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SOME COUNTER-EXAMPLES IN THE THEORY OF THE GALOIS MODULE STRUCTURE OF WILD EXTENSIONS

by Stephen M. J. WILSON (*)

Let \mathfrak{N} be a maximal order (over a Dedekind ring) in a (finite dimensional) semisimple algebra A then (with \tilde{K}_0 denoting rankless objects in K_0 , C a picard group and Z the centre of an order) we have a homomorphism

$$\text{Nrd}_0 : \tilde{K}_0(\mathfrak{N}) \rightarrow C(Z(\mathfrak{N}))$$

$$\text{by} \quad (\alpha \mathfrak{N}) \mapsto (\nu(\alpha) Z(\mathfrak{N})) \quad [4]$$

where α is an idele of A and $\nu(\alpha)$ is its reduced norm. (If M is an \mathfrak{N} -module and $M \otimes_{\mathfrak{N}} A \cong A^{(n)}$ then we put

$$(M) = [M] - n[\mathfrak{N}].$$

We note that if A is simple and split (so Nrd_0 is an isomorphism), P a minimal projective of \mathfrak{N} and α a non-zero ideal of $Z(\mathfrak{N})$ then

$$\text{Nrd}_0([\alpha P] - [P]) = \alpha. \quad (1)$$

We consider here the following problem. Let \mathcal{O} be the ring of integers in a number field. Let G be a group of automorphisms of \mathcal{O} and \mathfrak{N} a maximal order containing $\mathbf{Z}G$. It was thought possible that one of

$$x = \text{Nrd}_0(\mathcal{O} \otimes_{\mathbf{Z}G} \mathfrak{N}) \quad \text{or} \quad y = \text{Nrd}_0(\mathcal{O} \otimes_{\mathbf{Z}G} \mathfrak{N}/\text{torsion})$$

was zero or, at least, independent of \mathfrak{N} . (This is certainly true when $\mathcal{O}/\mathcal{O}^G$ is tamely ramified see [5] Theorem 3). Cougnard [2] has

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shown that y need not be zero and that x is often zero [2], [3]. However, we here present an example (with $\Theta^G = \mathbf{Z}$) where neither x nor y is even independent of the choice of \mathfrak{N} .

The presentation of this example falls into four parts. We use a group G (in fact $C_{23} \times D_{18}$) such that the images of $\mathbf{Z}G$ in the semisimple components of $\mathbf{Q}G$ are twisted group rings (with trivial cocycle) with the group action tamely ramified and faithful. Lemma 1 investigates the maximal orders containing such rings and lemmas 1 and 2 calculate the 'difference' between the extensions of a module to two different maximal orders, showing that this depends only on the local structure of the module at the primes where the maximal orders differ. Lemma 3 shows that in certain circumstances the local structure of a module over one twisted group ring may be determined from its local structure over another. (The point here is that the modules which we investigate are expressed as quotients of $\mathbf{Z}G$ -modules which are not themselves modules over the image of $\mathbf{Z}G$ in question. They are, however, modules over a different twisted group ring with the same group.) Lemma 4 and the work which follows construct the example. Lastly, the Theorem uses the four lemmas to expose the desired properties of the example.

If M is a subset of an A -module then $\Lambda_A(M)$ denotes $\{a \in A \text{ such that } Ma \subseteq M\}$.

LEMMA 1. — *Let S be a Dedekind ring with field of quotient L and group of automorphisms Γ . Suppose that S is tamely ramified over $S^\Gamma = R$. Let I and J be ambiguous ideals of S .*

- (i) $\mathfrak{N}(I) = \Lambda_{L\Gamma}(I)$ is a maximal order containing $S\Gamma$ and
- (ii) all such orders arise in this way. (Here $S\Gamma$ and $L\Gamma$ denote the appropriate twisted group rings.)
- (iii) The minimal projectives of $\mathfrak{N}(I)$ are (isomorphic to) the αI where α is an ideal of R .
- (iv) $\mathfrak{N}(I) = \mathfrak{N}(J)$ if and only if IJ^{-1} is extended from an ideal of R .
- (v) $J\mathfrak{N}(I) = J \otimes_{S\Gamma} \mathfrak{N}(I) = \alpha I$ where α is the minimal R -ideal such that $\alpha I \supseteq J$. i.e. $\alpha^{-1} = (IJ^{-1}) \cap K$, where $K = L^\Gamma$.

