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THE CLASS GROUP OF A ONE-DIMENSIONAL AFFINOID SPACE

by Marius van der PUT

Introduction.

The field k is supposed to be complete with respect to a non-archimedean valuation. Moreover we will assume that k is algebraically closed. An affinoid space Y over k is the set of maximal ideals of an affinoid algebra. The standard affinoid algebra is $k\langle T_1,\ldots,T_n\rangle =$ the set of all power series $\Sigma a_\alpha T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ converging on the closed polydisk

$$\{(t_1,\ldots t_n) \in k^n | \text{all} |t_i| \leqslant 1\}.$$

An affinoid algebra is a residue class ring of some $k\langle T_1,\ldots,T_n\rangle$. An algebraic variety over k can be studied locally by its analytic structure over k, that is by means of affinoid spaces.

We show that a one-dimensional, normal, connected affinoid space Y is an affinoid subset of a non-singular, complete curve C over k (Thm 1.1). If Y has a trivial classgroup then Y is in fact an affinoid subset of $\mathbf{P}^1$ (Thm 2.1). A curve is locally a unique factorization domain (U.F.D. for short) if and only the curve is a Mumford curve (i.e. can be parametrized by a Schottky group). In general the class group of Y can be expressed in terms of the Jacobi-variety of C (prop. 3.1).

Some examples show the connection between the class group of Y and the class group of the (stable) reduction of Y. For k-analytic spaces we refer to [2], [3]. I thank A. Escassut for bringing the problem on unique factorization on affinoid spaces to my attention. Related questions are treated in [1].

1. Affinoid subspaces of an algebraic curve.

A curve C (non-singular and complete) over k has a natural structure as (rigid) analytic space over k. This structure is given by a collection of subspaces Y of C, called affinoid, and a sheaf $\mathcal{O} = \mathcal{O}_C$ with respect to the Grothendieck topology of finite coverings by affinoids. For any Y, $\mathcal{O}(Y)$ is an affinoid algebra (1-dim. and normal) over k with $\mathrm{Sp}(\mathcal{O}(Y)) = Y$. We want to show :

1.1. — THEOREM. — *Every 1-dimensional, normal, connected affinoid space* $Y = \mathrm{sp}(A)$ *is an affinoid subspace of a non-singular complete curve.*

Proof. — Y is called connected and normal if the algebra A has no idempotents $\neq 0, 1$ and A is integrally closed. We use the notations $A^{o} = \{f \in A \mid \|f\| \leqslant 1\}$, $A^{oo} = \{f \in A \mid \|f\| < 1\}$ and $\bar{A} = A^{o}/A^{oo}$, where $\|f\| = \max\{|f(y)| \mid y \in Y\}$ is the spectral norm on Y. The algebra $\bar{A}$ is of finite type over $\bar{k}$ = the residue field of k and the algebraic variety $\bar{Y}_c = \mathrm{Max}(\bar{A})$ is called the canonical reduction of Y. There is a natural surjective map $R : Y \to \bar{Y}_c$, also called the canonical reduction. A pure covering of an analytic space X, is an allowed covering $\mathcal{U} = (U_i)$ by affinoid spaces, such that for every $i \neq j$ with $U_i \cap U_j \neq \emptyset$, the set $U_i \cap U_j$ is the inverse image of a Zariski open set V_{ij} in $\overline{(U_i)_c}$ under the map $U_i \to \overline{(U_i)_c}$. The reduction $\overline{X_{\mathcal{U}}}$ of X with respect to $\mathcal{U}$ is obtained by glueing the affine algebraic varieties $\overline{(U_i)_c}$ over the open sets V_{ij}. The result is an algebraic variety over $\bar{k}$. If X is separated then the $U_i \cap U_j$ are also affinoid, the V_{ij} are affine and equal to $\overline{(U_i \cap U_j)_c}$ and $\overline{X_{\mathcal{U}}}$ is separated. If X is non-singular, 1-dimensional, connected and if $\overline{X_{\mathcal{U}}}$ is complete then X is a non-singular complete curve over k (see [2] ch. IV 2.2).

