ANNALES DE L'INSTITUT FOURIER

JOEL H. SHAPIRO

Zeros of random functions in Bergman spaces

Annales de l'institut Fourier, tome 29, nº 4 (1979), p. 159-171
<http://www.numdam.org/item?id=AIF_1979__29_4_159_0>

Ann. Inst. Fourier, Grenoble
29, 4 (1979), 159-171.

ZEROS OF RANDOM FUNCTIONS IN BERGMAN SPACES

by Joel H. SHAPIRO (*)

1. Introduction.

Let μ be a finite, positive, rotation invariant Borel measure on the open unit disc Δ of the complex plane, and suppose that μ gives positive mass to each annulus $r < |z| < 1$. For $0 < p < \infty$ the *weighted Bergman space* A_μ^p is the collection of functions f holomorphic in Δ with

$$\|f\|_p^p = \int |f|^p \, d\mu < \infty .$$

Let $A_\mu^{p+} = \bigcup_{q>p} A_\mu^q$, so $A_\mu^{p+} \subset A_\mu^p$. For f holomorphic in Δ, let $Z(f)$ denote the *zero set* of f, with each zero counted according to its multiplicity. If f belongs to some class $\mathfrak{F}$ of holomorphic functions we frequently refer to $Z(f)$ as an *$\mathfrak{F}$-zerò set.*

Recently we showed [7] that for each such μ and p *there exists f in A_μ^p such that :*

(*a*) $Z(f)$ *is contained in no* A_μ^{p+} *zero set, and*

(*b*) $Z(f+1) \cup Z(f-1)$ *lies in no* $A_\mu^{(p/2)+}$ *zero set, hence in no* A_μ^p *zero set.*

These results continued the work of Charles Horowitz [2] and Walter Rudin [4]. Horowitz considered the special measures

$$d\mu(z) = (1-|z|)^\alpha \, dx \, dy \quad (\alpha > -1),$$

(*) Research partially supported by the National science Foundation.

and used infinite products to construct the desired functions; while Rudin got similar results for Hardy spaces on the unit ball and polydisc in $\mathbf{C}^n$ by means of an ingenious « multiplier argument ». The proof in [7] used Rudin's idea, with the desired function f constructed as a gap series.

The point of this paper is that Rudin's method (which we will describe in the next section) also works very naturally in the context of *random* power series. We show that *a Gaussian power series which almost surely lies in* $A^p_\mu \backslash A^{p+}_\mu$ *must almost surely have properties* (*a*) *and* (*b*) *listed above.*

More precisely, let $(\zeta_n)_0^\infty$ be a sequence of independent complex Gaussian random variables with mean zero and variance one [3; Ch. 9, sec. 3, p. 118]. Suppose $(a_n)_0^\infty$ is a sequence of complex numbers with

$$\limsup_{n\to\infty} |a_n|^{1/n} \leqslant 1, \tag{1.1}$$

and consider the random power series

$$f(z) = \sum_{n=0}^{\infty} \zeta_n a_n z^n . \tag{1.2}$$

Since almost surely $|\zeta_n| = 0(\sqrt{\log n})$ [3; Ch. XI, sec. 4, p. 121, Prop. 3], condition (1.1) insures that with probability one the series (1.2) converges uniformly on compact subsets of Δ to a holomorphic function. The quantity which controls the random behavior of f is its variance $\sigma_f^2(z)$, defined for $z \in \Delta$ by

$$\sigma_f^2(z) = \mathscr{E}\{|f(z)|^2\} = \sum_{n=0}^{\infty} |a_n|^2 |z|^{2n} . \tag{1.3}$$

The main result of this paper is the following.

THEOREM 1. — *Suppose* f *is defined by formulas* (1.1) *and* (1.2). *Then*

(*a*) *the following are equivalent :*

(i) $\sigma_f \in L^p(\mu)$ *but* $\notin L^{p+}(\mu)$.

(ii) *With probability one :* $f \in A^p_\mu$ *but* $\notin A^{p+}_\mu$.