Proof. — (i) Considering $L\Gamma$ as $\text{End}_K(L)$, we have $\Lambda_{L\Gamma}(I) = \text{End}_R(I)$ and so, locally, $\Lambda_{L\Gamma}(I) = \text{Mat}_{|\Gamma|}(R)$.

(ii) Let \mathfrak{N} be a maximal order containing $S\Gamma$. A minimal projective of \mathfrak{N} may be considered as a sublattice (and hence an ambiguous S -ideal) I of L . Clearly $\mathfrak{N}I = \mathfrak{N}$. Moreover if $J \subseteq L$ is another minimal projective of \mathfrak{N} the local \mathfrak{N} -isomorphisms from I to J extend to local $L\Gamma$ -isomorphisms from L to L . Together these may be expressed as an idele α of K and so $J = \alpha I$ where α is the ideal of R corresponding to α . As, clearly, $\mathfrak{N}I = \mathfrak{N}(\alpha I)$ for any R -ideal α we have (iii).

(iv) Is immediate as, from the proof of (i), we have that I is a minimal projective of $\mathfrak{N}(I)$.

(v) $J\mathfrak{N}(I)$ is a minimal projective of $\mathfrak{N}(I)$ containing J ...

With notation as in Lemma 1 we define $\det^I : K_0(S\Gamma) \rightarrow C(R)$ by $\det^I([M]) = (\alpha)$ where $M \otimes_{S\Gamma} \mathfrak{N}(I) \cong \bigoplus \alpha_i I$ (where the α_i are R -ideals) and $\alpha = \prod \alpha_i$. (In fact $M \otimes \mathfrak{N}(I) \cong \alpha I \oplus I \dots \oplus I$ in this case. We can also use the injection $C(R) \rightarrow C(S, \Gamma)$ to describe α as the equivariant Steinitz class : $\det_S(M \otimes_{S\Gamma} \mathfrak{N}(I) \cdot I^{-1}) \in C(S, \Gamma)$, see [7]). We define $\delta_I(x) = \det^I(x) \det^S(x)^{-1}$.

LEMMA 2. — *Let I and J be ambiguous ideals of S .*

(i) *If $x \in \tilde{K}_0(S\Gamma)$ then $\text{Nrd}_0(x \otimes_{S\Gamma} \mathfrak{N}(I)) = \det^I(x)$*

(ii) $\det^I(J) = ((IJ^{-1}) \cap K)^{-1}$

For every prime ideal $\mathfrak{p} \neq 0$ of R let $\phi(\mathfrak{p})$ be the maximal ambiguous S -ideal lying over \mathfrak{p} . Then $\mathfrak{p}S = \phi(\mathfrak{p})^{e(\mathfrak{p})}$.

(iii) $\delta_I(J) = \prod_{\phi(\mathfrak{p})|I} \mathfrak{p} \left[\frac{j(\mathfrak{p}) - i(\mathfrak{p})}{e(\mathfrak{p})} \right] - \left[\frac{j(\mathfrak{p})}{e(\mathfrak{p})} \right]$ where $I_{\mathfrak{p}} = \phi(\mathfrak{p})_{\mathfrak{p}}^{i(\mathfrak{p})}$ and $J_{\mathfrak{p}} = \phi(\mathfrak{p})_{\mathfrak{p}}^{j(\mathfrak{p})}$.

(Here $[x]$ denotes the greatest integer not greater than x .)

