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A NEW SETTING FOR POTENTIAL THEORY (Part 1)

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Introduction.

In Hunt's theory of Markov processes certain duality assumptions are made to generalize well known classical potential theoretic results such as F. Riesz representation theorem, uniqueness, existence of equilibrium potential etc. A standard treatment developed by several subsequent authors can be found in [1]. In a different direction, it was shown in [2] under simple analytic conditions that the equilibrium measure is inherently linked to the last exit distribution of the process. It thus appears feasible that this last result, namely on "equilibrium principle" may be made the starting point to which other major results are related.

In this paper we exploit the line of thought in [2] to derive some of these major results under the same set of conditions as in [2]. It turns out that these sets of conditions automatically imply the existence of a dual. However, this will not be proved here.

In § 1 we collect a number of simple consequences of our basic assumptions. In § 2 we construct a version of the potential density with "point supports" namely which is "harmonic off its pole". This is a general result not dependent on the specific conditions of the present paper. This good version of the potential density turns out

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to be indispensable in settling the question of uniqueness of representing measure in § 3 and § 4. § 4 also contains the Riesz representation theorem. These results automatically lead to a dual semigroup which is used to prove Hunt's Hypothesis (B) under mild supplementary conditions in § 5. Further development of these results is relegated to Part II.

1.

The basic assumptions are the same as in [2], viz. :

(i) The underlying process X is a Hunt process on a locally compact Hausdorff space E with countable base, which is transient in the following sense: for each compact K and every x we have

$$\lim_{t\to\infty} P^x\{T_K \circ \theta_t\} = 0. \tag{1}$$

(ii) The potential kernel is $U(x, dy) = u(x, y)\,\xi(dy)$ where ξ is a Radon measure and the potential density function u has the following properties:

(iia) $\forall x : y \longrightarrow \dfrac{1}{u(x, y)}$ is finite continuous

(iib) $u(x, y) = +\infty$ if and only if $x = y$.

To save constant repetition we shall fix the usage of certain symbols and terms below (unless explicitly contravened), as follows:

A is a Borel set, written also as $A \in \mathfrak{B}$;
D is an open set with compact closure, not empty;
G is an open set, not empty;
K is a compact set, not empty;

a function such as f or f_n is positive Borel measurable; the support of a function f or a measure μ is denoted by $\lfloor f$ or $\lfloor \mu$;

$P_t u(x, y) = \int P_t(x, dz)\, u(z, y)$, $P_A u(x, y) = \int P_A(x, dz)\, u(z, y)$,

where (P_t) is the (Borelian) semigroup of the process

(X_t) ; $P_A f(x) = E^x\{f(X_{T_A}) ; T_A < \infty\}$, $T_A = \inf\{t > 0 : X_t \in A\}$;

s is superaveraging iff $s \geqslant P_t s$ for every t; the excessive regularization of s is denoted by $\underline{s} = \lim_{t\to 0} P_t s$; s is excessive at x iff $s(x) = \underline{s}(x)$;

a potential s is an excessive function such that $\lim_{K \uparrow E} P_{K^c} s = 0$, ξ a.e.;

$A_n \downarrow\downarrow A$ means $A_n \supset \overline{A}_{n+1}$ for all n and $\bigcap_n A_n = A$;

$\mu_1 < \mu_2$ means μ_1 is absolutely continuous with respect to μ_2.

A measure is diffuse iff it does not charge any singleton.

We list here some simple consequences of the basic assumptions.

PROPOSITION 1. – *There exists* $h > 0$ *everywhere such that* $Uh \leqslant 1$ *everywhere.*

Proof. – This may be known, but observe that we make no assumption on lower semi-continuity of $x \longrightarrow u(x, y)$. Here is a proof. Let $D_k \uparrow E$, there exists t_k such that

$$P_{D_k} 1(x) - P_{t_k} P_{D_k} 1(x) > 0 \text{ on } A_k \subset D_k \text{ with } \xi(D_k - A_k) < \frac{1}{2^k}.$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{2^k t_k} (P_{D_k} 1 - P_{t_k} P_{D_k} 1)$$

converges everywhere and is strictly positive ξ-a.e. Put h to equal the sum where it is > 0; $= 1$ elsewhere.

It is convenient to put

$$\xi_0(dy) = h(y)\, \xi(dy). \tag{2}$$

For each x, the measure $u(x, y)\, \xi_0(dy)$ is then finite, whereas $u(x, y)\, \xi(dy)$ may not be a Radon measure. Since ξ_0 is equivalent to ξ, this is convenient for applications of Fubini's theorem.

PROPOSITION 2. – ξ *is a diffuse measure.*

Proof. – This follows from $\infty > U1_{\{x\}}(x) \geqslant u(x, x)\, \xi(\{x\})$.

PROPOSITION 3. – $\forall y : u(\cdot, y)$ *is superaveraging.*

Proof. – Since $P_t Uf \leqslant Uf$ for any f it follows that $\forall x$, $\exists N_x$ with $\xi(N_x) = 0$ such that $P_t u(x, y) \leqslant u(x, y)$ for $y \notin N_x$. Let $y_n \notin N_x$, $y_n \longrightarrow y$, then $\forall z : u(z, y_n) \longrightarrow u(z, y)$. Hence by Fatou's lemma,

$$P_t u(x, y) \leqslant \varliminf_n P_t u(x, y_n) \leqslant \varliminf_n u(x, y_n) = u(x, y). \qquad \square$$

For each y, we write $\underline{u}(\cdot, y)$ for the excessive regularization of u; $\underline{U}f = \int u(\cdot, y) f(y) dy$, $\underline{U}\mu = \int \underline{u}(\cdot, y)\, \mu(dy)$.

PROPOSITION 4. – *We have*

$$\forall f: \underline{U}f = Uf. \tag{3}$$

If $s = U\mu$ *where* μ *is any measure, then*

$$\underline{s} = \underline{U}\mu. \tag{4}$$

Proof. – (3) is true because both members are excessive and they are equal ξ-a.e. by Fubini; (4) contains (3) and is proved by Fubini and Fatou.

PROPOSITION 5. – *If* s *is excessive, then* $\exists f_n$ *such that* $f_n \leqslant n^2$, $Uf_n \leqslant n$, *and* $Uf_n \uparrow s$.

Proof. – This is well known but we indicate the proof. Let $K_n \uparrow\uparrow E$, $s_n = s \wedge (n P_{K_n} 1)$ and $f_n = n[s_n - P_{1/n} s_n]$.

The next two propositions are proved in [2], reviewed here for orientation and some quick applications.

PROPOSITION 6. – *For each* K, $\exists$ *a Radon measure* μ *such that*

$$P_K 1 = U\mu. \tag{5}$$

It is important to observe that the proof in [2] does not establish that $\lfloor\mu \subset K$. In fact it shows that

$$\int_A u(x, y)\, \mu(dy) = P^x\{\gamma_K > 0;\ X(\gamma_{K-}) \in A\} \tag{6}$$

where $\gamma_K = \sup\{t > 0: X_t \in K\}$. If there is a jump at γ_K, it is possible that $X(\gamma_{K-}) \notin K$. However, (6) does establish the next proposition.

PROPOSITION 7. – *For each* D, $\exists\mu$ *with* $\lfloor\mu \subset \overline{D}$ *such that*

$$P_D 1 = U\mu. \tag{7}$$

One of our principal results below is to prove that there exists μ with $\lfloor\mu \subset K$ and $\mu(Z) = 0$ (see (12) of § 3), for which (5) is true. This turns out to be equivalent to Hunt's Hypothesis (B) and

will be proved under an additional assumption. For the moment we note some consequences of the two preceding propositions.

PROPOSITION 8. – *If* s *is excessive and* $s \not\equiv \infty$, *then the set* $\{s = \infty\}$ *is polar, in particular* $s < \infty$ ξ-a.e.

Proof. – Let $K \subset \{s = \infty\}$, and $s(x_0) < \infty$. Then it is well known that $P_K 1(x_0) = 0$, namely by (5): $\int u(x_0, y)\, \mu(dy) = 0$. Since $u(x, y) > 0$ for all (x, y) by basic assumption (iia) this implies $\mu \equiv 0$ and so the preceding equation holds for any x instead of x_0. Thus K is polar, and so is $\{s = \infty\}$.

COROLLARY. – *If* $U\mu \not\equiv \infty$, *then* μ *is a Radon measure; moreover,* $U\mu < \infty$ ξ-a.e.

Proof. – Let $s = U\mu$. If $s(x_0) < \infty$, $\forall K$:

$$\infty > \int_K u(x_0, y)\, \mu(dy).$$

Since $\inf_{y \in K} u(x_0, y) > 0$ by (iia), it follows that $\mu(K) < \infty$. Hence μ is Radon. Now write $s = U\mu$, then $\underline{s} \not\equiv \infty$, hence by the proposition $\underline{s} < \infty$ ξ-a.e. Since $s = \underline{s}$ ξ-a.e., we have $s < \infty$ ξ-a.e.

PROPOSITION 9. – *If* s *is excessive and* $s \not\equiv 0$, *then* $s > 0$ *everywhere.*

Proof. – By Proposition 5, $\exists f_n$ such that $Uf_n \uparrow s$. Hence if $s(x_0) > 0$, then $\int u(x_0, y)\, f_n(y)\, \xi(dy) > 0$ for large n. Since $u > 0$, the same is then true if x_0 is replaced by any x. Hence $s(x) > 0$.