(iii) *With probability one :* $f \in A^p_\mu$ *but* $Z(f)$ *is not contained in any* A^{p+}_μ *zero set.*

(*b*) *If any (hence all) of the above conditions hold, then with probability one :* $Z(f+1)$ *and* $Z(f-1)$ *are* A_μ^p *zero sets, but their union is not even an* $A_\mu^{(p/2)+}$ *zero set.*

The most important of these results are (*b*), and the implication (i) → (iii) of (*a*) : these imply the corresponding results in [7]. For their proof we require only the most basic facts about Gaussian random variables. The other non-trivial implication in (*a*) is (ii) → (i), which follows from a beautiful result of X. Fernique concerning moments of vector valued Gaussian random variables. These matters, Rudin's multiplier argument, and some other preliminaries are reviewed in section 2. Theorem 1 is proved in the third section, and the paper closes with some remarks and open problems.

I want to thank my colleague Joel Zinn of Michigan State University for several interesting discussions, and especially for pointing out Fernique's theorem to me.

2. Preliminaries.

(*a*) *Rudin's multiplier argument.* As exploited in both this paper and [7], Rudin's idea is this : if the zero set of $f \in A_\mu^p \backslash A_\mu^{p+}$ is contained in some A_μ^{p+} zero set, then $fh \in A_\mu^{p+}$ for some h holomorphic in Δ. Since h decreases the growth of f, it must have relatively small values where f is large. Assuming (without loss of generality) that $h(0) = 1$, we obtain from the subharmonicity of h :

$$0 = \log |h(0)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log |h(re^{i\theta})| \, d\theta \tag{2.1}$$

for $0 \leqslant r < 1$, which forces h on the circle $|z| = r$ to balance out any small values with appropriate large ones. Therefore if $f \in A_\mu^p$ does not get into A_μ^{p+} because it has large values on substantial portions of certain circles $|z| = r_n (r_n \to 1-)$, then we should expect that no h holomorphic in Δ can multiply f into A_μ^{p+}. We will show in the next section that any Gaussian series (1.2) which almost surely lies in $A_\mu^p \backslash A_\mu^{p+}$ will almost surely be such an f. This complements the work in [7] where such f's were constructed as gap series.

(*b*) *Gaussian random variables.* The reference for all of this material is [3 ; Ch. XI, sec. 1-4]. From now on $(\zeta_n)_0^\infty$ denotes a sequence of independent complex Gaussian random variables with mean zero and variance one, defined on a probability space $(\Omega, \mathfrak{F}, \mathrm{Pr})$. In particular, (ζ_n) is an orthonormal sequence in $L^2(\Omega,\mathfrak{F},\Pr)$, and for each Borel subset B of the complex plane :

$$\Pr\{\zeta_n \in B\} = \frac{1}{2\pi}\iint_B e^{-(x^2+y^2)/2}\,dx\,dy.$$

From this it follows quickly that for $0 \leqslant \lambda < \infty$,

$$\Pr\{|\zeta_n| > \lambda\} = e^{-\pi\lambda^2}$$

[3; Ch. XI, sec. 4, p. 121, formula (3.1)]. A crucial property of the sequence (ζ_n) is that if (a_n) is a complex sequence with $\|a\|_2^2 = \Sigma\,|a_n|^2 < \infty$, and if $Z = \Sigma a_n\zeta_n$, then the random variable $Z/\|a\|_2$ has the same distribution as ζ_n. In particular :

$$\Pr\{|Z| > \lambda\|a\|_2\} = e^{-\pi\lambda^2}, \tag{2.2}$$

and for $0 < p < \infty$:

$$\mathscr{E}\{|Z|^p\} = C_p^p(\mathscr{E}\{|Z|^2\})^{p/2} = C_p^p\|a\|_2^p, \tag{2.3}$$

where C_p is independent of (a_n), and $\mathscr{E}$ denotes integration with respect to Pr. These are the only properties of (ζ_n) that we require for the main part of the proof of Theorem 1.

We remark in passing that the statement « f has property Q with probability one » (or « almost surely ») means that there exists $E \in \mathfrak{F}$ with $\Pr\{E\} = 1$ such that f has property Q for every $\omega \in E$. We do not require $\{\omega \in \Omega : f \text{ has property } Q\}$ to belong to $\mathfrak{F}$. Similar remarks apply to statements like « with probability $\geqslant \delta$, f has property Q ».

(c) *Interchanging measure and probability.* Some form of the next result occurs frequently in applications of probability to analysis.