(iv) *In particular if $I = \phi(\mathfrak{p})^i$ then $\delta_I(x)$ depends only on the image of x in $K_0(S_{\mathfrak{p}}\Gamma)$ —or, indeed, that in $K_0(S_{\mathfrak{P}}\Gamma(\mathfrak{P}))$, where \mathfrak{P} is a prime of S over \mathfrak{p} and $\Gamma(\mathfrak{P})$ its decomposition group.*

(v) *Specifically, if $|\Gamma| = e = 2$ and x goes to $n[S_{\mathfrak{p}}] + m[\phi(\mathfrak{p})_{\mathfrak{p}}]$ in $K_0(S_{\mathfrak{p}}\Gamma)$ then $\delta_{\phi(\mathfrak{p})}(x) = (\mathfrak{p}^{-n})$.*

Proof. — (i) If $x \in \widetilde{K}_0(S\Gamma)$ then

$$\begin{aligned} x \otimes \mathfrak{N}(I) &= \sum_{i=1}^n n_i [a_i I] \quad \text{where} \quad \sum n_i = 0 \\ &= \sum_i n_i ([a_i I] - [I]). \end{aligned}$$

The first expression gives $\det^I(x) = \Pi a_i^{n_i}$ and by (1) the second gives $\text{Nrd}(x \otimes \mathfrak{N}(I)) = \Pi a_i^{n_i}$.

(ii) Is immediate from Lemma 1 (v).

$$(iii) \quad (IJ^{-1}) = (\phi(\mathfrak{p})^{i-j})_{\mathfrak{p}} \quad \text{and} \quad \phi(\mathfrak{p})^n \cap K = \mathfrak{p}^{-\left[\frac{-n}{e(\mathfrak{p})}\right]}.$$

(iv) From (iii), if $I = \phi(\mathfrak{p})^i$ then $\delta_I(J)$ depends only on $j(\mathfrak{p}) \bmod e$ and as the distinct irreducible projectives of $S_{\mathfrak{p}}\Gamma$ are $\phi(\mathfrak{p})_{\mathfrak{p}}^i$, $i = 0, \dots, e-1$ and those of $S_{\mathfrak{p}}\Gamma(\mathfrak{P})$ are $\mathfrak{P}_{\mathfrak{p}}^i$, $i = 0, \dots, e-1$, see [6], $\delta_I(J)$ is determined by the image of J in $K_0(S_{\mathfrak{p}}\Gamma)$ or in $K_0(S_{\mathfrak{p}}\Gamma(\mathfrak{P}))$. The general result follows.

$$\begin{aligned} (v) \quad \delta_{\phi(\mathfrak{p})}(n[S] + m[\phi(\mathfrak{p})]) &= \mathfrak{p}^{\left(n\left(\left[\frac{0-1}{2}\right] - \left[\frac{0}{2}\right]\right) + m\left(\left[\frac{1-1}{2}\right] - \left[\frac{1}{2}\right]\right)} \\ &= (\mathfrak{p}^{-n}). \end{aligned}$$

LEMMA 3. — *Let S be a complete discrete valuation ring with maximal ideal \mathfrak{P} . Let Γ be a finite group of automorphisms of S with inertia subgroup $C = \langle \tau \rangle$ of order e prime to $|S/\mathfrak{P}|$. Then $P_i = \mathfrak{P}^i$, $i = 0, \dots, e-1$ are the distinct minimal projectives of $S\Gamma$ and $\mu_i = \mathfrak{P}^i/\mathfrak{P}^{i+1}$, $i = 0, \dots, e-1$ are the distinct simple $S\Gamma$ -modules. Choose a generator π of \mathfrak{P} such that $\pi^\tau/\pi = \eta$ is an e^{th} root of 1. Let R_0 be a sub-valuation-ring of S^C containing η with $\mathfrak{p}_0 = R_0 \cap \mathfrak{P}$ and suppose that $r = \text{rk}_{R_0}(S)$ and $f = \dim_{R_0/\mathfrak{p}_0}(S/\mathfrak{P})$ are finite.*

Let $S', \mathfrak{P}', \Gamma', C', P'_i, \mu'_i, \pi', \eta', R'_0, r', f'$ be another such set of data with $C' = C$ and $R'_0 = R_0$.