PROPOSITION 10. – *Each singleton is a polar set.*

Proof. – Let $D_n \downarrow\downarrow \{x_0\}$; by Proposition 7, $P_{D_n} 1 = U\mu_n$ where $\underline{\mu_n} \subset \overline{D}_n$. For each K and any fixed x:

$$1 \geqslant \int_K u(x, y)\, \mu_n(dy) \geqslant \inf_{y \in K} u(x, y)\, \mu_n(K).$$

Hence $\{\mu_n\}$ is vaguely bounded and a subsequence converges vaguely to μ, which must be supported by $\{x_0\}$, namely $\mu = \lambda \delta_{x_0}$ for

some $\lambda \geqslant 0$. Since for each x, $u(x, \cdot)$ is extended continuous, we have $\lambda u(x, x_0) = U\mu(x) \leqslant \varliminf_n U\mu_n(x) \leqslant 1$. For $x = x_0$ this yields $\lambda = 0$ by basic assumption (iib); thus $\mu \equiv 0$. Now let $x_1 \notin \overline{D}_1$, then $y \longrightarrow u(x_1, y)$ is finite continuous in $\overline{D}_1$. Hence we obtain

$$\lim_n P_{D_n} 1(x_1) = \lim_n \int_{\overline{D}_1} u(x_1, y)\, \mu_n(dy) = \int_{\overline{D}_1} u(x_1, y)\, \mu(dy) = 0$$

by vague convergence, and consequently $P_{\{x_0\}} 1(x_1) = 0$. By Proposition 9, $P_{\{x_0\}} 1 \equiv 0$, namely $\{x_0\}$ is polar.

The following lemma is known (see [1; p. 84]) but we supply a proof for the sake of completeness.

LEMMA. – *Let* $K_n \uparrow E$ *and* $T_n = T_{K_n^c}$, or $T_n = n$. *Then for any excessive function* f *we have*

$$P_{T_n} f \downarrow g \tag{8}$$

where $g = \underline{g}$ *on* $\{g < \infty\}$.

Proof. – The case $T_n = n$ is easy; so we treat only the case $T_n = T_{K_n^c}$. It is clear from (8) that g is superaveraging. Fix an x such that $g(x) < \infty$, then for $n \geqslant n_0(x)$ we have

$$\infty > P_{T_n}(f(x) \geqslant P_t P_{T_n} f(x) \geqslant E^x\{f(X(t + T_n \circ \theta_t)\,;\, t < T_n\}$$
$$= E^x\{f(X(T_n))\,;\, t < T_n\}.$$

It follows by subtraction that

$$P_{T_n} f(x) - P_t P_{T_n} f(x) \leqslant E^x\{f(X(T_n))\,;\, T_n \leqslant t\}. \tag{9}$$

For fixed t, the second member of (9) is decreasing in n by the supermartingale inequality. Hence for $n \geqslant k$, (9) is true when on the right side n is replaced by k. Now let $n \longrightarrow \infty$ on the left side and use domination to obtain for every k

$$g(x) - P_t g(x) \leqslant E^x\{f(X(T_k))\,;\, T_k \leqslant t\}. \tag{10}$$

For the fixed x there exists k such that $P^x\{T_k > 0\} = 1$. For this k in (10), as $t \downarrow 0$ the right member of (10) decreases to zero by domination; $P_{T_k} f(x) < \infty$. Hence $g(x) = \underline{g}(x)$. □

PROPOSITION 11. – *If s is potential, then*

$$\lim_{K \uparrow E} P_{K^c} s(x) = 0 \tag{11}$$

if $s(x) < \infty$. Hence the set of x for which (11) does not hold is polar.

Proof. – Apply the Lemma above to $\{P_{K_n^c} s\}$. Then we have by definition $g = \lim_n P_{K_n^c} s = 0$, ξ-a.e. Hence $\underline{g} \equiv 0$ and so by the Lemma, $g = 0$ on $\{g < \infty\} \supset \{s < \infty\}$. Standard arguments show that if $g(x) = 0$, then (11) is true.

This proposition will be crucial in the proof of Theorem 2, (δ) below.

2.

In this section we give a general construction of a good version of a given potential density function u. Of the latter we assume that

(a) $\forall y : u(\cdot, y)$ is excessive and finite ξ-a.e.

For the underlying process it is sufficient that it be a transient standard process satisfying the condition:

(b) every singleton is a polar set.

A function v measurable $\mathfrak{B} \times \mathfrak{B}$ is a version of u iff for every x and every f we have $\int v(x, y) f(y)\, \xi(dy) = \int u(x, y) f(y)\, \xi(dy)$.

THEOREM 1. – *Under the conditions stated above, there is a version w of u which has the following properties: for each y, $w(\cdot, y)$ is excessive; and for each (open) G we have*

$$P_G w(x, y) = w(x, y) \quad \textit{for all} \quad y \in G. \tag{1}$$

We shall refer to (1) as the "round" property of w.

Proof. – Let $\{B_n\}$ be a countable base of open sets of E such that each member of the sequence is repeated infinitely many times. Put $u_0(x, y) = u(x, y)$ for all (x, y); and define inductively for $n \geqslant 0$:

$$u_{n+1}(x, y) = \begin{cases} P_{B_{n+1}} u_n(x, y) & \text{if} \quad y \in B_{n+1}, \\ u_n(x, y) & \text{if} \quad y \notin B_{n+1}. \end{cases}$$

For each y, $u_n(\cdot, y)$ is excessive by assumption (a), and

$$u_n(x, y) \geqslant u_{n+1}(x, y)$$

for all (x, y). Let $u_\infty(x, y) = \lim_n \downarrow u_n(x, y)$. Then $u_\infty(\cdot, y)$ is superaveraging.

Recall the measure ξ_0 in (2) of § 1. We have $P_B U(1_B h) = U(1_B h)$ for open B, namely

$$\int_B P_B u(x, y)\, \xi_0(dy) = \int_B u(x, y)\, \xi_0(dy) < \infty. \tag{2}$$

(ξ_0 is used in lieu of ξ to ensure finiteness above.) Hence by Fubini there exists N with $\xi_0(N) = 0$ such that

$$P_B u(x, y) = u(x, y), \quad \text{for all} \quad y \notin N \quad \text{and} \quad y \in B \tag{3}$$

first for ξ_0-a.e. x, then for all x because both members of (3) are excessive, and ξ_0 is a reference measure as well as ξ. Applying (3) with $B = B_1$, we obtain $u_1(x, y)\, \xi_0(dy) = u(x, y)\, \xi_0(dy)$ and hence by induction $u_n(x, y)\, \xi_0(dy) = u(x, y)\, \xi_0(dy)$. Since both members above are Radon measures, it follows by monotone convergence that

$$u_\infty(x, y)\, \xi_0(dy) = u(x, y)\, \xi_0(dy). \tag{4}$$

Thus u_∞ is a version of u, but it may not be excessive.

For each y let us put $F_y = \{x \mid u_\infty(x, y) < \infty\} \backslash \{y\}$. Then $\xi(F_y^c) = 0$ by assumptions (a) and (b). Let B be one of the sequence $\{B_n\}$ and $y \in B$. Then by construction there exists a sequence $n_k \longrightarrow \infty$ such that $B_{n_k} = B$ and so $P_B u_{n_k - 1}(x, y) = u_{n_k}(x, y)$. It follows by monotone convergence with initial finiteness that

$$P_B u_\infty(x, y) = u_\infty(x, y) \quad \text{for all} \quad x \in F_y \quad \text{and} \quad y \in B. \tag{5}$$

Next, if $x \in F_y$, then $u_n(x, y) < \infty$ for all large n. Since $u_n(\cdot, y)$ is excessive, $u_n(X_t, y)$ is a right continuous supermartingale under P^x, for large n. Hence by Doob's stopping theorem

$$E^x\{u_n(X_t, y)\,;\ t \leqslant T_B\} \geqslant E^x\{u_n(X_{T_B}, y)\,;\ t \leqslant T_B\}. \tag{6}$$

Since for large n $E^x\{u_n(X_t, y)\} \leqslant u_n(x, y) < \infty$, we can let $n \longrightarrow \infty$ in (6) to obtain

$$E^x\{u_\infty(X_t, y)\ ;\ t \leqslant T_B\} \geqslant E^x\{u_\infty(X_{T_B}, y)\ ;\ t \leqslant T_B\} \tag{7}$$
$$= P_B u_\infty(x, y) - E^x\{u_\infty(X_{T_B}, y)\ ;\ t > T_B\}.$$

The last expectation in (7) is bounded by $E^x\{u_n(X_{T_B}, y)\ ;\ t > T_B\}$. Since $x \neq y$, we may choose B so that $x \notin \overline{B}$ so that $P^x\{T_B > 0\} = 1$. The expectation above then converges to zero as $t \longrightarrow 0$ because

$$E^x\{u_n(X_{T_B}, y)\ ;\ T_B < \infty\} = P_B u_n(x, y) \leqslant u_n(x, y) < \infty$$

for large n. Going back to (7), we see that if $x \in F_y$, then

$$\lim_{t \downarrow 0} P_t u_\infty(x, y) \geqslant \varliminf_{t \downarrow 0} E^x\{u_\infty(X_t, y)\ ;\ t \leqslant T_B\}$$
$$\geqslant P_B u_\infty(x, y) = u_\infty(x, y).$$

Thus $u_\infty(\cdot, y)$ is excessive at such an x, namely

$$\underline{u}_\infty(x, y) = u_\infty(x, y), \quad \text{for all} \quad x \in F_y. \tag{8}$$

In addition to equation (5), we have if B is any member of $\{B_n\}$:

$$P_B u_\infty(x, y) \leqslant u_\infty(x, y) \quad \text{for all} \quad (x, y), \tag{9}$$

because this is true when u_∞ is replaced by u_n, which implies (9) itself by Fatou's lemma. Now if $x \in F_y$, then the quantities in (9) are finite, and consequently in conjunction with $P_B(x, \{y\}) = 0$ due to the polarity of $\{y\}$, we have $P_B(x, F_y^c) = 0$. This relation and (8) imply that $P_B u_\infty(x, y) = P_B \underline{u}_\infty(x, y)$. Since $\underline{u}_\infty(\cdot, y)$ is excessive, we now have

$$u_\infty(x, y) \geqslant \underline{u}_\infty(x, y) \geqslant P_B \underline{u}_\infty(x, y) = P_B u_\infty(x, y) = u_\infty(x, y)$$

for all $y \in B$ and all $x \in F_y$. Since $\xi(F_y^c) = 0$, we conclude that

$$P_B \underline{u}_\infty(x, y) = \underline{u}_\infty(x, y) \quad \text{for all} \quad y \in B \tag{10}$$

first for ξ-a.e. x, then for all x. The validity of (10) for all members of a base of the topology implies its validity for every open set B.