LEMMA A [3; Ch. V, sec. 4, p. 42]. – *Suppose* $(\Omega,\mathfrak{F},P)$ *and* $(T,\mathfrak{B},m)$ *are probability spaces, and* $E \in \mathfrak{F} \otimes \mathfrak{B}$ *(product sigma-algebra). Define the usual cross-sections;*

$$E^\omega = \{t \in T : (\omega,t) \in E\} \quad (\omega \in \Omega)$$
$$E_t = \{\omega \in \Omega : (\omega,t) \in E\} \quad (t \in T),$$

and suppose $0 \leqslant \theta,\ \eta \leqslant 1$. *If* $P\{E_t\} \geqslant \eta$ *for* $[m]$ a.e. t *in* T, *then*

$$P\{\omega \in \Omega : m(E^\omega) \geqslant \theta\eta\} \geqslant \frac{(1-\theta)\eta}{1-\theta\eta}.$$

Proof. – Let $A = \{\omega \in \Omega : m(E^{\omega}) \geqslant \theta\eta\}$. Then by Fubini's theorem :

$$\begin{aligned}\eta &\leqslant \int_T P\{E_t\}\, dm(t)\\ &= \int_\Omega m(E^{\omega})\, dP(\omega)\\ &= \int_A + \int_{\Omega\setminus A} m(E^{\omega})\, dP(\omega)\\ &\leqslant P(A) + \theta\eta[1-P(A)],\end{aligned}$$

and the result follows upon solving for $P(A)$.

(*d*) *Fernique's Theorem.* For $f \in A^p_\mu$ let $\|f\| = \|f\|_p$ if $p \geqslant 1$, and $\|f\|^p_p$ if $0 < p < 1$. Then $\|.\|$ is a norm on A^p_μ if $p \geqslant 1$, and a « p-norm » if $0 < p < 1$ (that is, $\|af\| = |a|^p\|f\|$ when $0 < p < 1$). It is not difficult to use the subharmonicity of $|f.|^p$ to check that for each $z \in \Delta$ the linear functional of « evaluation at z »

$$f \to f(z) \quad (f \in A^p_\mu)$$

is continuous on A^p_μ $(0 < p < \infty)$. From this it follows that A^p_μ, in the metric induced by $\|.\|$, is complete, i.e., it is a Banach space when $p \geqslant 1$ and a « p-Banach space » when $0 < p < 1$. Even when $0 < p < 1$ there are enough continuous linear functionals to separate points (the point evaluations, for example), and the Borel structure induced on A^p_μ by the « norm » topology coincides with the one induced by the topology of uniform convergence on compact subsets of Δ (since the closed unit ball of A^p_μ is closed in this weaker topology).

From these considerations it follows routinely that if (u_n) is a sequence in A^p_μ for which the Gaussian series $Z = \Sigma\zeta_n u_n$ converges almost surely in A^p_μ, then, even when $0 < p < 1$, Z is an A^p_μ-valued Gaussian random variable in the following sense : if Z' and Z'' are independent and similar to Z, then so are $(Z' + Z'')/\sqrt{2}$ and $(Z' - Z'')\sqrt{2}$.

Thus X. Fernique's Theorem [1] (or more precisely when $0 < p < 1$, its proof) applies to Z, and shows that the tail distribution $\Pr\{\|Z\| > \lambda\}$ decays exponentially as $\lambda \to \infty$. In particular,

$$\mathscr{E}\{\|Z\|^p_p\} < \infty, \tag{2.4}$$

which yields the following characterization of Gaussian Taylor series which a.s. belong to A^p_μ.

LEMMA B. — *Suppose* f *and* σ_f *are given by* (1.1)-(1.3). *Then* $f \in A^p_\mu$ *almost surely if and only if* $\sigma_f \in L^p(\mu)$.

Proof. — For any f given by (1.1) and (1.2), we have from (2.3) and Fubini's Theorem :

$$\int \sigma_f^p \, d\mu = C_p^{-p} \int \mathscr{E}\{|f|^p\} \, d\mu = C_p^{-p} \, \mathscr{E}\{\|f\|_p^p\}.$$

Thus $\sigma_f \in L^p(\mu)$ implies $\mathscr{E}\{\|f\|_p^p\} < \infty$, hence $\|f\|_p^p < \infty$ a.s. Conversely, suppose $\|f\|_p^p < \infty$ a.s. We claim f is an A^p_μ-valued Gaussian random variable. Indeed, the fact that the integral means

$$\text{(2.5)} \qquad M_p^p(f; r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$$

increase with r [6; Theorem 17.6, p. 363], along with the monotone convergence theorem, show that the Taylor series (1.2) of f is a.s. Abel summable to f in A^p_μ. Thus by [3; Theorem 1, Ch. II, p. 11] (whose proof works even when $0 < p < 1$), the series converges a.s. in A^p_μ to f; hence f is Gaussian. From (2.4) we see $\mathscr{E}\{\|f\|_p^p\} < \infty$, hence by the calculations at the beginning of this proof, $\sigma_f \in L^p(\mu)$. This completes the proof.

