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CARTAN'S BALAYAGE THEORY FOR HYPERBOLIC RIEMANN SURFACES

par R. E. EDWARDS.

§ 1. — Introduction.

Cartan's general theory of balayage ([1], [2], [3]), as applied to Newtonian and Rieszian potentials in R^m, depends on foundations which split broadly into two categories

(I) The theorem of Evans-Vasilesco-Frostman, Frostman's Maximum Principle, Riesz's Representation Theorem, and various corollaries of these.

(II) The positive definite character of the energy integral, and the completeness of the space of positive measures of finite energy.

In [2] the two categories are not kept entirely separate; for example, the first part of (II) combines with the Riesz Theorem to give the Maximum Principle in (I). However, we shall find it convenient to make the separation, which is certainly legitimate.

The first part of (II) amounts to the validity of the Schwarz-like inequality

$$\text{(S)} \qquad (\mu, \nu) \leqslant ||\mu|| \cdot ||\nu||$$

for positive measures μ and ν, where

$$(\mu, \nu) = \int U^{\mu} d\nu = \int U^{\nu} d\mu = \iint g(x, y)\, d\mu(x)\, d\nu(y),$$
$$||\mu|| = +\sqrt{(\mu, \mu)},$$

and g is the appropriate kernel.

In the cases mentioned, (S) can be proved at the outset, independently of results of the category (I), by using the group structure and convolutions; concerning this, see also Ohtsuka [4].

The possibility of extending Cartan's method has been considered by several writers. In particular, Bader [5] and Parreau [6] assert that the theory can be extended to the case of Greenian potentials on a hyperbolic Riemann surface. The present writer feels the need for some details regarding this, and it is to this question that the present paper is devoted.

Results of category (I) do indeed go over without essential change. But it would appear that a fundamental change in tactics is required to cope with category (II). There is now apparent no method of establishing (S) at the outset. Failing anything better, it has been found necessary to develop a very primitive form of balayage theory, based upon (I) in a manner similar to the earlier theories of Frostman and de la Vallée Poussin, and to use this at the earliest possible moment to yield (S).

Thus, it will be assumed that results in category (I) have been established (see the beginning of § 2 for some comments on this) and a bee-line is then set for a proof of (S). We shall at no place turn aside to develop the initial balayage theory, since it is certain that Cartan's theory, once obtained, is more powerful.

For the case of quite general convolution kernels on R^m, Ninomiya [8], [9] has given a complete and elegant discussion of the positive-definite character of the energy integral in relation to results of category (I). There is apparent no reason why his methods should not be adapted to the case of Greenian potentials, thus deriving an alternative approach for this case. I am grateful to Mr. R. F. Hoskins for drawing my attention to Ninomiya's work.

§ 2. — Preliminaries.

In all that follows, X denotes a hyperbolic Riemann surface with Green's function g. The potential U^μ of a positive (Radon) measure μ on X is the function.

$$U^\mu(x) = \int g(x, y)\, d\mu(y),$$

the mutual energy (μ, ν) being defined as in § 1.

Regarding results in category (I), we refer the reader to the rapid survey in ([6], pp. 124-7); in relation to the Maximum Principle, see also Ninomiya's adaptation of Y. Yosida's proof ([7], pp. 2-3), which may in turn be adapted to the present case.

The minimising principles to which we revert depend only on perfectly general results in integration theory. Into the set of measures on X we introduce the vague topology ([2], § 1). It is well known that a set $\mathfrak{S}$ of measures is relatively vaguely compact if and only if, for each compact $K \subset X$,

$$\mathrm{Sup}_{\mu \in \mathfrak{S}} |\mu|(K) < +\infty.$$

Moreover, since g is lower semicontinuous on $X \times X$, the mutual energy (μ, ν) is lower semicontinuous for the product of the vague topology on the set of *positive* measures.