(i) *Choose $x \in G_0^f(S\Gamma)$ and $x' \in G_0^f(S'\Gamma')$ such that their images in $G_0^f(R_0 C)$ agree. If $x = \sum n_i [\mu_i]$ then $x' = \sum \frac{f}{f'} n_{it} [\mu'_t]$ where $\eta' = \eta^t$ (suffixes are taken modulo e).*

(ii) Let M be an $S\Gamma$ - and an $S'\Gamma'$ - module with identical R_0C actions and commuting S and S' actions.

If $[M] = \sum_0^{e-1} m_i [P_i]$ in $G_0(S\Gamma)$ and $\sum_0^{e-1} m'_i [P'_i]$ in $G_0(S'\Gamma')$

then $m'_i = m'_{i-1} + \frac{f}{f'} (m_{it} - m_{(i-1)t})$.

(iii) In particular if $|\Gamma| = |\Gamma'| = 2 = e$ and

$$[M]_{S\Gamma} = a([P_0] + [P_1]) + b[P_1]$$

then $[M]_{S'\Gamma'} = a'([P'_0] + [P'_1]) + \frac{bf}{f'} [P'_1]$

where $a' = a \frac{r}{r'} + \frac{b}{2} \left(\frac{r}{r'} - \frac{f}{f'} \right)$.

Proof. — (i) Let $\nu_i = (R_0/\mathfrak{p}_0) C/(\tau - \eta^i)$, $i = 0, \dots, e-1$, be the simple R_0C -modules. Then we have restriction isomorphisms

$$G_0^t(S\Gamma) \xrightarrow{\sim} G_0^t(R_0C) \xleftarrow{\sim} G_0^t(S'\Gamma'),$$

where $[\mu_i] \mapsto f[\nu_i]$ and $[\mu'_i] \mapsto f'[\nu_{it}]$. The result follows.

(ii) Let $M = M_1/M_2$ where M_1, M_2 are torsion-free $S \otimes_R S'C$ -modules (we assume $\Gamma = \Gamma' = C$ in view of the restriction isomorphisms between $K_0(S\Gamma)$ and $K_0(SC)$ etc.) and put $\tilde{M} = M_1\pi'/M_2\pi'$.

If $N \subseteq M_i$, $N \cong_{SC} P_j$ then $N\pi' \cong_{SC} P_{j+t}$. Hence in $G_0(SC)$

$$\begin{aligned} [M] - [M] &= [M_1] - [M_1\pi'] - [M_2] + [M_2\pi'] = \sum m_i ([P_i] - [P_{i+1}]) \\ &= \sum (m_i - m_{i-1}) [P_i]. \end{aligned}$$

Now if $[M_1/M_1\pi'] - [M_2/M_2\pi'] = \sum x_i [\mu_i]$ in $G_0^t(SC)$ then $[M] - [\tilde{M}] = \sum x_i ([P_i] - [P_{i+1}]) = \sum (x_i - x_{i-1}) [P_i]$ in $G_0(SC)$.

But, in $G_0^t(S'C)$, $[M_1/M_1\pi'] - [M_2/M_2\pi']$ is the 'semisimplification' of $[M_1] - [M_2]$ and so is $\sum m'_i [\mu'_i]$.

$$\text{So} \quad m'_i = \frac{f}{f'} x_{it} = m'_{i-1} + \frac{f}{f'} (m_{it} - m_{(i-1)t})$$

(iii) follows.

Let Δ be a finite group. The intersection, \mathcal{H} , of all maximal orders which contain $Z\Delta$ is the minimal hereditary order containing $Z\Delta$. I am indebted to Anne-Marie Bergé for the following cons-

truction of a Galois extension \hat{L} of \mathbf{Q}_p , p an odd prime, such that, with $\Delta = \text{Gal}(\hat{L}/\mathbf{Q}_p)$ and $\hat{S} = \text{int}(\hat{L})$, $\hat{S}\mathcal{H}_p \not\cong \mathcal{H}_p$. (The fact that this is so will emerge later although my proof, using the preceeding lemma, is somewhat different from that of Mlle Bergé.)