(*e*) *Two technical lemmas.* We close this section with two lemmas needed to deduce Theorem 1 from the essential probabilistic arguments, which will be isolated in Proposition 2 of the next section.

LEMMA C. — *Given* μ *as usual, and* $0 < p < \infty$, *there exists a finite positive rotation invariant Borel measure* ν *whose closed support is* $\{|z| \leqslant 1\}$, *and such that* $A^p_\mu = A^p_\nu$.

Proof. — For f holomorphic in Δ the integral mean $M_p^p(f; r)$ defined by (2.5) increases with r, so if $f \in A^p_\mu$ then

$$\text{(2.6)} \qquad 2\pi M_p^p(f; r) \leqslant \|f\|_p^p \, \mu(r)^{-1}$$

where

$$\mu(r) = \mu\{z : r \leqslant |z| < 1\}.$$

Our standing hypotheses on the measure μ insure that $\mu(r) > 0$ for each $0 < r < 1$, and $\mu(r) \downarrow 0$ as $r \uparrow 1$. In particular μ is a bounded, strictly positive, measurable function on $[0,1)$, hence the measure

$$d\nu(z) = d\mu(z) + \mu(|z|) \, dx \, dy/\pi$$

has closed support equal to $\{|z| \leqslant 1\}$, and dominates μ. Since (2.6) insures that

$$\int_\Delta |f|^p \, d\nu \leqslant 2\|f\|_p^p$$

we see that $A_\mu^p = A_\nu^p$, as desired.

LEMMA D. — *Suppose* γ *is a finite positive Borel measure on the interval* $[0,1)$ *which is either* (i) *purely atomic, or* (ii) *continuous with closed support equal to* $[0,1]$. *The for* $0 < \alpha < 1$:

$$\int_0^1 \gamma[r,1)^{-\alpha} \, d\gamma(r) < \infty . \tag{2.7}$$

Proof. — (i) Suppose γ is purely atomic, say with mass γ_n at r_n $(n = 1,2,\ldots)$, and no mass anywhere else. Let $\rho_n = \sum_{k \geqslant n} \gamma_n$. Then the integral in (2.7) is just the series $\sum \rho_n^{-\alpha} \gamma_n$, whose convergence for $0 < \gamma < 1$ is a standard exercise in advanced calculus (see [5; Ch. 3, pp. 79-80, problem 12(*b*)] for the special case $\alpha = 1/2$).

(ii) If γ is continuous with closed support $= [0,1]$, then the function

$$\gamma(r) = \gamma([r,1]) \quad (0 \leqslant r < 1)$$

is continuous and strictly decreasing on $[0,1)$. The integral in (2.7) can then be interpreted as the Riemann-Stieltjes integral

$$-\int_0^1 \nu(r)^{-\alpha} \, d\nu(r)$$

which, after making the change of variable $x = \nu^{-1}(r)$ (composition inverse), and paying due respect to the singularity at $r = 1$, becomes [5; Theorem 6.19, p. 132]

$$\int_0^1 x^{-\alpha} \, dx < \infty .$$

This completes the proof.

3. Proof of the Main Theorem.

We isolate the essential part of Theorem 1 in the following proposition, which we state in somewhat more generality than actually required. The following notations help the exposition. As in the proof of Lemma C, let

$$\mu(r) = \mu\{z \in \Delta : r \leqslant |z| < 1\}.$$

For b holomorphic in Δ and $0 \leqslant r < 1$, let

$$\mathrm{M}_\infty(b\,;r) = \max\{|b(z)| : |z| = r\}$$

and write

$$b_r(e^{i\theta}) = b(re^{i\theta}).$$

From now on, f always represents a Gaussian power series as given by (1.1) and (1.2), with σ_f given by (1.3). We also assume that the measure μ has total mass 1, so $0 < \mu(r) \leqslant 1$.