Finally, the proof of Proposition 4 of § 1 shows that $\underline{u}_\infty$ as well as u_∞ is a version of u. This is the w claimed in Theorem 1. □

To apply Theorem 1 to the case under consideration in this paper, we must start with $\underline{u}$ instead of u because of condition (a). Note that by (3) of § 1, $\underline{u}$ is a potential density of the given process. Proposition 10 of § 1 supplies the condition (b) required for Theorem 1.

We shall refer to the w just constructed from $\underline{u}$ as *the round version* of u. It will play a key role in what follows.

3.

In this section we give the principal convergence theorem for potentials of measures. It is an extension of the main result in [9] but will soon be strengthened to include a new feature relative to the round version w constructed in § 2.

THEOREM 2. – *Let* $\{\mu_n\}$ *be a sequence of diffuse measures, satisfying conditions (a), (b) and either* (c_1) *or* (c_2) *below:*

(a) $\forall n : U\mu_n \leqslant \sigma$ *where* σ *is excessive and* $\not\equiv \infty$;

(b) $\lim_n U\mu_n = s$ *everywhere;*

(c_1) $\forall n : \lfloor \mu_n$ *is contained in a fixed compact;*

(c_2) $\forall n : \mu_n \prec \xi$ *and* σ *is a potential.*

Then there exists a subsequence $\{n_j\}$ *and a Radon measure* μ *such that*

(α) μ_{n_j} *converges vaguely to* μ;

(β) *at each* x *where* $\sigma(x) < \infty$ *and* $s(x) = \underline{s}(x) < \infty$, *the measures* $u(x, y)\,\mu_{n_j}(dy)$ *converge weakly to* $u(x, y)\,\mu(dy)$, *in particular* $s(x) = U\mu(x)$;

(γ) *if the* s *in condition (b) is an excessive function* $\not\equiv \infty$, *then* $s = U\mu$ *everywhere.*

Proof. – The proof is essentially the same as that in [9], but the basic steps will be sketched.

Let $\sigma(x_0) < \infty$, then by (a), for each K,

$$\sigma(x_0) \geqslant [\inf_{y \in K} u(x_0, y)]\,\mu_n(K)$$

where the infimum is strictly positive. This implies (α). We shall write μ_n for μ_{n_j} below for simplicity. Put

$$L_n(x, dy) = u(x, y)\,\mu_n(dy)\,; \quad L_n(x, f) = \int L_n(x, dy)\,f(y)\,.$$

For each x for which $\sigma(x) < \infty$, we have by (a) $\infty > \sigma(x) \geqslant L_n(x, 1)$. Hence there exists $\{n_j\}$ and $L(x, \cdot)$ such that

$$L_{n_j}(x, \cdot) \longrightarrow L(x, \cdot) \quad \text{vaguely.} \tag{1}$$

Let φ be continuous with compact support, then since $\forall n$: $\mu_n(\{x\}) = 0$ by hypothesis and $u(x, y) < \infty$ for $y \neq x$, we have

$$\begin{aligned} \int \varphi(y)\, \mu(dy) &= \lim_j \int \varphi(y)\, \mu_{n_j}(dy) \qquad (2) \\ &= \lim_j \int \frac{\varphi(y)}{u(x, y)}\, L_{n_j}(x, dy) = \int \frac{\varphi(y)}{u(x, y)}\, L(x, dy). \end{aligned}$$

Thus we have for each x such that $\sigma(x) < \infty$:

$$\mu(dy) = \frac{L(x, dy)}{u(x, y)}. \tag{3}$$

This is true of any vaguely convergent subsequence of $L_n(x, \cdot)$, hence by (3) any two vague limits coincide off $\{x\}$. Now under condition (c_1) the vague convergence is also weak convergence, namely with $L_n(x, 1) \longrightarrow L(x, 1)$. On the other hand, under condition (c_2) we may write $\mu_n(dy) = f_n(y)\, \xi(dy)$, and so for each K

$$\int_{K^c} u(x, y)\, \mu_n(dy) \leqslant U(f_n 1_{K^c})(x) = P_{K^c} U(f_n 1_{K^c})(x) \leqslant P_{K^c} \sigma(x). \tag{4}$$

By Proposition 11 of § 1, if $\sigma(x) < \infty$, then the last term in (4) decreases to zero as $K \uparrow E$. Consequently we have

$$L_n(x, \cdot) \longrightarrow L(x, \cdot) \quad \text{weakly} \tag{5}$$

and $L(x, 1) = s(x)$ by condition (b).

Let $F = \{y : \sigma(y) < \infty\}$. Then F^c is a polar set and we have proved under (c_1) or (c_2) that (5) is true for all $x \in F$. It follows from (3) that $\mu(\{x\}) = 0$ and

$$L(y, \{x\}) = 0 \quad \text{if} \quad x \in F,\ y \in F,\ x \neq y. \tag{6}$$

The limit s in (b) is superaveraging by Fatou. We are going to show that if $s(x) = \underline{s}(x) < \infty$, then

$$L(x, \{x\}) = 0. \tag{7}$$

Let g be continuous and $0 \leqslant g \leqslant 1$. Using (5) we see that $L(\cdot, 1 - g)$ is superaveraging, hence

$$L(x, g) - P_t L(x, g) \leqslant L(x, 1) - P_t L(x, 1).$$

The right member converges to zero as $t \longrightarrow 0$ by hypothesis. Taking a sequence $g_n \downarrow 1_{\{x\}}$, we obtain

$$L(x, \{x\}) \leqslant \epsilon + P_t L(x, \{x\}) \tag{8}$$

for any $\epsilon > 0$ and sufficiently small $t > 0$. Now $F^c \cup \{x\}$ is a polar set by Proposition 10 of § 1, hence $P_t(x, \cdot)$ does not charge it and so the last term in (8) is equal to zero by (6). Thus (7) follows from (8). We can now conclude from (3) that

$$L(x, dy) = u(x, y)\, \mu(dy). \tag{9}$$

This and (5) establish the conclusion (β). Integrating, we obtain

$$s(x) = U\mu(x) \tag{10}$$

except possibly for a polar set. Under the hypothesis in (γ), this implies $s = \underline{U}\mu$ everywhere. But the lower semicontinuity of $u(x, \cdot)$ for each x and the vague convergence of μ_n to μ also implies that $s \geqslant U\mu$ everywhere. Thus $s = U\mu$ as asserted. □

COROLLARY 1. – *If* s *is excessive and* $\not\equiv \infty$, *then* $P_D s = U\mu$, *where* $\lfloor \mu \subset \overline{D}$.

Proof. – By a theorem of Hunt's [3], there exists f_n with $\lfloor f_n \subset D$ such that $Uf_n \uparrow P_D s$. Hence (a), (b) and (c_1) are satisfied with s and σ both equal to $P_D s$ here, since $P_D s < \infty$ ξ-a.e. by Proposition 8 of § 1. It is trivial that $\lfloor \mu \subset \overline{D}$ by vague convergence.

COROLLARY 2. – *If* s *is a potential, then there is a Radon measure* μ *such that* $s = U\mu$.

Proof. – By Proposition 5 of § 1, (a), (b) and (c_2) are satisfied if $\sigma = s$. Note that a potential is necessarily $\not\equiv \infty$, hence $< \infty$ ξ-a.e.

COROLLARY 3. – *For each* K, *we have* $P_K 1 = \underline{U}\mu$, *where* $\lfloor \mu \subset K$; *also,* $P_K 1 = U\mu$ *on* K^c.

Proof. – Let $D_n \downarrow\downarrow K$; by Corollary 2, we have $P_{D_n} 1 = U\mu_n$, where $\lfloor \mu_n \subset \overline{D}_n$. Put $s = \lim_n \downarrow P_{D_n} 1$ and apply Theorem 2 under (c_1) to obtain a subsequence $\{\mu_{n_j}\}$ converging vaguely to μ, such that $s = U\mu$ on $\{s = \underline{s}\}$. Then $\lfloor \mu \subset \bigcap_n \overline{D}_n = K$ and $P_K 1 = \underline{s} = \underline{U}\mu$.

If $x \notin K$, then $u(x, \cdot)$ is continuous in $\overline{D}_n$ for large n; on the other hand, $s(x) = \underline{s}(x)$. This establishes the second assertion of the lemma.

We can now prove a key property of the round version w of u.

THEOREM 3. – *For any* y *: either* $w(\cdot, y) \equiv u(\cdot, y)$ *or* $w(\cdot, y) \equiv 0$.