The proofs (though not the statements) of Theorems A and A′ below involve mixed measures of finite energy. These are handled with ease only *after* (S) is proved, prior to which their manipulation is somewhat tedious. The definition we work with initially is that a real measure λ has finite energy if and only if

$$\|\lambda^+\|, \qquad \|\lambda^-\|, \qquad (\lambda^+, \lambda^-)$$

are all finite, in which case we define

$$\|\lambda\|^2 = \|\lambda^+\|^2 + \|\lambda^-\|^2 - 2(\lambda^+, \lambda^-)$$

the positive measures λ^+ and λ^- being those obtained in the minimal decomposition of λ.

For our restricted purposes, we need only the concept of interior capacity $c(E)$ for Borel sets $E \subset X$:

$$c(E) = 1/\mathrm{Inf}\, \|\mu\|^2$$

μ ranging over all positive measures of total mass one which are concentrated on E; if this set of measures is empty, we define $c(E) = 0$. A property of points of X is said to hold p.p.p. if the set where it fails to hold has zero interior capacity.

§ 3. — Basic Theorems.

For the proofs of Theorems A and A′ we revert to Frostman's methods [10].

THEOREM A. — *Suppose* $K \subset X$ *is compact and* $c(K) > 0$, *and suppose given a measure* $\alpha \geqslant 0$ *with* U^α *continuous, together with a number* $m > 0$. *Then*

(i) $G^\alpha(\nu) = ||\nu||^2 - 2(\alpha, \nu)$ *realises its finite minimum* $g(\alpha, m, K)$ *on the set* $\mathfrak{M}_m(K)$ *of measures* $\nu \geqslant 0$ *which are concentrated on* K *and have total mass* m;

(ii) *if we define*

$$(3.\ 1) \quad k = k(\alpha, m, K) = m^{-1}\{g(\alpha, m, K) + (\alpha, \mu)\}$$

μ *being any minimising measure, then*

$$(3.\ 2) \qquad U^\mu \leqslant U^\alpha + k \qquad \textit{everywhere}$$
$$(3.\ 3) \qquad U^\mu = U^\alpha + k \qquad \textit{p.p.p. on } K.$$

THEOREM A′. — *Suppose* K *and* α *are as in Theorem* A. *Then*

(i) $G^\alpha(\nu)$ *realises its finite minimum* $g(\alpha, K)$ *on the set* $\mathfrak{M}(k)$ *of measures* $\nu \geqslant 0$ *concentrated on* K;

(ii) *if* μ *is any minimising measure,* (3. 2) *and* (3. 3) *hold provided we take* $k = 0$ *when* $\alpha = 0$ *and otherwise* $k = k(\alpha, m, K)$ *with* $m = \int d\mu$.

Proofs. Consider first the existence of minimising measures. In either case, $G^\alpha(\nu)$ is lower semicontinuous for the vague topology. $\mathfrak{M}_m(K)$ is vaguely compact. This suffices to ensure the existence of a minimising measure in Theorem A, and also the finiteness of $g(\alpha, m, K)$ and of $||\mu||^2$.

For Theorem A′, if we put $a = \mathrm{Sup}_{x \in K}\, U^\alpha(x)$ and $N = \int d\nu$, then

$$(3.\ 4) \qquad G^\alpha(\nu) \geqslant N^2/c(K) - 2aN \geqslant -a/^2c(K)$$

so that $g(\alpha, K) > -\infty$. Since also $c(K) > 0$, $g(\alpha, K) < +\infty$. Thus $g(\alpha, K)$ is finite. Take now any sequence (ν_p) for which $G^\alpha(\nu_p) \to g(\alpha, K)$. By (3. 4) it follows that $N_p = \int d\nu_p$ remains bounded. Consequently (ν_p) admits a vague limiting point μ. This μ is then minimising. Moreover, it is easily shown that, if $\alpha \neq 0$, then $\mu \neq 0$ and so $m = \int d\mu > 0$.

The proof of (3. 2) and (3. 3) is common to both theorems and proceeds exactly as in Section 17 of [10]. As has been said, the arguments involve mixed measures of finite energy and a little care must be taken to justify their manipulation.