LEMMA 4. — (i) Let $\hat{L}_0 = \mathbf{Q}_p[\sqrt{p}]$ and let \hat{L} be the extension of \hat{L}_0 with norm group $N = \langle \mathbf{Q}_0^*, \sqrt{p}, (1 + \sqrt{p})^{p^2} \rangle \subseteq \hat{L}_0^*$. Putting $H = \text{Gal}(\hat{L}/\hat{L}_0)$ we have ramification groups $H^0 = H^1 \simeq C_{p^2}$, $H^2 = H^3 \simeq C_p$, $H^4 = \{1\}$ and $\text{Gal}(\hat{L}/\mathbf{Q}_p)$ is dihedral.

(ii) $H_0 = H_1 \simeq C_p$, $H_2 = \dots = H_{2p+1} \simeq C_p$, $H_{2p+2} = \{1\}$.

(iii) Let σ be a generator of H and π_1 a prime of $\hat{S}_1 = \hat{S}^{(\sigma^p)}$ then $\text{tr}_{\sigma^p} \hat{S} = \pi_1^{2p-1} \hat{S}_1$.

Proof. — (i) \hat{L}/\hat{L}_0 is clearly cyclic of degree p^2 and, as $(1 + \sqrt{p})(1 - \sqrt{p}) = 1 - p \in N$, \hat{L}/\mathbf{Q}_p is dihedral. Moreover $1 + \sqrt{p} \in U^1 \setminus U^2$ and $(1 + \sqrt{p})^p = 1 + p\sqrt{p} + (\sqrt{p})^p + \dots \in U^3 \setminus U^4$ and the result follows by local class field theory.

(ii) $H^i = H_{\psi(i)}$ where $\psi(x) = \int_0^x |H^0 : H^u| du$ so $\psi(0) = 0$, $\psi(1) = 1$, $\psi(1+r) = 1 + rp$, $0 < r < 2$. Hence the result.

(iii) Put $H' = \text{Gal}(\hat{L}/\hat{L}^{(\sigma^p)})$. Then

$$H'_0 = H'_1 = \dots = H'_{2p+1} = C_p, H'_{2p+2} = \{1\}.$$

So the value of the different of \hat{S} over \hat{S}_1 is

$$(2p+2)(p-1) = 2p^2 - 2.$$

$$\text{Therefore } \text{tr}_{\sigma^p}(\hat{S}) = \pi_1^{\left[\frac{2p^2-2}{p}\right]} \hat{S}_1 = \pi_1^{2p-1} \hat{S}_1.$$

Choose a prime $p > 2$ and $n \in \mathbf{N}^*$ prime to p such that, with $\theta = \sqrt[n]{1}$, the prime \mathfrak{p}_0 over p in $\mathbf{Z}[\theta]$ is not principal (e.g. $p = 3$, $n = 23$). We put $G = C_n \times (C_{p^2} \times \Gamma)$ where $C_{p^2} = \langle \sigma \rangle$, $\Gamma = \langle \tau \rangle$ of order 2 and $\sigma\tau = \sigma^{-1}$. Put $T = \mathbf{Z}[\theta, \xi]$, where $\xi = \sqrt[p]{1}$, and $T_1 = \mathbf{Z}[\theta, \xi^p]$ and let $\mathfrak{p}_1, \mathfrak{p}_1, \mathfrak{p}, \mathfrak{p}$ be the primes over \mathfrak{p}_0 in $T_1, T_1^\Gamma, T, T^\Gamma$ where $\tau: \begin{cases} \theta \mapsto \theta \\ \xi \mapsto \xi^{-1} \end{cases}$. We note that all these primes are non-principal.

Choose a $D_{2p^2} (= C_{p^2} \times \Gamma)$ -extension L of \mathbf{Q} such that L_p is the \hat{L} in lemma 4 (we can do this as p is odd see [1] ch. 10

Thm 5), and an extension M of \mathbf{Q} disjoint from L , cyclic of degree n and non-ramified where L is ramified. Then $E = LM$ is a galois extension of \mathbf{Q} with group G and if S , V and W are the rings of integers of L , M and E respectively we have $W = SV = S \otimes_{\mathbf{Z}} V$.