Proof. – Fix y and let $D_n \downarrow\downarrow \{y\}$. Since $w(\cdot, y)$ is excessive and $< \infty$ ξ-a.e., we have by Corollary 1 to Theorem 2

$$P_{D_n} w(\cdot, y) = U\mu_n, \quad \text{where } \lfloor \mu_n \subset \overline{D}_n. \tag{11}$$

But the left member of (11) is just $w(\cdot, y)$. Hence Theorem 2 is applicable to the sequence $\{U\mu_n\}$ under condition (c_1), and we conclude that there is a subsequence μ_{n_j} converging vaguely to some μ such that $w(\cdot, y) = U\mu$. But μ must have support in $\bigcap_n \overline{D}_n = \{y\}$, thus $\exists \lambda : 0 \leqslant \lambda < \infty$, such that $w(\cdot, y) = u(\cdot, y)\lambda$. If $\lambda = 0$, then $w(\cdot, y) \equiv 0$. If $\lambda > 0$, then $u(\cdot, y) = \dfrac{w(\cdot, y)}{\lambda}$, and consequently $u(\cdot, y)$ is excessive and furthermore for any $G \ni y$, $P_G u(\cdot, y) = u(\cdot, y)$. The construction of Theorem 1 then yields successively $u_n(\cdot, y) \equiv u(\cdot, y)$, $u_\infty(\cdot, y) \equiv u(\cdot, y)$ and finally $w(\cdot, y) \equiv u(\cdot, y)$. □

We now introduce the all important exceptional set below:

$$Z = \{y : w(\cdot, y) \equiv 0\} = \{y : w(\cdot, y) \not\equiv u(\cdot, y)\}. \tag{12}$$

Clearly $Z \in \mathfrak{B}$. Note that $U\mu = W\mu$ if and only if $\mu(Z) = 0$.

PROPOSITION 12. – *We have* $\xi(Z) = 0$.

Proof. – Since w is a potential density, we have

$$0 = \int_Z w(x, y)\, \xi_0(dy) = \int_Z u(x, y)\, \xi_0(dy)$$

where ξ_0 is defined in (2) of § 1. But $u > 0$ everywhere, hence $\xi_0(Z) = 0$ which is the same as $\xi(Z) = 0$. □

Theorem 2 under condition (c_2) was stated in a restricted way because we needed its Corollary 1 to prove Theorem 3. We can now state Theorem 2 in a more complete form as follows.

THEOREM 2 (continued). – *If we impose the additional condition that* $\mu_n(Z) = 0$ *for all* n, *then in condition* (c_2) *we may remove the assumption that* $\mu_n \prec \xi$, *and the conclusions* (α), (β), (γ) *still hold; moreover, we have*

(δ) *for the* μ *in* (γ) *we have* $\mu(Z) = 0$.

Proof. – The proof of (α) requires no change. In the proof of (β), under condition (c_2), the inequality (4) is replaced as follows. Let $K_n \uparrow E$; since $\mu_n(Z) = 0$, we have

$$\int_{K_n^c} u(x, y)\, \mu_n(dy) = \int_{K_n^c} w(x, y)\, \mu_n(dy). \tag{13}$$

By the round property of w, $P_{K_n^c} w(\cdot, y) = w(\cdot, y)$ for all $y \in K_n^c$. Hence the second member of (13) does not exceed

$$P_{K_n^c}\left[\int_{K_n^c} w(\cdot, y)\, \mu_n(dy)\right] \leqslant P_{K_n^c}\sigma$$

which decreases to zero as $n \longrightarrow \infty$, on $\{\sigma < \infty\}$ by Proposition 11 of § 1. The rest of the proof of (β) and (γ) are the same as before. To prove the new conclusion (δ) put $F = \{x\colon \sigma(x) < \infty \text{ and } s(x) < \infty\}$, and let $\mu(\partial D) = 0$. By (9), we have $L(x, \partial D) = 0$; by (5), we have $L_n(x, D) \longrightarrow L(x, D)$; both provided $x \in F$. Now for any measure ν define its part in D by

$$\nu^D(A) = \nu(D \cap A). \tag{14}$$

Then we have just shown that on F

$$U\mu_n^D \longrightarrow U\mu^D. \tag{15}$$

By (γ)

$$s = U\mu^D + U\mu^{E-D}; \tag{16}$$

hence on $\{s < \infty\}$, both terms on the right are excessive, and consequently

$$U\mu^D = \underline{U}\mu^D. \tag{17}$$

Since $\mu_n(Z) = 0$, we have by the definition of Z

$$U\mu_n^D = W\mu_n^D \qquad [= \int w(x, y)\, \mu_n^D(dy)]. \tag{18}$$

Hence by the round property of w and the domination in condition (a), we have on F

$$P_D\, U\mu_n^D = U\mu_n^D. \tag{19}$$

Now F^c is polar by Proposition 8 of § 1, hence $P_D(x, F^c) = 0$ for every x. Therefore (15), valid on F, implies that

$$P_D U\mu_n^D \longrightarrow P_D U\mu^D \tag{20}$$

on F, by the domination in condition (a). It follows from (15), (19) and (20) that

$$P_D U\mu^D = U\mu^D \tag{21}$$

on F, hence by (17), using again $P_D(x, F^c) = 0$,

$$P_D \underline{U}\mu^D = \underline{U}\mu^D \tag{22}$$

everywhere by excessiveness. Thus for μ_D-a.e. y,

$$P_D \underline{u}(x, y) = \underline{u}(x, y) \tag{23}$$

first for ξ-a.e. x, then for all x. Let $\{D_n\}$ form a base of the topology such that $\mu(\partial D_n) = 0$. It follows from (23) that there exists N with $\mu(N) = 0$ such that if $y \notin N$, then for all n such that $y \in D_n$: $P_{D_n}\underline{u}(\cdot, y) = \underline{u}(\cdot, y)$; and therefore also for every G containing y, $P_G\underline{u}(\cdot, y) = \underline{u}(\cdot, y)$. For such a y the construction in Theorem 1 yields

$$\underline{u}(\cdot, y) = w(\cdot, y). \tag{24}$$

Since $\mu(N) = 0$, we have, using (γ):

$$s = U\mu = \underline{U}\mu = W\mu. \tag{25}$$

Since $u \geqslant \underline{u} \geqslant w$, (25) implies that $\mu(Z) = 0$. □

COROLLARY 4. – *The measure μ in Corollary 1 or 2 to Theorem 2 does not charge Z.*

By contrast, the method does not prove that the measure μ in Corollary 3 does not charge Z. This should be compared with (5) of § 1, where the measure μ is seen not to charge Z by the proof in [2] and the argument used in Theorem 2 to deduce (δ). Thus we have the anomaly of two representations of $P_K 1$ as potentials of measures both of which lack an essential feature. The next proposition clarifies the issue.

Recall that Hunt's Hypothesis (B) (see [3]) may be stated as follows: for every (open) G which contains (compact) K, we have

$$P_G P_K 1 = P_K 1. \tag{26}$$

There are several equivalent properties in terms of the sample function behavior of the process; see, e.g., [5]. It is known that the hypothesis is true under strong duality assumptions, see [5].

THEOREM 4. – *The following three propositions are equivalent.*

(a) Z *is polar;*

(b) *Hypothesis (B) is true;*

(c) $\forall K: P_K 1 = U\mu$ *where* $\lfloor\mu \subset K$ *and* $\mu(Z) = 0$.

Proof. – (a) $\Longrightarrow$ (b): Suppose Z is polar and K be given. Let L be compact, $L \subset K \cap Z^c$. By Corollary 3 to Theorem 2, we have $P_L 1 = \underline{U}\mu$ where $\lfloor\mu \subset L$. For each $y \in L$, we have $w(\cdot, y) = u(\cdot, y)$, hence also $= \underline{u}(\cdot, y)$. Thus we have $P_L 1 = W\mu$; and for any $G \supset K$

$$P_G P_L 1 = P_G W\mu = W\mu = P_L 1. \tag{27}$$

By Hunt's approximation theorem, $\exists L_n \subset K \cap Z^c$, $L_n \uparrow$, such that for each x, both P^x-a.s. and P^λ-a.s., where $\lambda(\cdot) = P_G(x, \cdot)$, we have $P_{L_n} 1 \uparrow P_{K \cap Z^c} 1$. The limit above is equal to $P_K 1$ because Z is polar. Taking such a sequence $\{L_n\}$ in (27), we obtain (26) by monotone convergence.

(b) $\Longrightarrow$ (c): Let $D_n \downarrow\downarrow K$; then by (b)

$$P_K 1 = P_{D_n} P_K 1 = U\mu_n, \tag{28}$$

where $\mu_n(Z) = 0$ and $\lfloor\mu_n \subset \overline{D}_n$ by Corollaries 1 and 4 to Theorem 2. Hence by Theorem 2 (δ) there exists $\{\mu_{n_j}\}$ converging vaguely to μ so that (c) is true.

(c) $\Longrightarrow$ (a): Let $K \subset Z$; then (c) clearly implies that $P_K 1 \equiv 0$. Thus K is polar and so is Z.

We shall prove later that under the additional assumption that $\forall x: \underline{u}(x, x) = +\infty$, Z is indeed a polar set. Let us remark here that it is easy to show that Z is left-polar, hence semipolar. For this purpose we define

$$S_A = \inf\{t > 0: X_{t-} \in A\}, \quad P_A^* 1(x) = P^x\{S_A < \infty\}. \tag{29}$$

Then the method of proof in [2] yields

$$P_K^* 1 = U\nu, \quad \lfloor\nu \subset K, \quad \nu(Z) = 0. \tag{30}$$

Hence the argument leading from (c) to (a) above shows that $P_Z^* 1 \equiv 0$.

