-1}(L_1) \). Then \( Y_1 \) is affinoid, also a U.F.D. and with canonical reduction \( L_1 \). We know by 2.3 and 2.4 that \( L_1 \) is Zariski-open in \( \mathbb{P}^1(k) \) and so \( Y_1 \) must be an affinoid subset of \( \mathbb{P}^1(k) \) of the form
\[
\{z \in k \mid |z| \leq 1, \quad |z - a_i| \geq 1 \quad (i = 1, \ldots, s)\}.
\]
Let \( a_{d+1}, \ldots, a_s \) correspond to the points of intersection of \( L \) with the other components of \( Y \). Let \( Y_2 = \{z \in k \mid |z| \leq 1 \text{ and } |z - a_i| \geq 1 \text{ for } i = d + 1, \ldots, s\} \). Then we glue \( Y_2 \) to \( Y \) over the open subset \( Y_1 \). The resulting analytic space \( Y \cup Y_2 \) has as reduction with respect to the covering \( \{Y, Y_2\} \) the space \( \tilde{Y} \cup \tilde{Y}_2 \). From [2] ch. IV (2.2) it follows that \( Z = Y \cup Y_2 \) is also affinoid and its canonical reduction is obtained by contracting the complete one of \( \tilde{Y} \cup \tilde{Y}_2 \) to a point. If we can show that \( Z \) is also a U.F.D., then (2.1) follows by induction on the number of components of \( \tilde{Y} \). Since
\[
H^1(Y,\mathcal{O}_Y^*) = H^1(Y_1,\mathcal{O}_{Y_1}^*) = H^1(Y_2,\mathcal{O}_{Y_2}^*) = 0
\]
we can calculate \( H^1(Z,\mathcal{O}_Z^*) \) = the class group of \( Z \), with respect to the covering \( \{Y_2, Y\} \). That \( Z \) is a U.F.D. is equivalent with \( H^1(Z,\mathcal{O}_Z^*) = 0 \) and will follow from the following
2.6. - **Lemma.** - The map \( \mathcal{O}(Y)^* \oplus \mathcal{O}(Y_2)^* \rightarrow \mathcal{O}(Y_1)^* \), given by \((f_1, f_2) \mapsto f_1f_2^{-1} \), is surjective.

**Proof.** - The norm on \( \mathcal{O}(Y_1)^* \) is multiplicative. So any \( f \in \mathcal{O}(Y_1)^* \) has the form \( f = cg \) with \( c \in k^* \) and \( g \in \mathcal{O}(Y_1)^0) \). Further the analogous map \( \mathcal{O}(Y)^* \oplus \mathcal{O}(Y_2)^* \rightarrow \mathcal{O}(Y_1)^* \) is clearly surjective. So \( g = f_1f_2^{-1} \) for certain \( f_1 \in \mathcal{O}(Y)^0 \) and \( f_2 \in \mathcal{O}(Y_2)^0 \). We are reduced to consider \( f \in \mathcal{O}(Y_1)^* \) of the form \( 1 + h \) with \( h \in \mathcal{O}(Y_1), \|h\| < 1 \). We want to write \( f \) as \((1 + h_1)(1 + h_2)^{-1} \) with \( h_1 \in \mathcal{O}(Y), \ h_2 \in \mathcal{O}(Y_2) \) and \( \|h_1\| < 1, \|h_2\| < 1 \). This amounts to showing that \( \beta : \mathcal{O}(Y)^0 \oplus \mathcal{O}(Y_2)^0 \rightarrow \mathcal{O}(Y_1)^0 \), given by \((h_1, h_2) \mapsto h_1 - h_2 \), is surjective. By [2], ch. IV (2.2), we know that the cokernel of \( \beta \) is a finitely generated \( k^0 \)-module \( M \). Moreover \( M \otimes \bar{k} = 0 \) since \( \mathcal{O}(Y) \oplus \mathcal{O}(Y_2) \rightarrow \mathcal{O}(Y_1) \) is surjective. So \( M = 0, \beta \) is surjective and the Lemma is proved.

2.7. - **Corollary.** - Let \( X \) be a complete non-singular curve over \( k \). Then \( X \) is a Mumford curve (i.e. can be parametrized by a Schottky group) if and only if \( X \) is locally a U.F.D.

**Proof.** - Locally a U.F.D. means that \( X \) has an affinoid covering \( (X_i)_{i=1}^s \) such that each \( \mathcal{O}(X_i) \) is a unique factorization domain. According to (2.1) this implies \( X_i \subset P^1(k) \). According to [2], ch. IV (5.1), this is equivalent with \( X \) is a Mumford curve.

3. **Class groups.**

\( X \) will denote a normal, connected, 1-dimensional affinoid space. The class group of \( X \) (i.e. the group of isomorphy-classes of projective, rank 1, \( \mathcal{O}(X) \)-modules) is equal to the analytic cohomology group \( H^1(X, \mathcal{O}_X^*) \). This follows from the bijective correspondence between projective, rank 1, \( \mathcal{O}(X) \)-modules and invertible sheaves on \( X \).