We put $L_1 = L^{(\sigma^p)}$ and $L_0 = L^{(\sigma)}$ with rings of integers S_1 , S_0 and we choose π , π_1 , π_0 , prime elements of \hat{L} , \hat{L}_1 , \hat{L}_0 so that $\pi^\tau = -\pi$ etc.

Let χ , χ_1 , be the characters of G induced from the C_{np^2} characters $\rho \mapsto \theta$, $\sigma \mapsto \zeta$ and $\rho \mapsto \theta$, $\sigma \mapsto \zeta^\rho$ and let A_χ , A_{χ_1} be the corresponding factors of $\mathbf{Q}G$. Choose \mathfrak{N} , \mathfrak{N}_1 , maximal orders containing the images of $\mathbf{Z}G$ in the complements of A_χ and A_{χ_1} . Note that the projections of $\mathbf{Z}G$ into A_χ and A_{χ_1} are, respectively, $T\Gamma$ and $T_1\Gamma$. We recall that $\mathfrak{N}(T)$ is a maximal order containing $T\Gamma$ etc.

THEOREM. — *With the above data*

- (i) $\text{Nrd}_0(W \otimes_{\mathbf{Z}G} (\mathfrak{N} \oplus \mathfrak{N}(\mathfrak{P}))) \neq \text{Nrd}_0(W \otimes_{\mathbf{Z}G} (\mathfrak{N} \oplus \mathfrak{N}(T)))$.
- (ii) $\text{Nrd}_0(W \otimes_{\mathbf{Z}G} (\mathfrak{N}_1 \oplus \mathfrak{N}(\mathfrak{P}_1))) \neq \text{Nrd}_0(W \otimes_{\mathbf{Z}G} (\mathfrak{N}_1 \oplus \mathfrak{N}(T_1)))$.

Proof. — (i) We write $\text{tr} = \text{tr}_{L/L_1}$. As

$$\mathbf{Z}[\xi] = \mathbf{Z}[\sigma]/(1 + \sigma^p + \dots + \sigma^{p(p-1)}),$$

$$\begin{aligned} [S \otimes_{\mathbf{Z}C_{p^2}} \mathbf{Z}_p[\xi]]_{\hat{S}_0\Gamma} &= [\hat{S}/\text{tr}\hat{S}] = [\hat{S}] - [\text{tr}\hat{S}] = [\hat{S}] - [\pi_1^{2p-1} S_1] \\ \text{by lemma 4} \quad &= [\hat{S}_0] + [\pi\hat{S}_0] + \dots + [\pi^{p^2-1}\hat{S}_0] \\ &\quad - [\pi_1^{2p-1} S_0] - \dots - [\pi_1^{3p-2} S_0] \\ &= \frac{p^2 - p + 2}{2} [\hat{S}_0\Gamma] - 2[\hat{\Phi}_0] \end{aligned}$$

where $\hat{\Phi}_0 = \pi_0\hat{S}_0$ as $\pi^i\hat{S}_0 \cong_{S_0\Gamma} \pi_1^i\hat{S}_0 \cong \hat{\Phi}_0$ and $\hat{S}_0\Gamma \cong_{S_0\Gamma} \hat{S}_0 \oplus \hat{\Phi}_0$.

Also $V \otimes_{\mathbf{Z}C_n} \mathbf{Z}[\theta]_{\mathfrak{p}_0} \cong \mathbf{Z}[\theta]_{\mathfrak{p}_0}$ as V/\mathbf{Z} is tame at p ($(n, p) = 1$).