Our proof consists of glueing affinoid spaces $Y_1, \ldots, Y_s$ to Y such that the reduction of $X = Y \cup Y_1 \cup \ldots \cup Y_s$ with respect to the pure covering $\{Y, Y_1, \ldots, Y_s\}$ is complete. Then clearly Y is an affinoid domain of the algebraic curve X. The 1-dimensional space $\overline{Y_c}$ lies in a complete 1-dimensional Z such that $F = Z - \overline{Y_c}$ is a finite set of non-singular points. Suppose that we can find for every $p \in F$ an affinoid space Y_p with canonical reduction $R_p : Y_p \to \overline{(Y_p)_c} \subset Z$ where $\overline{(Y_p)_c}$ is a neighbourhood of p and such that

$$Y_p \supset R_p^{-1}(\overline{(Y_p)_c} \cap \overline{Y_c}) \simeq R^{-1}(\overline{(Y_p)_c} \cap \overline{Y_c}) \subset Y.$$

Then we can glue Y_p to Y. The space $X = YU \cup Y_p$ has reduction Z which is complete. So the glueing has to be done locally on Y and $\overline{Y_c}$. The component C of Z on which p lies can be projected into $\mathbf{P}^2(\bar{k})$ such that (the image of) p is still non-singular. A good projection onto $\mathbf{P}^1$ maps p onto o and o is an unramified point for the projection. Replacing Y and $\overline{\mathrm{Y}_c}$ by neighbourhoods of p we may therefore suppose :

$$\overline{\mathcal{O}(\mathrm{Y})} = \mathcal{O}(\overline{\mathrm{Y}_c}) = \bar{k}[t,(t,e(t))^{-1},s]/(\mathrm{P}),$$

where

1) $e(t) = (t-\overline{a_1})\dots(t-\overline{a_s})$ with $\overline{a_1},\dots,\overline{a_s}$ different points of $\bar{k}^*$; they are the residues of $a_1,\dots,a_s \in k^0$.

2) P is a monic irreducible polynomial of degree n with coefficients in $\bar{k}[t]$.

3) $\dfrac{d\mathrm{P}}{ds}$ is invertible as element of $\bar{k}[t,(e(t))^{-1},s]/(\mathrm{P})$.

4) the point « p » corresponds to $t = 0$.

Then $\mathcal{O}(\mathrm{Y})^0$ has the form $k^0\langle \mathrm{T,U,S}\rangle/(\mathrm{TE(T)U}-1,\mathrm{Q})$ where

$$\mathrm{E(T)} = (\mathrm{T}-a_1)\dots(\mathrm{T}-a_s) \qquad \text{and} \qquad \bar{\mathrm{Q}} = \mathrm{P}.$$

Since Q is general with respect to the variable S, we can apply Weierstrass-division and assume that Q is a monic polynomial of degree n in S with coefficients in $k^0\langle \mathrm{T,U}\rangle/(\mathrm{TE(T)U}-1)$. Suppose that we can find a monic polynomial Q^* of degree n in S and coefficients in $k^0\langle \mathrm{T,V}\rangle/(\mathrm{E(T)V}-1)$ such that

$$k^0\langle \mathrm{T,U,S}\rangle/(\mathrm{TE(T)U}-1,\mathrm{Q}^*) \simeq \mathcal{O}(\mathrm{Y})^0.$$

Then $\mathrm{Y}_p = Sp(k\langle \mathrm{T,V,S}\rangle/(\mathrm{E(T)V}-1,\mathrm{Q}^*))$ has the required properties. So we have to get rid of the negative powers of T in the coefficients of

$$\mathrm{Q} = \mathrm{S}^n + a_{n-1}\mathrm{S}^{n-1} + \cdots + a_0.$$

1.2. – Lemma. – *If* $\mathrm{Q}^* = \mathrm{S}^n + a^*_{n-1}\mathrm{S}^{n-1} + \cdots + a^*_0$ *has coefficients in* $\mathrm{A} = k^0\langle \mathrm{T,U}\rangle/(\mathrm{TE(T)U}-1)$ *and* $\overline{\mathrm{Q}^*} = \overline{\mathrm{Q}} = \mathrm{P}$, *then*

a) Q^* *is irreducible*

b) Q^* *has a zero in* $\mathcal{O}(\mathrm{Y})^0$

c) $k\langle \mathrm{T,U,S}\rangle/(\mathrm{TE(T)U}-1,\mathrm{Q}^*) \simeq \mathcal{O}(\mathrm{Y})$.