The next step is to use Theorem A to establish a uniqueness theorem, which will in turn reflect upon Theorems A and A'.

THEOREM B. — *Let* K *be as in Theorem* A, *and let* λ *be any positive measure with* $U^\lambda \not\equiv +\infty$. *Suppose that* $\mu_i (i = 1, 2)$ *are positive measures of finite energy concentrated on* K *which satisfy*

(i) $\int d\mu_1 = \int d\mu_2$;

(ii) *there are numbers* $k_i (i = 1, 2)$ *such that*

$$U^{\mu_i} = U^\lambda + k_i \qquad (i = 1, 2) \tag{3. 5}$$

p.p.p. on K.

Then

$$\mu_1 = \mu_2.$$

Proof. On the basis of Theorem A we select, for each $\alpha \geqslant 0$ having a compact support and continuous potential, a minimising measure α', m being taken to be unity. (Actually, it is enough to do this for countably many suitably selected α'*s*.) We write $k_\alpha = k(\alpha, 1, K)$, $m = \int d\mu_i$. Since $\int d\alpha' = 1$, (3. 3) and (3. 5) yield

$$\begin{aligned}\int U^\alpha d\mu_i &= \int U^{\alpha'} d\mu_i + k_\alpha \int d\mu_i = \int U^{\mu_i} d\alpha' + mk_\alpha \\ &= \int U^\lambda d\alpha' + k_i + mk_\alpha,\end{aligned}$$

all the integrals being finite.

By subtraction, this yields

$$\int (U^{\mu_1} - U^{\mu_2})\, d\alpha = k_1 - k_2.$$

Choose a point $a \in X$ at which both U^{μ_i} are finite, and take $\alpha = \varepsilon_{a,r}$, the Greenian measure on the level curve with équation $g(x, a) = 1/r$ defined by

$$d\varepsilon_{a,r} = (2\pi)^{-1} \partial g(x, a)/\partial\nu \,.\, ds.$$

As a corollary of the Riesz Representation Theorem one may show that

$$\int U^{\mu_i} d\varepsilon_{a,r} \to 0 \quad (r \to \infty).$$

It appears thus that $k_1 = k_2$, and so

$$\int U^{\alpha}\, d(\mu_1 - \mu_2) = 0$$

for all α, and this suffices to show that $\mu_1 - \mu_2 = 0$.

COROLLARY. — *The minimising measure in Theorem A is uniquely determined by* K, α *and* m.

As another application of Theorem A we may introduce the *capacitary measure* $\varkappa$ of a compact K : if $c(\mathrm{K}) = 0$, we define $\varkappa = 0$; if $c(\mathrm{K}) > 0$, we apply Theorem A with $\alpha = 0$ and $m = 1$ to derive a (unique) minimising measure μ, and we define $\varkappa = c(\mathrm{K}).\mu$. Thus in all cases $\varkappa$ is concentrated on K, $U^{\varkappa} \leqslant 1$ everywhere, $U^{\varkappa} = 1$ p.p.p. on K, and $\|\varkappa\|^2 = c(\mathrm{K}) = \int d\varkappa$. In terms of this $\varkappa$, the above three theorems may be supplemented by three more assertions :

(*a*) *If in Theorem B the hypothesis* (i) *is dropped, the conclusion is that* $\mu_1 - \mu_2$ *is a multiple of* $\varkappa$.

(*b*) *Any minimising measure in Theorem* A′ *differs from that in Theorem* A *by a multiple of* $\varkappa$.

(*c*) *In place of* (3. 1) *one may write*

$$(3.\ 1') \qquad k(\alpha, m, \mathrm{K}) = c(\mathrm{K})^{-1}\Big\{m - \int U^{\varkappa}\, d\alpha\Big\}.$$

§ 4. — Balayage.

We shall need no more than the possibility of balayage onto compact or open subsets of X of measures having continuous potentials. The compact case is dealt with first, and this as a direct application of Theorem A.