PROPOSITION 2. — *Suppose that*

$$\limsup_{r\to 1-} \frac{\sigma_f(r)\mu(r)^{1/p}}{-\log \mu(r)} > 0. \tag{3.1}$$

Then the following holds with probability one : for each positive integer N, *every* b *holomorphic in* Δ *with*

$$\limsup_{r\to 1-} \mathrm{M}_\infty(b\,;r)\mu(r)^{\mathrm{N}/p} < 1,$$

and every h *holomorphic in* Δ, *we have*

$$(f^{\mathrm{N}} + b)h \notin \mathrm{A}_\mu^{p/\mathrm{N}}.$$

Remark. — For part (*a*) of Theorem 1 we need only the case $\mathrm{N} = 1$, $b \equiv 0$, while for part (*b*) we require $\mathrm{N} = 2$, $b \equiv -1$. However, these special cases are no easier to prove than the general proposition, which gives some further information regarding remark (i), section 5 of [7].

Proof. — Let $\mathbf{T}$ denote the unit circle $\{|z| = 1\}$, m normalized Lebesgue measure on $\mathbf{T}$, and μ_1 the unique finite positive Borel measure on $[0,1)$ such that

$$\int_\Delta g\,d\mu = \int_{[0,1)} \left\{\int_{\mathbf{T}} g(rt)\,dm(t)\right\} d\mu_1(r)$$

for each $g \in \mathrm{C}_0([0,1))$.

Fix $k > 0$. We are going to show that the desired result holds with probability at least $k/(k+1)$; hence with probability one, since k is arbitrary. In this regard the reader should note that although the set $\{\omega : f^N \notin A_\mu^{p/N}\}$ is a tail event, the set we are interested in :

$$\{\omega : (f^N + b)h \notin A_\mu^{p/N} \text{ for all } h, b \text{ as in the Proposition}\}$$

is *not* (in fact it is not even clear that it is an event), so the zero-one law does not apply.

According to the hypothesis (3.1) there is a positive number δ and a positive sequence $r_n \to 1-$ such that

$$\sigma_f(r_n)\mu(r_n)^{1/p} \geqslant -\delta \log \mu(r_n).$$

Let $\lambda_n^{-1} = \sigma_f(r_n)\mu(r_n)^{1/p}$, so

$$0 < \lambda_n \leqslant \frac{1}{-\delta \log \mu(r_n)} \to 0. \tag{3.2}$$

For each positive integer n, (2.2) insures that for all $t \in \mathbf{T}$ we have

$$\Pr\{|f(r_n t)| > \lambda_n \sigma_f(r_n)\} = e^{-\pi\lambda_n} = \eta_n, \tag{3.3}$$

where $\eta_n \to 1$ because $\lambda_n \to 0$. Let

$$E(n) = \{(\omega,t) \in \Omega \times \mathbf{T} : |f(r_n t)| > \lambda_n \sigma_f(r_n)\}.$$

Then using the notation of Lemma A, equation (3.3) asserts that $\Pr\{E_t(n)\} = \eta_n$ for every t in $\mathbf{T}$; hence Lemma A, with $\theta = \eta_n^k$, shows that with probability at least

$$\beta_n = \frac{\eta_n(1-\eta_n^k)}{1-\eta_n^{k+1}}$$

we have $m\{E^\omega(n)\} \geqslant \eta_n^{k+1}$ $(n = 1,2,\ldots)$. Since $\beta_n \to k/(k+1)$ as $n \to \infty$, it follows that with probability $\geqslant k/(k+1)$:

$$m\{E^\omega(n)\} \geqslant \eta_n^{k+1} \text{ for infinitely many } n. \tag{3.4}$$

Let $F = \{\omega \in \Omega : (3.4) \text{ holds}\}$; then $\Pr\{F\} \geqslant k/(k+1)$. We are going to show that for each $\omega \in F$;

$$(f^N + b)h \notin A_\mu^{p/N}$$

whenever h, b, N are as in the hypothesis of the proposition. This will complete the proof.