Proof. – *a)* Let Q^* be reducible over the quotient field of A. Since A is normal, Q^* is a product of monic polynomials with coefficients in A. This contradicts the irreducibility of $\overline{\mathrm{Q}^*} = \mathrm{P}$.

b) First we show that $\left\{Q^*, \frac{dQ^*}{dS}\right\}$ generates the unit ideal in $A[S]$. Let $\mathfrak{m}$ be a maximal ideal containing Q^* and $\frac{dQ^*}{dS}$. If $\mathfrak{m} \cap k^0 \neq 0$ then $\mathfrak{m}$ induces a maximal ideal of $\bar{k}[t,(te(t))^{-1}][S] = \bar{A}[S]$ containing P and $\frac{dP}{dS}$. This contradicts our assumptions on P. So $\mathfrak{m}$ corresponds to a maximal ideal $\mathfrak{m}_1$, of $k\langle T,U\rangle/(TE(T)U-1)[S]$, containing Q^* and $\frac{dQ^*}{dS}$.

If $\mathfrak{m}_1 \cap k\langle T,U\rangle/(TE(T)U-1) \neq 0$ then $\mathfrak{m}_1$, is the kernel of a homomorphism in k given by $T \longmapsto \lambda_1 \in k$, $S \longmapsto \lambda_2 \in k$ with

$$|\lambda_1| \leqslant 1, \qquad |\lambda_1 E(\lambda_1)| = 1, \qquad |\lambda_2| \leqslant 1$$

since $Q^*(\lambda_2) = 0$. From $\left(P, \frac{dP}{dS}\right) = \bar{k}[t,(te(t))^{-1},S]$ it follows that

$$Z_1(S)Q^* + Z_2(S)\frac{dQ^*}{dS} = 1 + \sum_{i>0} a_i S^i$$

for certain Z_1, $Z_2 \in A[S]$ and $a_i \in A$ with $\|a_i\| < 1$. The substitution $T \longmapsto \lambda_1$; $S \longmapsto \lambda_2$ makes $0 = 1 + \sum_{i>0} a_i(\lambda_1)\lambda_2^i$, which is impossible. So $\mathfrak{m}$ and $\mathfrak{m}_1$ correspond to an ideal of $L[S]$ with L the quotient field of A. Since Q^* is irreducible, this means that $\frac{dQ^*}{dS} = 0$. This is obviously in contradiction with $\left(P, \frac{dP}{dS}\right) = \bar{k}[t,(te(t))^{-1}]$.

We conclude the existence of Z_1, $Z_2 \in A[S]$ with

$$1 = Z_1(S)Q^* + Z_2(S)\frac{dQ^*}{dS}.$$

By Newton's method we will show that Q^* has a zero in $\mathcal{O}(Y)^0$. Let $\eta \in \mathcal{O}(Y)^0$ satisfy $\|Q^*(\eta)\| < 1$ (e.g. η is the residue of S mod Q in $\mathcal{O}(Y)^0$). Then $1 - Z_1(\eta)Q^*(\eta) = Z_2(\eta)\frac{dQ^*}{dS}(\eta)$ and since

$\|Z_1(\eta)Q^*(\eta)\| < 1$ it follows that $\dfrac{dQ^*}{dS}(\eta)$ is invertible. Put $\eta_1 = \eta - Q^*(\eta)\left(\dfrac{dQ^*}{dS}(\eta)\right)^{-1}$. Then $\|Q^*(\eta_1)\| \leqslant \|Q^*(\eta)\|^2$. The usual procedure and the completeness of $\mathcal{O}(Y)^0$ show the existence of a root of Q^* in $\mathcal{O}(Y)^0$.

c) The quotient field of $A[S]/Q^*$ is contained in that of $A[S]/Q$, because of (*b*). Both fields are extensions of degree n of the quotient field of A. So they are equal. The rings $k\langle T,U,S\rangle/(TE(T)U-1,Q^*)$ and $\mathcal{O}(Y)$ are both the integral closure of $k\langle T,U\rangle/(TE(T)U-1)$ in that field. So they are equal.

End of the proof of 1.1. — We choose Q^* with coefficients in $k^0\langle T,V\rangle/(VE(T)-1)$ and $Q^* = P$.

1.3. — COROLLARY. — *Let* Y *be as in* (1.1) ; *then* Y *is affinoid in a curve* X (*complete non-singular*) *such that* $\bar{X} - \bar{Y}_c$ *is a finite set of non-singular points.*

2. Unique factorization.

We want to show the following :

2.1. — THEOREM. — *Let* $Y = Sp\, A$ *be a* 1-*dimensional connected affinoid space. Then* A *has unique factorization if and only if* Y *is an affinoid subspace of* $\mathbf{P}^1(k)$.