3.1. - **Proposition.** - Let \( X \) be embedded in a complete non-singular curve \( C \). Then \( H^1(X, \mathcal{O}_X^*) \simeq J(C)/H \) where \( J(C) \) is the Jacobi-variety of \( C \) and \( H \) is the subgroup consisting of the images of the divisors of degree zero on \( C \) with support in \( C - X \). The group \( H \) is an open subgroup in the topology of \( J(C) \) induced by the topology of \( k \).

**Proof.** - The restriction map \( \text{Div}_0(C) \rightarrow \text{Div}(X) \) induces a surjective homomorphism \( \text{Div}_0(C)/\text{P}(C) \rightarrow \text{Div}(X)/\text{P}(X) \) where \( \text{P}(C) \) denotes the principal divisors on \( C \) and \( \text{P}(X) = \{(f) \mid f \text{ is a principal divisor on } X \} \).
meromorphic on $X$}. It is easily seen that $H^1(X, \mathcal{O}_X^*) = \text{Div}(X)/\text{P}(X)$. Let $D \in \text{Div}_0(C)$ have image 0 in $H^1(X, \mathcal{O}_X^*)$, then there exists a meromorphic function $f$ on $X$ with $(f) = D$ on $X$. As one can calculate (see [2], ch. III (1.18.5) and on) any divisor of a holomorphic (or meromorphic) function on $C$ restricted to $X$ is the divisor of a rational function on $C$ restricted to $X$. So there is a rational function $g$ on $C$ with $(g) = D$ on $X$. Then $D - (g)$ is a divisor of degree 0 with support in $C - X$. This proves the first assertion. The map $C \times \ldots \times C \longrightarrow J(C)$ given by $(x_1, \ldots, x_g) \longmapsto \sum_{i=1}^g x_i - gx_0$ (where $x_0 \in C - X$ is fixed) is surjective and induces the algebraic structure and topology on $J(C)$. The map is almost bijective and open. So the image of $(C - X) \times \ldots \times (C - X)$ is open and $H$ is open.

**Remark.** In general it seems to be rather difficult to calculate explicitly $H^1(X, \mathcal{O}_X^*)$. However using (3.1) one can work out the following special cases.

3.2. **Example.** Let the curve $C$ have a reduction $R : C \longrightarrow \overline{C}$ such that $\overline{C}$ is rational and has one ordinary double point $p$. Take $p_1, \ldots, p_s$ points in $\overline{C} - \{p\}$ and put $X = R^{-1}(\overline{C} - \{p_1, \ldots, p_s\})$. Then $X$ is affinoid and its canonical reduction is $C - \{p_1, \ldots, p_s\}$. The curve $C$ is a Tate-curve and $\simeq k^*/\langle q \rangle$ with $0 < |q| < 1$. The points $p_1, \ldots, p_s$ correspond to open discs of radii 1 around points $1 = a_1, a_2, \ldots, a_s \in k$ with all $|a_i| = 1$ and $|a_i - a_j| = 1$ if $i \neq j$. Using (3.1) one finds an exact sequence:

$$1 \longrightarrow k^*/\langle a_1, \ldots, a_s \rangle \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow |k^*/\langle q \rangle| \longrightarrow 1$$

where $\langle a_2, \ldots, a_s \rangle$ is the subgroup of $k^*$ generated by $a_2, \ldots, a_s$; $|k^*|$ is the value group of $k$ and $\langle q \rangle$ its subgroup generated by $|q|$. Note further that $k^*/\langle a_2, \ldots, a_s \rangle = H^1(X, \mathcal{O}_X^*)$.

3.3. **Example.** Let $C$ be a Mumford curve of genus $g \geq 1$ and let $R : C \longrightarrow \overline{C}$ be its stable reduction. (The components of $C$ are rational, the only singularities are ordinary double points.) The Jacobi-variety of $C$ is a holomorphic torus $(k^*)^g/\Lambda$ where $\Lambda$ is a lattice in $(k^*)^g$. Take ordinary points $p_1, \ldots, p_s \in \overline{C}$ and put $X = R^{-1}(\overline{C} - \{p_1, \ldots, p_s\})$. Then $X$ is affinoid and using (3.1) one calculates an exact sequence:

$$1 \longrightarrow (k^*)^g/S \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow |k^*/\Lambda| \longrightarrow 1$$
where
\[ |\Lambda| = \{ (|\lambda_1|, |\lambda_2|, \ldots, |\lambda_g|) : (\lambda_1, \ldots, \lambda_g) \in \Lambda \} \]
and \( S \) is a finitely generated subgroup of \((k^*)^g\). The group \((k^*)^g\) is in fact the Jacobi-variety of \( C \) and the subgroup \( S \) is the subgroup of the divisors of degree 0 on \( C \) with support in \( \{ p_1, \ldots, p_s \} \). So \((k^*)^g/S\) is again \( H^1(X_S, \mathcal{O}^*) \) where \( X_S \) denotes the stable reduction of \( X \).

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Marius van der PUT,
Mathematisch Instituut
Universiteit van Groningen
WSN-Gebouw, Postbns 800
Groningen (Pays-Bas).

&
U.E.R. de Mathématiques
et d'Informatique
Université de Bordeaux I
F-33405 Talence Cedex.