To this end, fix $\omega \in F$ and b, N, and h. Suppose, as we may, that $h(0) = 1$, and choose $0 < \varepsilon < 1$ so that

$$\limsup_{r \to 1-} M_\infty(b; r)\mu(r)^{N/p} < \varepsilon.$$

Letting $R_n = \{r_n \leqslant |z| < 1\}$ we have :

$$\int_{R_n} |(f^N + b)h|^{p/N}\, d\mu = \int_{[r_n,1)} \left\{ \int_T |(f_r^N + b_r)h_r|^{p/N}\, dm \right\} d\mu_1(r)$$

$$\geqslant \int_{[r_n,1)} \exp \left\{ (p/N) \int_T \log |(f_r^N + b_r)h_r| dm \right\} d\mu_1(r)$$

by the arithmetic-geometric mean inequality. Let $I(r)$ denote the integral inside the braces. Then using (2.1) and the fact that $\int \log |g_r|\, dm$ increases with r for any holomorphic function g on Δ [6; Theorems 17.3 and 17.5, pp. 362-363] we obtain for $r_n \leqslant r < 1$:

$$\begin{aligned} I(r) &\geqslant \int_T \log |f_r^N + b_r|\, dm \\ &\geqslant \int_T \log |f_{r_n}^N + b_{r_n}|\, dm \\ &\geqslant \int_{E^\omega(n)} \log ||f_{r_n}|^N - |b_{r_n}||\, dm. \end{aligned}$$

Since $\omega \in F$, this yields for infinitely many n :

$$\begin{aligned} I(r) &\geqslant \int_{E^\omega(n)} \log |[\lambda_n \sigma_f(r_n)]^N - M_\infty(b\,; r_n)| dm \\ &\geqslant m\{E^\omega(n)\} \log [\mu(r_n)^{-N/p} - \varepsilon\mu(r_n)^{-N/p}] \\ &\geqslant \eta_n^{k+1} \log [(1-\varepsilon)\mu(r_n)^{-N/p}] \quad \text{(by (3.4))} \end{aligned}$$

whenever $r_n \leqslant r < 1$. Thus, infinitely often :

$$\begin{aligned} \int_{R_n} |(f^N + b)h|^{p/N}\, d\mu &\geqslant \int_{[r_n,1)} \exp\{(p/N)I(r)\}\, d\mu_1(r) \\ &\geqslant (1-\varepsilon)^{p/N} \mu(r_n)^{1-\eta_n^{k+1}}. \end{aligned}$$

Recalling the definition of η_n :

$$1 - \eta_n^{k+1} = 1 - e^{-(k+1)\pi\lambda_n} \leqslant (k+1)\pi\lambda_n,$$

so

$$\begin{aligned} \mu(r_n)^{1-\eta_n^{k+1}} &\geqslant [\mu(r_n)^{\lambda_n}]^{(k+1)\pi} \\ &\geqslant \{\mu(r_n)^{1/\log \mu(r_n)}\}^{-\delta(k+1)\pi} \quad \text{(by 3.2)} \\ &= e^{-\delta(k+1)\pi}. \end{aligned}$$

Thus for each $\omega \in F$:

$$\limsup_{n\to\infty} \int_{R_n} |(f^N+b)h|^{p/N}\, d\mu \geqslant (1-\varepsilon)^{p/N} e^{-\delta(k+1)\pi} > 0,$$

hence $(f^N+b)h \notin A_\mu^{p/N}$. This completes the proof.

Deduction of Theorem 1. – In part (*a*) the equivalence (i) → (ii) is immediate from Lemma B (section 2), and the implication (iii) → (ii) is trivial. So it remains to show that (i) implies both (iii) and (*b*). In view of Lemma B it is enough to show that if $\sigma_f \notin L^{p+}(\mu)$, then with probability one : $Z(f)$ is not contained in any A_μ^{p+} zero set, and $Z(f+1) \cup Z(f-1)$ is not contained in any $A_\mu^{(p/2)+}$ zero set.

So suppose $\sigma_f \notin L^{p+}(\mu)$. We will show in a moment that this implies

$$\limsup_{r\to 1-} \sigma_f(r)\mu(r)^{1/q} = \infty \tag{3.5}$$

for each $q > p$, which yields :

$$\limsup_{r\to 1-} \frac{\sigma_f(r)\mu(r)^{1/q}}{-\log \mu(r)} = \infty$$

for each $q > p$. Thus Proposition 2 (with q replacing p) guarantees that for each $q > p$ it is almost sure that

$$(f^N+b)h \notin A_\mu^{q/N}$$

for every h holomorphic in Δ, b constant, and $N = 1,2,\ldots$. Since a countable intersection of sets of probability one again has probability one, it follows upon quoting the above result for a sequence $q_n \downarrow p$ that almost surely : $(f^N+b)h \notin A_\mu^{(p/N)+}$ for all b, h, N as above.