Remarks. — 1) Since A has dimension 1 the condition « A has unique factorization » is equivalent to « A is a principal ideal domain ».

2) It seems that this theorem has also been proved by M. Raynaud.

A connected affinoid subspace Y of $\mathbf{P}^1(k)$ has clearly a U.F.D. as affinoid algebra. Before we start the proof of 2.1, we like to state its algebraic analogue. It is :

2.2. — PROPOSITION. — *Let* A *be a finitely generated algebra over an algebraically closed field* k. *Suppose that* A *is* 1-*dimensional and a U.F.D. Then* A *is isomorphic to the coordinate ring of a Zariski-open subset of* $\mathbf{P}^1(k)$.

Proof. — A is the coordinate ring of a Zariski-open subset X of some non-singular complete curve C; put $X = C - \{p_1,\dots,p_s\}$. Let D be a divisor of degree 0 on C; since A is a U.F.D. there is a rational function f on C with $D = (f)$ on X. This means that the map $\left\{ \sum_{i=1}^{s} n_i p_i | n_i \in \mathbf{Z} \text{ and } \Sigma n_i = 0 \right\} \longrightarrow J(C) =$ the Jacobi-variety of C, is surjective. If C is not a rational curve then its Jacobi variety (or better its points in k) is not a finitely generated group. Hence $C \simeq \mathbf{P}^1(k)$.

We prove the theorem in some steps.

2.3. — LEMMA. — *Suppose that* $\mathcal{O}(Y)$ *is a U.F.D. and that* $\bar{Y}$ *is irreducible, then* $H^1(\bar{Y},\mathcal{O}^*_{\bar{Y}}) = 0$.

Proof. — $\bar{Y}$ denotes the canonical reduction of Y. An element $\xi \in H^1(\bar{Y},\mathcal{O}^*)$ corresponds to a projective, rank 1, $\mathcal{O}(\bar{Y})$-module N; let F be a free $\mathcal{O}(\bar{Y})$-module, $\sigma : F \longrightarrow F$ an idempotent endomorphism with $\operatorname{im} \sigma = N$. Then F, σ lift to similar things over $\mathcal{O}(Y)^0$ since $\mathcal{O}(Y)^0$ is complete and $\mathcal{O}(\bar{Y}) = \mathcal{O}(Y)^0 \otimes \bar{k}$. So we find a projecture, rank 1, $\mathcal{O}(Y)^0$-module M with $M \otimes \bar{k} = N$.

Further $M \otimes \mathcal{O}(Y) \simeq \mathcal{O}(Y)$ since $\mathcal{O}(Y)$ is a U.F.D. There exists a Zariski-open covering of $\bar{Y}$ such that N is free on the sets of this covering. That implies the existence of $f_1, \ldots, f_s \in \mathcal{O}(Y)^0$ such that

a) each $\|f_i\| = 1$ and $(f_1, \ldots, f_s)\mathcal{O}(Y)^0 = \mathcal{O}(Y)^0$.

b) $M \otimes \mathcal{O}(X)^0\langle S\rangle/(Sf_i - 1)$ is a free $\mathcal{O}(X)^0\langle S\rangle/(Sf_i - 1)$-module.

We identify M with $M \otimes \mathcal{O}(Y)^0 \subset \mathcal{O}(Y)$ and we may suppose that $M \subset \mathcal{O}(Y)^0$; $\max\{\|m\| \mid m \in M\} = 1$ and $M \supset \lambda\mathcal{O}(Y)^0$ for certain $\lambda \in k^0$, $\lambda \neq 0$. Then

$$M \otimes \mathcal{O}(Y)^0\langle S\rangle/(Sf_i - 1) \subseteq \mathcal{O}(Y)^0\langle S\rangle/(Sf_i - 1)$$

is generated by one element h. This element has norm 1 and it has no zeros is $\{y \in Y \mid |f_i(Y)| = 1\} = Y_i$. So h is invertible in $\mathcal{O}(Y_i)$. Its inverse h^{-1} has also norm 1 since $\overline{Y_i}$ is irreducible and the norm on $\mathcal{O}(Y_i)$ is, as a consequence, multiplicative. Hence $M\mathcal{O}(Y_i)^0 = \mathcal{O}(Y_i)^0$. It follows that some power of f_i lies in M. Since $(f_1, \ldots, f_s) = \mathcal{O}(Y)^0$ we find that $M = \mathcal{O}(Y)^0$. So N is free and $\xi = 0$.

