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A note on the paper “The Poulsen Simplex” of Lindenstrauss, Olsen and Sternfeld


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A NOTE ON THE PAPER
« THE POULSEN SIMPLEX »
OF LINDENSTRAUSS, OLSEN AND STERNFELD
by Wolfgang LUSKY

It was shown in [5] that there is only one metrizable Poulsen simplex $S$ (i.e. the extreme points $\text{ex } S$ are dense in $S$) up to affine homeomorphism. Thus, $S$ is universal in the following sense: Every metrizable simplex is affinely homeomorphic to a closed face of $S$ ([5], [6]).

The Poulsen simplex can be regarded as the opposite simplex to the class of metrizable Bauer simplices ([5]). There is a certain analogy in the class of separable Lindenstrauss spaces (i.e. the preduals of $L_1$-spaces); the Gurarij space $G$ is uniquely determined (up to isometric isomorphisms) by the following property: $G$ is separable and for any finite dimensional Banach spaces $E \subset F$, linear isometry $T : E \rightarrow G$, $\varepsilon > 0$, there is a linear extension $\hat{T} : F \rightarrow G$ of $T$ with $(1 - \varepsilon)\|x\| \leq \|\hat{T}(x)\| \leq (1 + \varepsilon)\|x\|$ for all $x \in F$. ([3], [7]).

$G$ is universal: Any separable Lindenstrauss space $X$ is isometrically isomorphic to a subspace $X \subset G$ with a contractive projection $P : G \rightarrow X$ ([9], [6]).

Furthermore $G$ is opposite to the class of separable $C(K)$-spaces. There is another interesting property of $G$:

For any smooth points $x, y \in G$ there is a linear isometry $T$ from $G$ onto $G$ with $T(x) = y$. ($x \in G$ is smooth point if $\|x\| = 1$ and there is only one $x^* \in G^*$ with $x^*(x) = 1 = \|x^*\|$).
In their last remark the authors of [5] point out that here the analogy between G and $A(S) = \{f: S \to \mathbb{R} \mid f$ affine continuous$\}$ seems to break down.

The purpose of this note is to show that under the aspect of rotation properties there is still some kind of analogy between G and $A(S)$.

Take $s_0 \in \text{ex} S$ and consider

$$A_0(S; s_0) = \{f \in A(S) \mid f(s_0) = 0\},$$

for any normed space $X$ let $B(X) = \{x \in X \mid \|x\| \leq 1\}$ and $\partial B(X) = \{x \in X \mid \|x\| = 1\}$. In particular

$$\partial B(A(S))_+ = \{f \in \partial B(A(S)) \mid f \geq 0\}.$$

We show:

**Theorem.**

(a) Let $f, g \in \partial B(A(S))_+$ so that $f, 1 - f, g, 1 - g$ are smooth points of $A(S)$. Then there is an isometric isomorphism $T$ from $A(S)$ onto $A(S)$ with

(i) $T(f) = g$

(ii) $T(A_0(S; s_0)) = A_0(S; s_1)$ where $f(s_0) = 0 = g(s_1)$

(iii) $T(1) = 1$

(b) Let $f \in \partial B(A_0(S; s_0))_+$ and $g \in \partial B(A_0(S; s_1))$ so that neither $g \leq 0$ nor $g \geq 0$ hold. Then there is no isometric isomorphism $T$ from $A(S)$ onto $A(S)$ with $T(f) = g$.

(c) The elements $f \in A_0(S; s_0)$, so that $f, 1 - f$ are smooth points of $A(S)$, form a dense subset of $\partial B(A_0(S; s_0))_+$.

The proof of the Theorem which is based on a method used in [5] and [7] is a consequence of the following lemmas and proposition 6. From now on let $s_0 \in \text{ex} S$ be fixed and set $A_0(S) = A_0(S; s_0)$. We shall retain a notation of [5]:

By a peaked partition we mean positive elements $e_1, \ldots, e_n \in A_0(S)$ so that $\sum_{i=1}^n \lambda_i e_i = \max_{1 \leq i \leq n} |\lambda_i|$ for all $\lambda_i \in \mathbb{R}; i \leq n$. Notice that this definition just means « peaked partition of unity in $A(S)$ » ([5]) if we add $e_0 = 1 - \sum_{i=1}^n e_i$.