Hence

$$\begin{aligned} [W \otimes_{\mathbf{Z}C_{np^2}} T_{\mathfrak{P}}]_{\hat{S}_0\Gamma} &= [(V \otimes_{\mathbf{Z}} S) \otimes_{\mathbf{Z}C_n \times C_{p^2}} (\mathbf{Z}[\theta]_{\mathfrak{p}_0} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\xi])] \\ &= [(V \otimes_{\mathbf{Z}C_n} \mathbf{Z}[\theta]_{\mathfrak{p}_0}) \otimes_{\mathbf{Z}_p} (S \otimes_{\mathbf{Z}C_{p^2}} \mathbf{Z}_p[\xi])] \\ &= f\left(\frac{p^2 - p + 2}{2}\right) [\hat{S}_0\Gamma] - 2f[\hat{\Phi}_0], \end{aligned}$$

where $f = \text{rk}_{\mathbf{Z}}(\mathbf{Z}[\theta]_{\mathfrak{p}_0}) = \dim_{\mathbf{F}_p}(\mathbf{Z}[\theta]_{\mathfrak{p}_0}) = \dim_{\mathbf{F}_p}(\mathbf{T}/\mathfrak{P})$. Now $\dim_{\mathbf{F}_p}(\hat{S}_0/\hat{\Phi}_0) = 1$ so, by lemma 3 (iii),

$$\begin{aligned} [W \otimes_{\mathbf{Z}_{np^2}} \mathbf{T}_{\mathfrak{P}}]_{\mathbf{T}_{\mathfrak{P}}\Gamma} &= s[\mathbf{T}_{\mathfrak{P}}\Gamma] - 2[\mathfrak{P}_{\mathfrak{P}}] \quad \text{for some } s \\ &= 2[\mathbf{T}_{\mathfrak{P}}] \quad \text{as its T-rank is 2.} \end{aligned}$$

Hence

$$\begin{aligned} \text{Nrd}_0(W \otimes_{\mathbf{Z}_G} (\mathfrak{N} \otimes \mathfrak{N}(\mathfrak{P}))) &\times \text{Nrd}_0(W \otimes_{\mathbf{Z}_G} (\mathfrak{N} \oplus \mathfrak{N}(\mathbf{T})))^{-1} \\ &= \text{Nrd}_0(W \otimes_{\mathbf{Z}_G} \mathfrak{N}(\mathfrak{P})) \times \text{Nrd}_0(W \otimes_{\mathbf{Z}_G} \mathfrak{N}(\mathbf{T}))^{-1} \\ &= \delta_{\mathfrak{P}}([W \otimes_{\mathbf{Z}_G} \mathbf{T}\Gamma] - [\mathbf{T}\Gamma]) \quad \text{by Lemma 2 (i)} \\ &= \delta_{\mathfrak{P}}([W \otimes_{\mathbf{Z}_{np^2}} \mathbf{T}] - [\mathbf{T}\Gamma]) = \delta_{\mathfrak{P}}([\mathbf{T}] - [\mathfrak{P}]) \\ &\quad \text{by Lemma 2 (iv)} \\ &= (\mathfrak{p}^{-1}) \quad \text{by Lemma 2 (v)} \\ &\neq 0. \end{aligned}$$

(ii) The kernel of the epimorphism, $s \otimes 1 \mapsto (\text{tr } s)$

$$S \otimes_{\mathbf{Z}_{np^2}} \mathbf{Z}[\zeta^p] \rightarrow \text{tr } S / (S_0 \cap \text{tr } S)$$

is a torsion group as the ranks of the two modules are equal. The image module is, however, torsion-free and so

$$\begin{aligned} [(S \otimes_{\mathbf{Z}_{p^2}} \mathbf{Z}[\zeta^p]) / (\text{torsion})]_{\hat{S}_0\Gamma} &= [\text{tr } \hat{S} / \text{tr } \hat{S} \cap \hat{S}_0] = [\pi_1^{2p-1} \hat{S}_1] + [\pi^2 \hat{S}_0] \\ &= \frac{p-3}{2} [\hat{S}_0\Gamma] + 2[\hat{\Phi}_0] \dots \quad (2) \end{aligned}$$