2.4. — LEMMA. — *Let* L *be affine, 1-dimensional and irreducible over* $\bar{k}$. *If* $H^1(L,\mathcal{O}^*_L) = 0$ *then* L *is rational and non-singular.*

Proof. – Let $\pi : L_1 \longrightarrow L$ be the normalization of L. We have an exact sequence of sheaves on $L : 0 \longrightarrow \mathcal{O}_L^* \longrightarrow \pi_* \mathcal{O}_{L_1}^* \longrightarrow F \longrightarrow 0$ where F is the skyscraper sheaf with stalks, $F_p = \tilde{\mathcal{O}}_{L,p}^*/\mathcal{O}_{L,p}^*$ and $\tilde{\mathcal{O}}_{L,p}$ is the integral closure of $\mathcal{O}_{L,p}$.

One finds an exact sequence

$$0 \longrightarrow \mathcal{O}(L)^* \longrightarrow \mathcal{O}(L_1)^* \longrightarrow H^0(F) \longrightarrow H^1(L,\mathcal{O}_L^*) \longrightarrow H^1(L_1,\mathcal{O}_{L_1}^*) \longrightarrow 0.$$

So clearly (by 2.2) $L_1 = \mathbf{P}^1(\bar{k}) - \{p_1,\ldots,p_s\}$ and the group $\mathcal{O}(L_1)^*$ is isomorphic to $\bar{k}^* \oplus N$ where N is a subgroup of $\mathbf{Z}^{s-1}$.

So we find that $H^0(F)$ is a finitely generated $\mathbf{Z}$-module.

If L has a singular point p then $H^0(F)$ has $\tilde{\mathcal{O}}_{L,p}^*/\mathcal{O}_{L,p}^*$ as component. The last group has $\bar{k}$ or $\bar{k}^*$ as quotient group. It is not finitely generated. So we conclude that L is non-singular, and hence a Zariski-open subset of $\mathbf{P}^1(\bar{k})$.

2.5. – *Continuation of the proof of* 2.1.

We have to consider the case where $\bar{Y}$, the canonical reduction of Y, has more than one component. Let L be a component and $L_1 = L - \{$the intersection of L with the other components$\}$; $Y_1 = R^{-1}(L_1)$. Then Y_1 is affinoid, also a U.F.D. and with canonical reduction L_1. We know by 2.3 and 2.4 that L_1 is Zariski-open in $\mathbf{P}^1(\bar{k})$ and so Y_1 must be an affinoid subset of $\mathbf{P}^1(k)$ of the form

$$\{z \in k \mid |z| \leqslant 1, \qquad |z - a_i| \geqslant 1 \qquad (i=1,\ldots,s)\}.$$

Let $a_{d+1},\ldots,a_s$ correspond to the points of intersection of L with the other components of $\bar{Y}$. Let $Y_2 = \{z \in k \mid |z| \leqslant 1$ and $|z - a_i| \geqslant 1$ for $i = d+1, \ldots, s\}$. Then we glue Y_2 to Y over the open subset Y_1. The resulting analytic space $Y \cup Y_2$ has as reduction with respect to the covering $\{Y,Y_2\}$ the space $\bar{Y} \cup \bar{Y}_2$. From [2] ch. IV (2.2) it follows that $Z = Y \cup Y_2$ is also affinoid and its canonical reduction is obtained by contracting the complete one of $\bar{Y} \cup \bar{Y}_2$ to a point. If we can show that Z is also a U.F.D., then (2.1) follows by induction on the number of components of $\bar{Y}$. Since

$$H^1(Y,\mathcal{O}_Y^*) = H^1(Y_1,\mathcal{O}_{Y_1}^*) = H^1(Y_2,\mathcal{O}_{Y_2}^*) = 0$$

we can calculate $H^1(Z,\mathcal{O}_Z^*)$ = the class group of Z, with respect to the covering $\{Y_2,Y\}$. That Z is a U.F.D. is equivalent with $H^1(Z,\mathcal{O}_Z^*) = 0$ and will follow from the following