4.

The main result of this section is that $U\mu$ uniquely determines μ provided that $\mu(Z) = 0$. This will be proved in a series of lemmas beginning with one due to Mokobodzki, which is essential. This is his result on excessivization valid for any discrete potential kernel; see [6] and [8]. The application to our case is made through standard techniques via resolvents, see [4]. Recall our convention in § 1 that all functions are positive measurable.

A "strong order" is defined as follows: $f \ll g$ iff $\exists$ an excessive function φ such that $f + \varphi = g$. For any measurable f (not necessarily positive), there exists a "least excessive majorant" f^* such that f^* is excessive, $f^* \geqslant f$, and for any excessive $\varphi \geqslant f$ we have $\varphi \geqslant f^*$. Mokobodzki's main result may be stated as follows:

$$\text{If} \quad f \ll g, \quad \text{then} \quad f^* \ll g^*. \tag{1}$$

We need also the following result, due to Mokobodzki (see another proof by Getoor in [7]).

(2) If φ is excessive and $\varphi \ll Ug$, then $\exists f$ such that $\varphi = Uf$.

Finally we need the following elementary uniqueness result.

$$\text{If} \quad Uf = Ug, \quad \text{then} \quad f = g \quad \xi\text{-a.e.} \tag{3}$$

This follows from a uniqueness theorem for additive functionals (see [1], p. 157), according to which the hypothesis in (3) implies that we have $u(x, y) f(y) \xi(dy) = u(x, y) g(y) \xi(dy)$ as measures for each x for which $Uf(x) < \infty$. Multiply both sides by $\dfrac{1}{u(x, y)}$ we obtain the conclusion in (3). For a general argument see [1].

Our application of these results is contained in the next lemma, which does not depend on the specific setting of this paper.

LEMMA 1. — *Let s be an excessive function which is finite ξ-a.e.; and let $Uf_n < \infty$ and*

$$Uf_n \uparrow s \tag{4}$$

everywhere. Suppose that we have

$$s = s_1 + s_2 \tag{5}$$

where s_1 *and* s_2 *are excessive. Then* $\exists g_n$ *and* h_n *such that*

$$f_n = g_n + h_n\,, \quad \xi\text{-a.e.}, \tag{6}$$

and

$$Ug_n \uparrow s_1\,, \quad Uh_n \longrightarrow s_2\,. \tag{7}$$

Proof. – Define

$$\varphi_n = (Uf_n - s_2)^+\,. \tag{8}$$

Then $\varphi_n \uparrow$ and

$$Uf_n = \varphi_n + (Uf_n \wedge s_2)\,. \tag{9}$$

Thus $\varphi_n \ll Uf_n$ and so by (1), $\varphi_n^* \ll Uf_n$. By (2), $\exists g_n$ and h_n such that $\varphi_n^* = Ug_n$ and

$$Uf_n = Ug_n + Uh_n\,. \tag{10}$$

This implies (6) by (3). Since $\varphi_n \leqslant s_1$ by (5) and (8), we have $\varphi_n^* \leqslant s_1$. Comparing (9) and (10), we see that $Uh_n \leqslant s_2$ because $\varphi_n \leqslant Uf_n$. Since $\varphi_n \uparrow$ so does $\varphi_n^* = Ug_n$. Let

$$\lim_n \uparrow Ug_n = \varphi\,; \quad \underline{\lim_n}\, Uh_n = \psi\,. \tag{11}$$

Then φ is excessive and ψ is superaveraging. Letting $n \longrightarrow \infty$ in (10) and using (4) and (11), we see that

$$s = \varphi + \psi\,. \tag{12}$$

But $\varphi \leqslant s_1$ and $\psi \leqslant s_2$, hence in view of (5) we must have

$$\varphi = s_1\,, \quad \psi = s_2 \quad \text{on} \quad \{s < \infty\}\,. \tag{13}$$

Since $s < \infty$ ξ-a.e., this implies that $\underline{\psi} = s_2$, hence also $\psi = s_2$. Finally, $\overline{\lim_n}\, Uh_n \leqslant s_2$, hence $\lim_n Uh_n$ exists and $= s_2$. □

LEMMA 2. – *Let* s *be excessive,* $s \not\equiv \infty$, *then*

$$P_D s = U\mu \tag{14}$$

where $\mu(Z) = 0$, $\lfloor\mu \subset \overline{D}$; *and this* μ *has the following splitting property. If* s_1 *and* s_2 *are excessive and*

$$P_D s = s_1 + s_2\,, \tag{15}$$

then $\exists \mu_1$ *and* μ_2 *with* $\mu_1(Z) = \mu_2(Z) = 0$, *and*

$$\mu = \mu_1 + \mu_2\,, \quad s_1 = U\mu_1\,, \quad s_2 = U\mu_2\,. \tag{16}$$

Proof. – Except for the splitting property this has been stated in Corollaries 1 and 4 to Theorem 2. By Corollary 1 to Theorem 2, we have $Uf_n \uparrow P_D s$ such that

$$f_n(y)\,\xi(dy) \longrightarrow \mu(dy), \qquad P_D s = U\mu. \tag{17}$$

Hence by Lemma 1, $\exists\{g_n\}$ and $\{h_n\}$ satisfying (6) and (7). We can now apply Theorem 2 to $\{Ug_n\}$ and $\{Uh_n\}$ to obtain $\{n_j\}$ such that

$$\begin{aligned} g_{n_j}(y)\,\xi(dy) &\longrightarrow \mu_1(dy), \quad s_1 = U\mu_1, \quad \mu_1(Z) = 0; \\ h_{n_j}(y)\,\xi(dy) &\longrightarrow \mu_2(dy), \quad s_2 = U\mu_2, \quad \mu_2(Z) = 0. \end{aligned} \tag{18}$$

It is clear from (6), (17) and (18) that $\mu = \mu_1 + \mu_2$.

LEMMA 3. – *For any* y_0 *and* $x \notin \overline{D}$, *we have*

$$P_D \underline{u}(x, y_0) < \infty. \tag{19}$$

Proof. – By Corollary 1 to Theorem 2,

$$P_D \underline{u}(x, y_0) = \int u(x, y)\,\mu(dy), \quad \lfloor\mu \subset \overline{D}.$$

If $x \notin \overline{D}$, then $\sup_{y \in \overline{D}} u(x, y) < \infty$; from which (19) follows since μ is a Radon measure.

LEMMA 4. – *Suppose* $U\mu = U\nu \not\equiv \infty$, *where* $\mu(Z) = \nu(Z) = 0$. *If* $\lfloor\mu \subset K$, *then* $\lfloor\nu \subset K$.

Proof. – By Corollary to Proposition 8 in § 1, $U\mu < \infty$ ξ-a.e. and both μ and ν are Radon measures. Let $F = \{x \mid U\mu(x) < \infty\}$, then $\xi(F^c) = 0$. We have $U\mu = W\mu$, $U\nu = W\nu$. For any $D \supset K$, we have $P_D W\mu = W\mu$ since $\lfloor\mu \subset K$; hence also

$$P_D W\nu = W\nu. \tag{20}$$

Now if $x \in F$, we can apply Fubini to infer from (20) that

$$P_D w(x, y) = w(x, y) \tag{21}$$

for ν-a.e. y. Again by Fubini, $\exists N$ with $\nu(N) = 0$ such that if $y \notin N$, then (21) holds for ξ-a.e. x, hence for all x because of excessiveness. If $\nu(\overline{D}^c) > 0$, then $\exists y_0 \in (\overline{D} \cup N \cup Z)^c$ for which (21) holds with $x = y = y_0$. Since $y_0 \notin Z$, $w(y_0, y_0) = \infty$, this would contradict Lemma 3. Thus $\lfloor\nu \subset \overline{D}$ and so $\lfloor\nu \subset K$ since D is arbitrary.

2 It is manifestly false that $W\mu = W\nu \not\equiv \infty$ and $\lfloor \mu \subset K$ implies $\lfloor \nu \subset K$.

THEOREM 5. – *Suppose that* $U\mu = U\nu \not\equiv \infty$ *where* $\mu(Z) = \nu(Z) = 0$; *then* $\mu \equiv \nu$.

Proof. – Suppose first that $\lfloor \mu \subset L$ where L is compact; then for any $D \supset L$ we have $U\mu = W\mu = P_D W\mu$. Hence by Lemma 2, $\exists$ a measure λ such that $\lambda(Z) = 0$,

$$W\mu = W\lambda \tag{22}$$

and moreover λ has the splitting property. We are going to show that $\mu \equiv \lambda$. The same argument then applies to ν by virtue of Lemma 4, and so $\nu \equiv \lambda \equiv \mu$.

Consider the Radon-Nikodym derivative $f = \dfrac{d\mu}{d(\mu + \lambda)}$ and $K \subset \left\{ f > \dfrac{1}{2} \right\}$. Recall the notation (14) in § 3. By Lemma 2, $\exists$ measure λ_1 such that $\lambda_1 \leqslant \lambda$ setwise and

$$W\mu^K = W\lambda_1; \tag{23}$$

by Lemma 4, $\lfloor \lambda_1 \subset K$. Were it possible that $(\mu + \lambda)(K) > 0$, then it would follow that

$$W\lambda^K \geqslant W\lambda_1^K = W\mu^K > \int_K w(x, y) \frac{1}{2} [\mu(dy) + \lambda(dy)] \tag{24}$$

and consequently by subtraction, on the set where $W\lambda < \infty$,

$$\int_K w(x, y)\, \lambda(dy) > \int_K w(x, y)\, \mu(dy) > \int_K w(x, y)\, \lambda(dy). \tag{25}$$

This contradiction shows that $f \leqslant \dfrac{1}{2}$, $(\mu + \lambda)$-a.e.; which means $\mu \leqslant \lambda$. Together with (22) and $w(x, y) = u(x, y) > 0$ for $y \notin Z$, we conclude $\mu = \lambda$ as desired.