THEOREM C. — *Let* $\mathrm{K} \subset \mathrm{X}$ *be compact, and let* α *be a positive measure with* U^{α} *continuous. There exists precisely one positive measure* $\alpha' = \alpha'_{\mathrm{K}}$ *with the following properties :*

(B_1) α' *is concentrated on* K and $\|\alpha'\| < +\infty$;

(B_2) $U^{\alpha'} = U^{\alpha}$ *p.p.p. on* K;

(B_3) $\int d\alpha' = \int U^{\varkappa}\, d\alpha$.

This α' *satisfies also*

(B_4) $U^{\alpha'} \leqslant U^{\alpha}$ *everywhere.*

Proof. — If $c(K) = 0$, the only possible choice is $\alpha' = 0$. Otherwise, apply Theorem A with m taken equal to $\int U^{x} d\alpha$ and recall (3. 1'). Uniqueness is settled by Theorem B.

Possession of Theorem C alone is sufficient to enable us to prove (S). However, the proof of completeness of the set of positive measures of finite energy has not been made independent of the concept of balayage onto an open set. The minimum requirements in this connection will now be given.

Let Ω be any non-empty open set in X, and let α be a positive measure with a continuous potential. We shall define α'_{Ω}, the result of balayage of α onto Ω, by the requirement that $U^{\alpha'_{\Omega}}$ shall be the upper envelope of $U^{\alpha'_{K}}$ as K ranges over all compact subsets of Ω. This definition is justified by use of the Riesz Representation Theorem: any positive superharmonic function which minorises a potential $\not\equiv +\infty$ is itself a potential. Moreover using properties (B_2) and (B_4), together with elementary mean-value properties of superharmonic functions, it is seen that

$$U^{\alpha'_{\Omega}} \leqslant U^{\alpha} \quad \text{everywhere,} \qquad U^{\alpha'_{\Omega}} = U^{\alpha} \quad \text{on} \quad \Omega.$$

It is also clear that, for a fixed α, $U^{\alpha'_{\Omega}}$ is increasing with Ω.

The only other properties of the balayage process we shall need relate to regular points. If A is either compact or open, a point p of X is said to be *regular* (for the balayage onto A) if and only if

$$U^{\alpha'_{A}}(p) = U^{\alpha}(p)$$

for all α. It is easily seen that the regular points necessarily belong to $\bar{A}$, and that each interior point of A is regular. Local criteria for regularity of frontier points (Wiener) may be established, whence it appears that all frontier points are regular provided the frontier is sufficiently smooth. In this case, assuming still that U^{α} is continuous, it follows that $U^{\alpha'_{A}}$ will be continuous.

§ 5. — Proof of (S).

Let $\mathcal{E}$ denote the set of positive measures of finite energy, $\mathcal{E}(K)$ the subset of those concentrated on a given compact set $K \subset X$.

The following proof of (S) is dependent on the possibility of balayage onto compact sets only.

Let K be any compact with $c(\mathrm{K}) > 0$: the emphasis is in fact on « arbitrarily large » K. We apply Theorem A′, denoting by $\alpha^\star$ any one minimising measure so obtained. According to (*b*) of § 3, $\alpha^\star = \alpha' + c\varkappa$ for some number c. We begin by verifying that: *if* $\alpha \in \mathcal{E}$, *then* $\alpha^\star = \alpha'$. For one may now write $\mathrm{G}^\alpha(\nu) = ||\nu - \alpha||^2 - ||\alpha||^2$, so that $\alpha^\star$ minimises $||\nu - \alpha||$ as ν ranges over $\mathcal{E}(\mathrm{K})$. Since $\alpha' \in \mathcal{E}(\mathrm{K})$, it follows that

$$||\alpha' - \alpha||^2 \geqslant ||\alpha^\star - \alpha||^2.$$

The right hand side here is $||\alpha' + c\varkappa - \alpha||^2$, which can be verified to equal

$$||\alpha' - \alpha||^2 + 2c(\alpha' - \alpha,\ \varkappa) + c^2||\varkappa||^2$$

Thus

$$2c(\alpha' - \alpha,\ \varkappa) + c^2||\varkappa||^2 \leqslant 0.$$

Now (B_2) shows that $(\alpha' - \alpha, \varkappa) = 0$. Hence $c^2||\varkappa||^2 \leqslant 0$ and so $c = 0$. Thus $\alpha^\star = \alpha'$.