2.6. — LEMMA. — *The map* $\mathcal{O}(\mathrm{Y})^* \oplus \mathcal{O}(\mathrm{Y}_2)^* \longrightarrow \mathcal{O}(\mathrm{Y}_1)^*$, *given by* $(f_1,f_2) \longrightarrow f_1 f_2^{-1}$, *is surjective.*

Proof. — The norm on $\mathcal{O}(\mathrm{Y}_1)$ is multiplicative. So any $f \in \mathcal{O}(\mathrm{Y}_1)^*$ has the form $f = cg$ with $c \in k^*$ and $g \in (\mathcal{O}(\mathrm{Y}_1)^0)^*$. Further the analoguous map $\mathcal{O}(\bar{\mathrm{Y}})^* \oplus \mathcal{O}(\bar{\mathrm{Y}}_2)^* \longrightarrow \mathcal{O}(\bar{\mathrm{Y}}_1)^*$ is clearly surjective. So $\bar{g} = \bar{f}_1 \bar{f}_2^{-1}$ for certain $f_1 \in (\mathcal{O}(\mathrm{Y})^0)^*$ and $f_2 \in (\mathcal{O}(\mathrm{Y}_2)^0)^*$. We are reduced to consider $f \in \mathcal{O}(\mathrm{Y}_1)^*$ of the form $1 + h$ with $h \in \mathcal{O}(\mathrm{Y}_1)$, $\|h\| < 1$. We want to write f as $(1+h_1)(1+h_2)^{-1}$ with $h_1 \in \mathcal{O}(\mathrm{Y})$, $h_2 \in \mathcal{O}(\mathrm{Y}_2)$ and $\|h_1\| < 1$, $\|h_2\| < 1$. This amounts to showing that $\beta : \mathcal{O}(\mathrm{Y})^0 \oplus \mathcal{O}(\mathrm{Y}_2)^0 \longmapsto \mathcal{O}(\mathrm{Y}_1)^0$, given by $(h_1,h_2) \longmapsto h_1 - h_2$, is surjective. By [2], ch. IV (2.2), we know that the cokernel of β is a finitely generated k^0-module M. Moreover $\mathrm{M} \otimes \bar{k} = 0$ since $\mathcal{O}(\bar{\mathrm{Y}}) \oplus \mathcal{O}(\bar{\mathrm{Y}}_2) \longrightarrow \mathcal{O}(\bar{\mathrm{Y}}_1)$ is surjective. So $\mathrm{M} = 0$, β is surjective and the Lemma is proved.

2.7. — COROLLARY. — *Let* X *be a complete non-singular curve over* k. *Then* X *is a Mumford curve* (*i.e. can be parametrized by a Schottky group*) *if and only if* X *is locally a U.F.D.*

Proof. — Locally a U.F.D. means that X has an affinoid covering $(\mathrm{X}_i)_{i=1}^s$ such that each $\mathcal{O}(\mathrm{X}_i)$ is a unique factorization domain. According to (2.1) this implies $\mathrm{X}_i \subset \mathbf{P}^1(\mathrm{k})$. According to [2], ch. IV (5.1), this is equivalent with X is a Mumford curve.

3. Class groups.

X will denote a normal, connected, 1-dimensional affinoid space. The class group of X (i.e. the group of isomorphy-classes of projective, rank 1, $\mathcal{O}(\mathrm{X})$-modules) is equal to the analytic cohomology group $\mathrm{H}^1(\mathrm{X},\mathcal{O}_{\mathrm{X}}^*)$. This follows from the bijective correspondance between projective, rank 1, $\mathcal{O}(\mathrm{X})$-modules and invertible sheaves on X.

3.1. — PROPOSITION. — *Let* X *be embedded in a complete non-singular curve* C. *Then* $\mathrm{H}^1(\mathrm{X},\mathcal{O}_{\mathrm{X}}^*) \simeq \mathrm{J(C)/H}$ *where* J(C) *is the Jacobi-variety of* C *and* H *is the subgroup consisting of the images of the divisors of degree zero on* C *with support in* C − X. *The group* H *is an open subgroup in the topology of* J(C) *induced by the topology of* k.