Coming to the general case of the theorem, we write

$$\begin{aligned} \varphi &= W\mu = W\mu^D + W\mu^{E-D} \\ P_D\varphi &= W\mu^D + P_D W\mu^{E-D} = W\mu^D + W\mu'_D \end{aligned} \tag{26}$$

where μ'_D is given by Corollaries 1 and 4 to Theorem 2, with $\lfloor \mu'_D \subset \overline{D}$. A similar expression holds when μ is replaced by ν in (26); and

so we obtain

$$W(\mu^D + \mu'_D) = W(\nu^D + \nu'_D). \tag{27}$$

Both measures here have support in $\overline{D}$, hence by what we have proved above,

$$\mu^D + \mu'_D \equiv \nu^D + \nu'_D. \tag{28}$$

Now on the set $F = \{x: W\mu(x) < \infty\}$, we have $W\mu^{E-D} \downarrow 0$ as $D \uparrow E$; a fortiori,

$$\lim_{D \uparrow E} W\mu'_D = \lim_{D \uparrow E} W\nu'_D = 0. \tag{29}$$

On the other hand, we obtain from (28) for each K

$$\int_K w(x, y)\, \mu^D(dy) + \int_K w(x, y)\, \mu'_D(dy) = \int_K w(x, y)\, \nu^D(dy) + \int_K w(x, y)\, \nu'_D(dy). \tag{30}$$

If $x \in F$, the second terms on both sides above converge to zero as $D \uparrow E$ by (29), whereas there is monotone convergence for the first terms. Therefore we have

$$\int_K w(x, y)\, \mu(dy) = \int_K w(x, y)\, \nu(dy),$$

and since K is arbitrary, the finite measures $w(x, y)\, \mu(dy)$ and $w(x, y)\, \nu(dy)$ coincide for $x \in F$. Fix such an x, and remember that w may be replaced by u which is strictly positive everywhere. We reach the final conclusion that $\mu \equiv \nu$. □

COROLLARY 5 to THEOREM 2. – *If* $\mu_n(Z) = 0$ *for all* n, *then conclusion* (α) *may be strengthened to read:* μ_n *converges vaguely to* μ.

This follows because all vague limits are the same by uniqueness.

We proceed to Riesz's decomposition. A function h is harmonic iff for every (compact) K

$$h = P_{K^c} h. \tag{31}$$

THEOREM 6. – *Let* f *be excessive,* $\not\equiv \infty$. *Then there exists a Radon measure* μ *with* $\mu(Z) = 0$ *and a harmonic function* h *such that*

$$f = U\mu + h. \tag{32}$$

If $f = U\mu_1 + h_1$ *is another such representation (with* $\mu_1(Z) = 0$), *then* $\mu \equiv \mu_1$ *and* $h \equiv h_1$.

Proof. – Let $K_n \uparrow E$ and $T_n = T_{K_n^c}$. Put

$$h = \lim_n P_{T_n} f \tag{33}$$

$$g = \begin{cases} f - h & \text{on } \{f < \infty\}, \\ \infty & \text{on } \{f = \infty\}. \end{cases} \tag{34}$$

Then we have

$$f = g + h. \tag{35}$$

It is clear from (33) that for each n,

$$h = P_{T_n} h \quad \text{on } \{h < \infty\}; \tag{36}$$

h is superaveraging and $\{h \neq \underline{h}\} \subset \{h < \infty\} \subset \{f < \infty\}$ by (33) and the Lemma in § 1. Hence $\{h \neq \underline{h}\}$ is polar by Proposition 8 of § 1. Consequently $P_{T_n} h = P_{T_n} \underline{h}$, and so $\underline{h} = P_{T_n} \underline{h}$ except for a polar set and therefore everywhere. Thus $\underline{h}$ is harmonic.

Next we have from (35) and (36)

$$P_{T_n} f = P_{T_n} g + h \quad \text{on } \{h < \infty\};$$

and so by (33),

$$h = \lim_n P_{T_n} g + h. \tag{37}$$

This shows that the limit in (37) is equal to zero on $\{h < \infty\}$, hence ξ-a.e. Assuming for a moment that g is superaveraging, then we have $\lim_n P_{T_n} \underline{g} = 0$ ξ-a.e., and this implies by standard arguments that $\underline{g}$ is a potential. Hence by Corollaries 1 and 4 of Theorem 2, $\underline{g} = U\mu$ with μ as asserted in the theorem. From (35) we have $f = \underline{g} + \underline{h} = U\mu + \underline{h}$ which is (32) except $\underline{h}$ is written there as h. The uniqueness is immediate by Theorem 5.

It remains to show that g is superaveraging. This is usually done via a result by Dynkin (see, e.g., [1], p. 273), but here is a shorter direct proof. Since h is the decreasing limit of excessive functions, we have $P_T h \leqslant P_S h$ if S and T are optional times such that $S \leqslant T$. In particular, we have by (31)

$$\forall t \geqslant 0: \quad h = P_{T_n \wedge t} h, \quad \text{on } \{h < \infty\}. \tag{38}$$

Now we have for each t

$$P_t g(x) \leqslant P_{T_n \wedge t} g(x) + E^x\{g(X_t); T_n \leqslant t\}. \tag{39}$$

Fix an x such that $f(x) < \infty$. Then the last term in (39) is bounded by $P^x\{f(X_t); T_n \leqslant t\}$ which converges to zero as $n \longrightarrow \infty$ since $P_t f(x) < \infty$. Furthermore, $P_{T_n \wedge t}(x, .)$ does not charge $\{f = \infty\}$ and so by (39) and the definition of g, we have

$$P_t g(x) \leqslant \varliminf_n P_{T_n \wedge t} g(x) = \varliminf_n [P_{T_n \wedge t} f(x) - P_{T_n \wedge t} h(x)]$$
$$= \varliminf_n P_{T_n \wedge t} f(x) - h(x) \leqslant f(x) - h(x) \leqslant g(x).$$

Thus $P_t g \leqslant g$ on $\{f < \infty\}$; hence everywhere since $g = \infty$ on $\{f = \infty\}$. □

PROPOSITION 13. – *For each y except possibly a polar set, we have*

$$\lim_{K \uparrow E} P_{K^c} u(x, y) = 0, \quad \text{for} \quad x \neq y; \tag{40}$$

$$\lim_{t \to \infty} P_t u(x, y) = 0, \quad \text{for} \quad x \neq y. \tag{41}$$

Proof. – Applying the Lemma in § 1, with $f = u(\cdot, y)$ and using the notation there, we have for fixed y

$$\lim_{n \to \infty} P_{T_n} u(x, y) = g(x, y)$$

where $g(x, y) = \underline{g}(x, y)$ if $x \neq y$. By Proposition 9 of § 1, either $\underline{g}(\cdot, y) \equiv 0$, or $\underline{g}(\cdot, y) > 0$ everywhere. Let $K \subset \{y : \underline{g}(\cdot, y) \not\equiv 0\}$. By Corollary 3 of Theorem 2, $\exists \mu$ with $\lfloor \mu \subset K$ such that

$$P_K 1(x) = \int u(x, y)\, \mu(dy) \quad \text{if} \quad x \notin K.$$

Hence by transience and dominated convergence,

$$0 = \lim_n P_{T_n} P_K 1(x) = \int g(x, y)\, \mu(dy).$$

But if $y \in K$, $g(x, y) \geqslant \underline{g}(x, y) > 0$. Hence $\mu \equiv 0$ and so K is polar by the cited Corollary 3. This proves (40), and (41) is similar.

An excessive function s is called "purely excessive" iff

$$\lim_{t \to \infty} P_t s = 0 \quad \xi\text{-a.e.} \tag{42}$$

A "pure potential" is a potential which is purely excessive. A result analogous to Proposition 11 of § 1 shows that (42) implies actually the limit there is zero on $\{s < \infty\}$, hence except possibly a polar set.

The following remark is important. If s is purely excessive, then we have everywhere

$$Uf_n \uparrow s, \quad \text{where} \quad f_n = n(s - P_{1/n}s). \tag{43}$$

To see this, recall that standard arguments show that Uf_n is increasing and converges to s on the set where (42) holds. Hence the limit is an excessive function which is equal to s ξ-a.e., therefore it coincides with s.

5.

Hunt's Hypothesis (B), which is equivalent to

$$Z \text{ is a polar set} \tag{1}$$

by Theorem 4, will now be proved under the additional assumption below:

$$\forall y: \underline{u}(y, y) = +\infty. \tag{2}$$

This is satisfied if $\underline{u} = u$, namely if $u(\cdot, y)$ is excessive for each y. The latter condition is in turn satisfied if $u(\cdot, y)$ is lower semi-continuous since it is superaveraging by Fatou's lemma. At a crucial point we need also the assumption

$$\xi \text{ is an excessive measure;} \tag{3}$$

namely $\xi \geqslant \xi P_t$ for every $t \geqslant 0$. This assumption is usually made for a reference measure.

Define the Borel set

$$Q = \{y : \underline{u}(\cdot, y) \quad \text{is a pure potential}\}. \tag{4}$$

According to Proposition 13 in § 4, Q^c is a polar set. We shall prove that $Q \subset Z^c$; then $Z \subset Q^c$ so that (1) is true.