Hence

$$\begin{aligned} [W \otimes_{\mathbf{Z}_{np^2}} \mathbf{T}_{1, \mathfrak{P}_1} / (\text{torsion})]_{\hat{S}_0\Gamma} &= [\mathbf{Z}[\theta]] \otimes_{\mathbf{Z}} \left(\frac{p-3}{2} [\hat{S}_0\Gamma] + 2[\hat{\Phi}_0] \right) \\ &\quad \text{(cf (1))} \\ &= \frac{p-3}{2} f[\hat{S}_0\Gamma] + 2f[\hat{\Phi}_0]. \end{aligned}$$

Hence

$$\begin{aligned} [W \otimes_{\mathbf{Z}_{np^2}} \mathbf{T}_{1, \mathfrak{P}_1} / (\text{torsion})]_{\mathbf{T}_1 \Gamma_{\mathfrak{P}_1}} &= s[(\mathbf{T}_1\Gamma)_{\mathfrak{P}_1}] + 2[\mathfrak{P}_{1, \mathfrak{P}_1}] \\ &\quad \text{by Lemma 3 (iii)} \\ &= 2[\mathfrak{P}_{1, \mathfrak{P}_1}] \quad \text{as the rank is 2} \dots \quad (3) \end{aligned}$$

Hence

$$\begin{aligned} &\text{Nrd}_0(W \cdot (\mathfrak{N}_1 \oplus \mathfrak{N}(\mathfrak{P}_1))) \cdot \text{Nrd}_0(W \cdot (\mathfrak{N}_1 \oplus \mathfrak{N}(\mathbf{T}_1)))^{-1} \\ &= \text{Nrd}_0(W \otimes_{\mathbf{Z}_G} \mathfrak{N}(\mathfrak{P}_1) / \text{torsion}) \cdot \text{Nrd}_0(W \otimes_{\mathbf{Z}_G} \mathfrak{N}(\mathbf{T}_1) / \text{torsion})^{-1} \end{aligned}$$

$$\begin{aligned}
&= \delta_{\mathfrak{P}_1}([W \otimes_{\mathbb{Z}C_{np^2}} T/\text{torsion}] - [T\Gamma]) \quad (\text{as } \otimes_{\text{TR}} \mathfrak{M}(I) \text{ is exact}) \\
&= \delta_{\mathfrak{P}_1}([\mathfrak{P}_1] - [T_1]) = (\mathfrak{p}_1) \quad \text{by Lemma 2 (iv) and (v)} \\
&\neq 0.
\end{aligned}$$

Note that, from (2), we deduce easily, using lemma 3, that $S \cdot \mathbb{Z}[\xi^p] \cong_{\mathbb{Z}[\xi^p]\Gamma} (1 - \xi) \oplus (1 - \xi) \not\cong \mathbb{Z}[\xi^p]\Gamma$ showing that, in the notation above with $\Delta = D_{2p^2}$, $\hat{S} \cdot \mathcal{H}_p \not\cong \mathcal{H}_p$. Also, of course, from (3) we have that, with $\Delta = G$, $W \cdot \mathcal{H}_p \not\cong \mathcal{H}_p$.

BIBLIOGRAPHY

- [1] E. ARTIN and J. TATE, *Class Field Theory*, Benjamin, New York, 1967.
- [2] J. COUGNARD, Un Contre-exemple à une conjecture de J. Martinet, in 'Algebraic Number Fields', ed. Fröhlich, Acad. Press, New York, 1977.
- [3] J. COUGNARD, Une propriété de l'anneau des entiers des extensions galoisiennes non-abéliennes de degré pq des rationnels, to appear in *Compositio Mathematica*.
- [4] A. FROHLICH, Locally free modules over arithmetic orders, *J. Reine Angew. Math.*, 274/275 (1975), 112-138.
- [5] A. FROHLICH, Galois Module Structure, in *Algebraic Number Fields*, ed. Fröhlich, Acad. Press 1977.
- [6] I.N. ROSEN, Representations of twisted group rings, Thesis, Princeton University, 1963.
- [7] S.M.J. WILSON, K-Theory for twisted group rings, *Proc. London Math. Soc.*, (3) 29 (1974), 257-271.

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