Proof. — The restriction map $\mathrm{Div}_0(\mathrm{C}) \longrightarrow \mathrm{Div(X)}$ induces a surjective homomorphism $\mathrm{Div}_0(\mathrm{C})/\mathrm{P(C)} \longrightarrow \mathrm{Div(X)/P(X)}$ where P(C) denotes the principal divisors on C and $\mathrm{P(X)} = \{(f)$ on $\mathrm{X}|f$

meromorphic on $X\}$. It is easily seen that $H^1(X,\mathcal{O}_X^*) = \mathrm{Div}(X)/P(X)$. Let $D \in \mathrm{Div}_0(C)$ have image 0 in $H^1(X,\mathcal{O}_X^*)$, then there exists a meromorphic function f on X with $(f) = D$ on X. As one can calculate (see [2], ch. III (1.18.5) and on) any divisor of a holomorphic (or meromorphic) function on X is the divisor of a rational function on C restricted to X. So there is a rational function g on C with $(g) = D$ on X. Then $D - (g)$ is a divisor of degree 0 with support in $C - X$. This proves the first assertion. The map $C \times \ldots \times C \longrightarrow J(C)$ given by $(x_1,\ldots,x_g) \longmapsto \sum_{i=1}^{g} x_i - gx_0$ (where $x_0 \in C - X$ is fixed) is surjective and induces the algebraic structure and topology on $J(C)$. The map is almost bijective and open. So the image of $(C-X) \times \ldots \times (C-X)$ is open and H is open.

Remark. – In general it seems to be rather difficult to calculate explicitely $H^1(X,\mathcal{O}_X^*)$. However using (3.1) one can work out the following special cases.

3.2. – *Example.* – Let the curve C have a reduction $R : C \longrightarrow \bar{C}$ such that $\bar{C}$ is rational and has one ordinary double point p. Take $p_1, \ldots, p_s$ points in $\bar{C} - \{p\}$ and put $X = R^{-1}(\bar{C} - \{p_1,\ldots,p_s\})$. Then X is affinoid and its canonical reduction is $C - \{p_1,\ldots,p_s\}$. The curve C is a Tate-curve and $\simeq k^*/\langle q \rangle$ with $0 < |q| < 1$. The points $p_1, \ldots, p_s$ correspond to open discs of radii 1 around points $1 = a_1, a_2, \ldots, a_s \in k$ with all $|a_i| = 1$ and $|a_i - a_j| = 1$ if $i \neq j$. Using (3.1) one finds an exact sequence :

$$1 \longrightarrow \bar{k}^*/\langle \overline{a_2}, \ldots, \overline{a_s} \rangle \longrightarrow H^1(X,\mathcal{O}_X^*) \longrightarrow |k^*|/\langle |q| \rangle \longrightarrow 1$$

where $\langle \overline{a_2},\ldots,\overline{a_s} \rangle$ is the subgroup of $\bar{k}^*$ generated by $\overline{a_2}, \ldots, \overline{a_s}$; $|k^*|$ is the value group of k and $\langle |q| \rangle$ its subgroup generated by $|q|$. Note further that $\bar{k}^*/\langle \overline{a_2},\ldots,\overline{a_s} \rangle = H^1(\bar{X},\mathcal{O}_{\bar{X}}^*)$.

3.3. – *Example.* – Let C be a Mumford curve of genus $g \geqslant 1$ and let $R : C \longrightarrow \bar{C}$ be its stable reduction. (The components of C are rational, the only singularities are ordinary double points.) The Jacobi-variety of C is a holomorphic torus $(k^*)^g/\Lambda$ where Λ is a lattice in $(k^*)^g$. Take ordinary points $p_1, \ldots, p_s \in \bar{C}$ and put $X = R^{-1}(\bar{C} - \{p_1,\ldots,p_s\})$. Then X is affinoid and using (3.1) one calculates an exact sequence :

$$1 \longrightarrow (\bar{k}^*)^g/S \longrightarrow H^1(X,\mathcal{O}_X^*) \longrightarrow |k^*|^g/|\Lambda| \longrightarrow 1$$

where

$$|\Lambda| = \{(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_g|) | (\lambda_1, \ldots, \lambda_g) \in \Lambda\}$$

and S is a finitely generated subgroup of $(\bar{k}^*)^g$. The group $(\bar{k}^*)^g$ is in fact the Jacobi-variety of $\bar{C}$ and the subgroup S is the subgroup of the divisors of degree 0 on $\bar{C}$ with support in $\{p_1, \ldots, p_s\}$. So $(\bar{k}^*)^g/S$ is again $H^1(\bar{X}_S, \mathcal{O}^*)$ where $\bar{X}_S$ denotes the stable reduction of X.

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