Let $y \in Q$. Then for each $t \geqslant 0$, $P_t\underline{u}(\cdot, y)$ is a potential. Hence by Corollaries 1 and 4 of Theorem 2, there exists a Radon measure, to be denoted by $\hat{P}_t(\cdot, y)$, which does not charge Z, such that

$$P_t\underline{u}(\cdot, y) = U\hat{P}_t(\cdot, y) = \int u(\cdot, z)\, \hat{P}_t(dz, y). \tag{5}$$

Here we have adopted the left-handed notation appropriate for the dual symbolism. Note that we can replace u by $\underline{u}$ in the last term

above, by Proposition 4 of § 1. It follows from (5) that if $y \in Q$, then $\hat{P}_t(\cdot, y)$ does not charge Q^c because the first member of (5) is a pure potential.

THEOREM 7. – *$\{\hat{P}_t, t \geqslant 0\}$ is a semigroup of kernels on $Q \times Q$. We have for each $y \in Q$*

$$\hat{P}_t(\cdot, y) \longrightarrow \hat{P}_0(\cdot, y) \quad \textit{vaguely.} \tag{6}$$

If we define a kernel $\hat{U}$ on $Q \times Q$ as follows:

$$\hat{U}(dx, y) = \xi(dx)\, u(x, y), \tag{7}$$

then for any f $(\geqslant 0)$ for which $f\hat{U}(y) < \infty$ we have

$$f\hat{U}(y) = \int_0^\infty \hat{P}_t(f, y)\, dt. \tag{8}$$

Proof. – The key is the uniqueness Theorem 5. We have for $w \in Q$, $t \geqslant 0$ and $s \geqslant 0$, since $\hat{P}_s(\cdot, w)$ is concentrated on Q, by (5):

$$\begin{aligned}\int [u(x, z)\, \hat{P}_t(dz, y)]\, \hat{P}_s(dy, w) &= \int P_t \underline{u}(x, y)\, \hat{P}_s(dy, w) \\ &= \int P_t(x, d\eta) \left[\int \underline{u}(\eta, y)\, \hat{P}_s(dy, w) \right] \\ &= \int P_t(x, d\eta)\, P_s \underline{u}(\eta, w) = P_{t+s} \underline{u}(x, w) \\ &= \int u(x, z)\, \hat{P}_{t+s}(dz, w).\end{aligned}$$

Hence

$$\int \hat{P}_t(dz, y)\, \hat{P}_s(dy, w) = \hat{P}_{t+s}(dz, w)$$

which establishes the semigroup property. Let $t_n \downarrow 0$ and apply Theorem 2 to the sequence $U\hat{P}_{t_n} = P_{t_n} \underline{u}$, with $\sigma = s = \underline{u}(\cdot, y)$ and under condition (c_2), we obtain $\underline{u}(\cdot, y) = U\mu$ where μ is a vague limit of $\hat{P}_{t_n}$, with $\mu(Z) = 0$. Hence $\mu = \hat{P}_0$ by Theorem 5, and (6) follows.

Next we have by Fubini,

$$\begin{aligned}U[\underline{u}(\cdot, y) - P_t \underline{u}(\cdot, y)] &= \int_0^t P_s \underline{u}(\cdot, y)\, ds \qquad (9) \\ &= \int_0^t ds \int u(\cdot, z)\, \hat{P}_s(dz, y) = \int u(\cdot, z) \int_0^t \hat{P}_s(dz, y)\, ds.\end{aligned}$$

The first term above is equal to

$$\int u(\cdot, z)\, [\underline{u}(z, y) - P_t \underline{u}(z, y)]\, \xi(dz);$$

hence by the uniqueness theorem we have

$$[\underline{u}(z, y) - P_t \underline{u}(z, y)]\, \xi(dz) = \int_0^t \hat{P}_s(dz, y)\, ds. \tag{10}$$

For any f $(\geqslant 0)$ such that

$$\int \xi(dz)\, f(z)\, u(z, y) < \infty \tag{11}$$

we have

$$\int \xi(dz)\, f(z)\, [\underline{u}(z, y) - P_t \underline{u}(z, y)] = \int_0^t f \hat{P}_s(y)\, ds,$$

and so letting $t \longrightarrow \infty$

$$\int \xi(dz)\, f(z)\, \underline{u}(z, y) = \int_0^\infty f \hat{P}_s(y)\, ds. \tag{12}$$

Note that $\forall y$: $\underline{u}(z, y) = u(z, y)$ for ξ-a.e. z, hence it is immaterial whether u or $\underline{u}$ is written in (11) or (12).

Remark. – It can be shown that if y is not in a certain polar set, then (11) holds for any bounded f with compact support.

Theorem 7 requires an essential complement which is stated separately to stress the point. We need first a lemma, the only place where the excessiveness of ξ is used. We write $\xi(f)$ for $\int f(x)\, \xi(dx)$.

LEMMA 5. – *If*

$$Uf < \infty \quad \textit{and} \quad Uf \leqslant \varliminf_n Ug_n \quad \xi\text{-a.e.}, \tag{13}$$

then

$$\xi(f) \leqslant \varliminf_n \xi(g_n).$$

Proof. – We prove first that if $Uf < \infty$ and $Uf \leqslant Ug$, ξ-a.e., then $\xi(f) \leqslant \xi(g)$. For this purpose we may suppose $\xi(g) < \infty$, hence $\xi(U^\lambda g) < \infty$ for $\lambda > 0$ because $\lambda \xi U^\lambda \leqslant \xi$. Write $P_t^\lambda = e^{-\lambda t} P_t$; then we have for any fixed $\lambda > 0$:

$$\frac{1}{t}\, (\xi - \xi P_t^\lambda)\, (U^\lambda g) = \frac{1}{t} \int_0^t (\xi P_s^\lambda)\, (g)\, ds \uparrow \xi(g) \tag{14}$$

as $t \downarrow 0$. Hence if

$$U^\lambda f \leqslant U^\lambda g, \tag{15}$$

then $\xi(f) \leqslant \xi(g)$. [We learned this argument from M.J. Sharpe.] Unfortunately (15) is not part of our hypothesis; whereas (14) need not hold for $\lambda = 0$. The remedy is as follows. Let $0 < a < 1$ and

put for $n \geqslant 1$:

$$A_n = \{x \in E \mid aU^{1/n} f(x) \leqslant U^{1/n} g(x)\}.$$

Since $U^{1/n}$ increases to U as $n \longrightarrow \infty$, our hypotheses imply that $\xi(E - \liminf_n A_n) = 0$. Now we have on A_n

$$aU^{1/n}(f 1_{A_n}) \leqslant U^{1/n} f \leqslant U^{1/n} g\ ;$$

hence the inequality holds everywhere in E by the domination principle for $U^{1/n}$ (see [4; p. 245]). Therefore the argument above with $\lambda = \frac{1}{n}$ yields $a\xi(f 1_{A_n}) \leqslant \xi(g)$. Letting $n \longrightarrow \infty$, then $a \uparrow 1$, we obtain $\xi(f) \leqslant \xi(g)$.

Now suppose that (13) is true. For $0 < a < 1$ put

$$B_m = \{x \in E \mid aUf(x) < \inf_{n \geqslant m} U(g_n)\}.$$

Then $B_m \uparrow$ and $\xi(E - \bigcup_m B_m) = 0$. We have for each $n \geqslant m$: $aU(f 1_{B_m}) \leqslant U(g_n)$ on B_m; hence the inequality holds everywhere by the domination principle for U. It follows from the first part of the proof that $a\xi(f 1_{B_m}) \leqslant \xi(g_n)$. Letting $n \longrightarrow \infty$, we infer that $a\xi(f 1_{B_m}) \leqslant \varliminf_{n \to \infty} \xi(g_n)$. Letting $m \longrightarrow \infty$, then $a \uparrow 1$, we obtain the conclusion of the lemma.

THEOREM 8. – $\{\hat{P}_t, t \geqslant 0\}$ *on* $Q \times Q$ *is a submarkovian semigroup, i.e.,* $\forall y \in Q$: $\hat{P}_t(Q, y) \leqslant 1$.

Proof. – Fix $y \in Q$, $t \geqslant 0$, and put for $\delta > 0$

$$f_\delta(x) = \frac{1}{\delta}\,[P_t \underline{u}(x, y) - P_{t+\delta} \underline{u}(x, y)].$$

Since $P_t \underline{u}(\cdot, y)$ is purely excessive, we have

$$Uf_\delta(x) \leqslant P_t \underline{u}(x, y) \tag{16}$$

$$\lim_{\delta \downarrow 0} \uparrow Uf_\delta(x) = P_t \underline{u}(x, y) \tag{17}$$

for all x, by (43) of § 4. Hence by Theorems 2 and 5 we have

$$f_{1/n}(z)\, \xi(dz) \longrightarrow \hat{P}_t(dz, y) \quad \text{vaguely.} \tag{18}$$

Let $D_n \downarrow\downarrow \{y\}$, and ($\xi$ charges each open set)

$$g_n(z) = \frac{1_{D_n}(z)}{\xi(D_n)} \tag{19}$$

so that

$$\forall n\colon \xi(g_n) = 1\,. \tag{20}$$

For each x, $u(x,\cdot)$ is lower semicontinuous; since $g_n(z)\,\xi(dz)$ converges vaguely to the unit mass at y, we have

$$u(x,y) \leqslant \varliminf_n Ug_n(x)\,. \tag{21}$$

Therefore we have in conjunction with (16), for each $\delta > 0$,

$$Uf_\delta \leqslant u(\cdot\,,y) \leqslant \varliminf_n Ug_n\,. \tag{22}$$

It follows by Lemma 5 and (20) that

$$\xi(f_\delta) \leqslant 1 \tag{23}$$

for every δ, and consequently by (18) that $\hat{P}_t(Q,y) \leqslant 1$. □

A function s defined on Q is called "co-superaveraging" iff $s \geqslant s\hat{P}_t$ for every $t > 0$ and is "co-excessive" iff in addition $s = \lim_{t\downarrow\downarrow 0} s\hat{P}_t$. We cannot yet define a "co-potential", but we can define a co-excessive s to be "purely co-excessive" iff $\lim_{t\to\infty} s\hat{P}_t = 0$, ξ-a.e.