Taking $N = 1, b \equiv 0$ we see from the discussion of section 2 (*a*) that a.s. $Z(f)$ is contained in no A_μ^{p+} zero set, which proves (iii). Taking $N = 2$ and $b \equiv -1$ we see that a.s.

$$Z(f^2-1) = Z(f+1) \cup Z(f-1)$$

lies in no $A_\mu^{(p/2)+}$ zero set, which proves (*b*).

It remains only to prove that (3.5) holds for each $q > p$. Suppose not. Then for some $q > p$:

$$\sigma_f(r) = 0(\mu(r)^{-1/q}) \quad (r \to 1-). \tag{3.6}$$

Fix $p < s < q$. We will show that $\sigma_f \in L^s(\mu)$, contrary to the hypothesis on σ_f. By Lemma C we may assume that the measure μ has $\{|z| \leqslant 1\}$ as its closed support, hence the closed support of μ_1 is the interval $[0,1]$. Thus $\mu_1 = \gamma_1 + \gamma_2$, where γ_1 is purely atomic and γ_2 is continuous with closed support $[0,1]$. By (3.6) we have

$$\sigma_f(r) = 0(\gamma_i[r,1)^{-1/q}) \quad (r \to 1 -)$$

for $i = 1,2$; hence by Lemma D,

$$\int_0^1 \sigma_f(r)^s \, d\gamma_i(r) < \infty \quad (i = 1,2),$$

hence $\sigma_f \in L^s(\mu)$: a contradiction. This completes the proof of Theorem 1.

4. Concluding Remarks.

Lemma B suggests that Proposition 2 should be capable of improvement.

CONJECTURE. — *If f is not* a.s. *in* A^p_μ (hence by the zero-one law, a.s. not in A^p_μ), *then* a.s. $(f^N + b)h \notin A^{p/N}_\mu$ *for all* b, h, N *as in the statement of Proposition* 2.

The arithmetic-geometric mean inequality seems to give away too much to get this result : In the case $N = 1$, $b \equiv 0$, Fernique's inequality might be a possibility. It is not difficult to check that if $fh \in A^p_\mu$ a.s. for some fixed holomorphic h in Δ, then fh is an A^p_μ-valued Gaussian random variable. Then Fernique's inequality, the rotational symmetry of $\sigma_f(z)$, and the monotonicity of $M^p_p(h,r)$ yield :

$$\begin{aligned}
\infty &> \mathscr{E}\{\|fh\|^p_p\} \\
&= \int \mathscr{E}\{|fh|^p\} \, d\mu \\
&= \int \mathscr{E}\{|f|^p\}|h|^p \, d\mu \\
&= C^p_p \int \sigma^p_f \, |h|^p \, d\mu \\
&= C^p_p \int_0^1 \sigma_f(r)^p M^p_p(h\,;r) \, d\mu_1(r) \\
&\geqslant C^p_p \int \sigma^p_f \, d\mu,
\end{aligned}$$

hence $f \in A^p_\mu$ a.s. But this merely shows that :

$$f \notin A^p_\mu \text{ a.s.} \Rightarrow \forall h \quad \text{holomorphic in } \Delta; \quad fh \notin A^p_\mu \text{ a.s.}$$

whereas the desired result is :

$$f \notin A^p_\mu \text{ a.s.} \Rightarrow \text{a.s.}: \quad fh \notin A^p_\mu \ \forall h \quad \text{holomorphic in } \Delta.$$

BIBLIOGRAPHY

[1] X. FERNIQUE, Intégrabilité des vecteurs gaussiens, *C.R. Acad. Sci.*, Paris, 270 (1970), 1698-1699.

[2] Ch. HOROWITZ, Zeros of functions in Bergman spaces, *Duke Math. J.*, 41 (1974), 693-710.

[3] J.P. KAHANE, Some Random Series of Functions, D.C. Heath and Co., Lexington, MA, 1968.

[4] W. RUDIN, Zeros of holomorphic functions in balls, *Indag. Math.*, 38 (1976), 57-65.

[5] W. RUDIN, Principles of Real Analysis, 3rd ed., McGraw-Hill, New York, 1976.

[6] W. RUDIN, Real and Complex Analysis, McGraw-Hill, New York, 1974.

[7] J. H. SHAPIRO, Zeros of functions in weighted Bergman spaces, *Michigan Math. J.*, 24 (1977), 243-256.

Manuscrit reçu le 5 juin 1979.

Joel H. SHAPIRO,
Department of Mathematics
Michigan State University
East Lansing, MI 48824 (USA).