It appears then that

$$||\nu - \alpha|| \geqslant ||\alpha' - \alpha|| \qquad \text{for} \qquad \nu \in \mathcal{E}(\mathrm{K}).$$

Thanks to (B_4), $||\alpha' - \alpha|| \geqslant 0$. Hence

$$||\nu - \alpha|| \geqslant 0$$

if $\alpha, \nu \in \mathcal{E}(\mathrm{K})$ and U^α is continuous. Replacing α by $a\alpha$ ($a = a$ number $\geqslant 0$), this gives

$$||\nu||^2 - 2a(\alpha,\ \nu) + a^2||\alpha||^2 \geqslant 0,$$

and this is trivially true for $a < 0$. Accordingly

$$(\alpha,\ \nu)^2 \leqslant ||\alpha||^2 \,.\, ||\nu||^2$$

Thus (S) is established for measures in $\mathcal{E}(\mathrm{K})$ having continuous potentials, K being any sufficiently large compact set. Both restrictions are easily removed by approximating a general U^μ ($\mu \in \mathcal{E}$) by a monotone sequence of potentials U^{μ_n} which are continuous and generated by measures μ_n having compact supports.

§6. — Completeness of $\mathcal{E}$.

Having proved (S), we now know that the space $\overline{\mathcal{E}}$ of mixed measures of finite energy is a pre-Hilbert space with the scalar product (μ, ν). The cornerstone of Cartan's theroy is the theroem that $\mathcal{E}$ is complete, even though $\overline{\mathcal{E}}$ is not generally so. To prove the theorem in the present case, one may modify the arguments given by Cartan (see especially § 5 of [2]). Of these arguments, all save perhaps one require only verbal changes. The miscreant is the assertion that *the* $\mu \in \overline{\mathcal{E}}$, *for which* U^μ *is continuous and has a compact support, are dense in* $\overline{\mathcal{E}}$. Cartan's proof of this depends on the fact the appropriate kernel function tends to zero nicely at infinity (loc. cit., passage preceding the converse of Proposition 3). This is no longer true in general for the Green's function of X. Nevertheless, the result remains true. The following proof uses the possibility of balayage onto non-r elatively-compact open sets, and it would be desirable to have a more direct proof.

Cartan's Proposition 4 remains true, the proof being exactly the same. It will suffice to show that any $\mu_0 \in \mathcal{E}$ is the (strong) limit of $\mu \in \overline{\mathcal{E}}$ for which U^μ is continuous and has a compact support. Now, by Proposition 4 just cited, μ_0 is certainly the limit of μ having compact supports and continuous potentials. Hence it will certainly suffice to show that if $\mu \in \mathcal{E}$ and U^μ is continuous, then μ is the limit of $\mu_n \in \overline{\mathcal{E}}$ for which U^{μ_n} is continuous and has a compact support. For this, exhaust X by an increasing sequence of compact sets K_n with smooth frontiers, and put $\Omega_n = X - K_n$. Let ν_n be the measure obtained by balayage of μ onto Ω_n, $\mu_n = \mu - \nu_n$. Then U^{ν_n} is continuous (see end of § 4 above). Since $U^\mu = U^{\nu_n}$ on Ω_n, U^{μ_n} has a compact support. It remains only to show that $||\nu_n|| \to 0$, i.e., in view of Cartan's Proposition 4, that $U^{\nu_n} \to 0$. Now $V = \lim\limits_{n \to \infty} U^{\nu_n}$ is harmonic and positive on X, and it is clearly majorised by U^μ. It follows that $V = 0$ ([6], Théorème 1 *bis*), and the proof is complete.

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