The following lemma is the co-version of a remark at the end of § 4, and is spelled out here because of its importance in the proof of Theorem 10 below.

LEMMA 6. – *If φ is purely co-excessive, then*

$$\psi_n \hat{U} \uparrow \varphi, \quad \textit{where} \quad \psi_n = n(\varphi - \varphi \hat{P}_{1/n}) \tag{24}$$

everywhere.

Proof. – Just as in (43) of § 4, $\psi_n \hat{U} \uparrow \widetilde{\varphi}$, where $\widetilde{\varphi}$ is co-excessive and $\widetilde{\varphi} = \varphi$ ξ-a.e. Now we see from (10) that the measure $\int_0^t \hat{P}_s(\cdot\,,y)\,ds$, $y \in Q$, is absolutely continuous with respect to ξ. Hence

$$\frac{1}{t}\int_0^t \widetilde{\varphi}\hat{P}_s\,ds = \frac{1}{t}\int_0^t \varphi \hat{P}_s\,ds\,. \tag{25}$$

Letting $t \downarrow 0$, we obtain $\widetilde{\varphi} = \varphi$. □

THEOREM 9. – *For each x, the function $\underline{u}(x,\cdot)$ on Q is purely co-excessive; so is*

$$\varphi_x(\cdot) = 1 - e^{-\underline{u}(x,\cdot)}\,. \tag{26}$$

Proof. – Using Proposition 4 of § 1, we see that (5) may be written as

$$\underline{u}_x \hat{P}_t(y) = P_t \underline{u}_x(y) \tag{27}$$

where $\underline{u}_x(y) = \underline{u}(x, y)$. This shows at once that u_x is purely co-excessive. Let $\Phi(\theta) = 1 - e^{-\theta}$, $\theta \geqslant 0$; then Φ is concave and $\varphi_x = \Phi \circ u_x$. Hence by Jensen's inequality

$$\varphi_x \geqslant \Phi \circ u_x \hat{P}_t \geqslant (\Phi \circ u_x)\hat{P}_t = \varphi_x \hat{P}_t , \tag{28}$$

namely φ_x is co-superaveraging. Now u_x is lower semicontinuous, so are successively: $P_t u_x$, $\underline{u}_x$ and φ_x. Therefore φ_x is co-excessive. It is purely so by (28), because u_x is purely co-excessive and $\Phi(0) = 0$. □

THEOREM 10. – $Q \subset Z^c$.

Proof. – Fix a $y \in Q$, and put $\varphi(z) = 1 - e^{-\underline{u}(y,z)}$. By Theorem 9, φ is purely co-excessive; hence by Lemma 6, $\exists \psi_n$ such that

$$\forall z: \ \psi_n \hat{U}(z) \uparrow \varphi(z) . \tag{29}$$

Now by integrating the fundamental representation formula (5) with $t = 0$, we have

$$\int \xi(dx)\, \psi_n(x)\, \underline{u}(x, y) = \int \left[\int \xi(dx)\, \psi_n(x)\, \underline{u}(x, z) \right] \hat{P}_0(dz, y) ,$$

namely $\psi_n \hat{U}(y) = \int \psi_n \hat{U}(z)\, \hat{P}_0(dz, y)$. Letting $n \longrightarrow \infty$ and using (29), we obtain

$$\varphi(y) = \int \varphi(z)\, \hat{P}_0(dz, y) . \tag{30}$$

Observe that $\varphi(y) = 1$ by our new assumption (2), and $\varphi(z) < 1$ for $z \neq y$ by our old assumption. Hence (20) together with $\hat{P}_0(Q, y) \leqslant 1$ (Theorem 8) forces

$$\hat{P}_0(\{y\}, y) = 1 . \tag{31}$$

But $\hat{P}_0(Z, y) = 0$ by Theorem 2, as recalled before (5). Therefore $y \in Z^c$. □

Finally, we give a generalization of Theorem 10 by weakening the condition (2) as follows:

(32) the family of excessive functions $\underline{u}(\cdot, y)$ on E, indexed by $y \in E$, are all distinct.

Such a condition is meaningfull in the general theory of excessive functions. It is implied by (2) because $\underline{u}(y, y') < \infty = \underline{u}(y, y)$ if $y' \neq y$. Our basic assumption (iib) in § 1 is thus thrown into relief.

THEOREM 11. – *Under our basic assumptions (i) and (ii) stated at the beginning of § 1, with ξ an excessive measure, if (32) is true, then Hunt's Hypothesis (B) is true.*

Proof. – Fix $y \in Q$ and write for brevity's sake μ for the measure $\hat{P}_0(\cdot, y)$. As an obvious generalization of (30), we have for each $\lambda > 0$:

$$1 - e^{-\lambda \underline{u}(x,y)} = \int [1 - e^{-\lambda \underline{u}(x,z)}]\, \mu(dz). \tag{33}$$

It follows from Proposition 9 of § 1 that $\underline{u}(x, y) > 0$ for all (x, y). Hence letting $\lambda \uparrow \infty$ above, we obtain $\mu(E) = 1$. Recall that μ is concentrated on Z^c as well as on Q. Next we have by (5) with $t = 0$:

$$\underline{u}(x, y) = \int \underline{u}(x, z)\, \mu(dz). \tag{34}$$

If we put $\lambda = 1$ in (33), the resulting equation may be written with our previous notation Φ as follows:

$$\Phi(\underline{u}(x, y)) = \int \Phi(\underline{u}(x, z))\, \mu(dz). \tag{35}$$

Now Φ is strictly concave, whereas we have the equality case of Jensen's inequality in (35). This forces the measure μ to concentrate on the set of z where the integrand $u(x, z)$ in (34) takes a constant value, which must then be $\underline{u}(x, y)$ because $\mu(E) = 1$. [We were unable to unearth a reference to the required proposition in the case of a general probability measure μ, but Michael Steele was kind enough to supply an elegant short proof on request.] Namely, if we put $B_x = \{z \in Z^c \mid \underline{u}(x, z) = \underline{u}(x, y)\}$, then $\mu(B_x) = 1$ for each $x \in E$. Now put also $C_x = \{z \in E \mid u(x, z) = \underline{u}(x, y)\}$. Since $z \longrightarrow u(x, z)$ is extended continuous, C_x is a closed set. We have $B_x = C_x \cap Z^c$, because if $z \in Z^c$, then $\underline{u}(x, z) = u(x, z)$ for all $x \in E$. Let $B = \bigcap_{x \in E} B_x$, $C = \bigcap_{x \in E} C_x$. Since $C_x \supset B_x$, we have $\mu(C_x) = 1$. Since C is closed, it follows that $\mu(C) = 1$ a cute little exercise in measure theory. Therefore, $B = C \cap Z^c$ belongs to $\mathfrak{B}$ and $\mu(B) = 1$. Thus B is not empty. Let $y' \in B$,

then we have by the definition of B: $\forall x \in E: \underline{u}(x, y') = \underline{u}(x, y)$. Our new condition (32) entails $y' = y$. Thus $y \in Z^c$ as we concluded at the end of the proof of Theorem 10. [In fact, $B = \{y\}$ and (31) follows.]

In closing, let us remark that the preceding proof of Hypothesis (B) is "perilously close", as Hunt would have said. More than one attempt was made to simplify it, but the efforts failed on rather delicate details. Hunt said [3, p. 81], "I have not found simple and general conditions on the transition measures to ensure the truth of Hypothesis (B)." it is implied by the usual duality assumptions, see [5]. It would be extremely interesting to know whether a simpler proof exists in the setting of this paper.

BIBLIOGRAPHY

[1] R.M. BLUMENTHAL and R.K. GETOOR, Markov Processes and Potential Theory, Academic Press, 1968.

[2] K.L. CHUNG, Probabilistic approach in potential theory to the equilibrium problem, *Ann. Inst. Fourier,* 23, 3 (1973), 313-322.

[3] G.A. HUNT, Markoff processes and potentials I, *Illinois J. Math.*, 1 (1957), 43-93.

[4] P.A. MEYER, Probabilités et potentiel, Hermann, 1966.

[5] P.A. MEYER, Processus de Markov : la frontière de Martin, *Lecture Notes in Mathematics* No. 77, Springer-Verlag, (1968).

[6] P.A. MEYER, Deux petits résultats de théorie du potentiel, Séminaire de Probabilités V, *Lecture Notes in Mathematics* No. 191, Springer-Verlag (1971), 211-212.

[7] P.A. MEYER, Le Retournement du temps, d'après Chung et Walsh, Séminaire de Probabilités V, *Lecture Notes in Mathematics* No. 191, Springer-Verlag (1971), 213-236.

[8] G. MOKOBODZKI, Densité relative de deux potentiels comparables, Séminaire de Probabilités IV, *Lecture Notes in Mathematics* No. 124, Springer-Verlag (1970), 170-194.

[9] K.M. RAO, Excessive functions as potentials of measures, *J. London Math. Soc.*, 16 (1977), 165-171.

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