Call a $l^\infty$-subspace $E \subset A_0(S)$ ([6]) positively generated if $E$ is spanned by a peaked partition. If $l^{n+1}_\infty \cong \tilde{E} \subset A(S)$
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is spanned by the peaked partition of unity \( \{f_0, f_1, \ldots, f_m\} \) and contains \( e_0, e_1, \ldots, e_n \) then we may arrange the indices \( j = 0, 1, \ldots, m \) so that

\[
(*) \quad e_i = f_i + \sum_{j=1}^{m-n} k_j f_{j+n} ; \quad i = 0, 1, \ldots, n ;
\]

where \( k_j \geq 0 \) for all \( j \) and \( \sum_{j=1}^{m-n} k_j \leq 1 \) ([6] Lemma 1.3 (i)).

**Lemma 1.** — Let \( E, F \subseteq A_0(S) \) be finite dimensional subspaces so that \( E \) is a positively generated \( l^m \)-space. For any \( \varepsilon > 0 \) there is a positively generated \( l^m \)-space \( \hat{E} \subseteq A_0(S) \) so that \( E \subseteq \hat{E} \) and \( \inf \{||x - y|| \mid y \in \hat{E}\} \leq \varepsilon ||x|| \) for all \( x \in F \).

**Proof.** — We may assume without loss of generality that \( F \) is spanned by positive elements. Let \( \{e_1, \ldots, e_n\} \) be the peaked partition which spans \( E \). Add \( e_0 \) as above. By [3] Theorem 3.1. there is \( l^m \cong \hat{E} \subseteq A(S) \) with \( E \subseteq \hat{E} \) and \( \inf \{||x - y|| \mid y \in \hat{E}\} \leq \varepsilon ||x|| \) for all \( x \in F \). Hence \( \hat{E} \) is positively generated by a peaked partition of unity \( \{f_0, f_1, \ldots, f_m\} \)

By (*) \( f_j(s_0) = 0 ; \ 1 \leq i \leq m \). Set \( \hat{E} = \text{linear span} \ \{f_1, \ldots, f_m\} \). \( \square \)

**Lemma 2.** — Let \( l^m \cong E \subseteq F \cong l^m \) be positively generated subspaces of \( A_0(S) \). Let \( \Phi \in E^* \) be positive. Then there is a positive extension \( \hat{\Phi} \in F^* \) of \( \Phi \) with \( ||\hat{\Phi}|| = ||\Phi|| \).

**Proof.** — Let \( \{e_i \mid 1 \leq i \leq n\} \) and \( \{f_j \mid 1 \leq j \leq m\} \) be peaked partitions spanning \( E \) and \( F \) respectively, so that (*) holds. Define then \( \hat{\Phi}(f_i) = \Phi(e_i) \) for all \( i = 1, \ldots, n \) and \( \hat{\Phi}(f_j) = 0 \) for all \( j = n + 1, \ldots, m \). \( \square \)

**Lemma 3.** — Let \( \{e_{i,n} \in A_0(S) \mid 1 \leq i \leq n\} \) be a peaked partition. Suppose that there is a positive \( \Phi \in \text{ex } B(A_0(S)^*) \) so that \( \sum_{i=1}^n \Phi(e_{i,n}) < 1 \). Then there is a peaked partition \( \{e_{i,n+1} \in A_0(S) \mid 1 \leq i \leq n + 1\} \) with

\[
e_{i,n} = e_{i,n+1} + \Phi(e_{i,n})e_{n+1,n+1}
\]

for all \( i = 1, \ldots, n \).
Proof. — Let \( \Phi_0 \in \text{ex } B(A(S)^*) \) be an element satisfying \( \Phi_0(y) = 0 \) for all \( y \in A_0(S) \). Consider furthermore
\[
\Phi_i \in \text{ex } B(A(S)^*); \quad i = 1, \ldots, n ;
\]
with
\[
\Phi_i(e_{j,n}) = \begin{cases} 1 & i = j; \\ 0 & i \neq j. \end{cases} \quad j = 1, \ldots, n.
\]
Define the affine \( \omega^* \)-continuous function \( f: H \rightarrow \mathbb{R} \) by
\[
f(\pm \Phi_i) = 0; \quad i = 0, 1, \ldots, n; \quad f(\pm \Phi) = \pm 1
\]
where \( H = \text{conv}\{\pm \Phi_i | i = 0, 1, \ldots, n\} \cup \{\pm \Phi\} \). Set
\[
h_1(y^*) = \min \left\{ \frac{1 - \sum_{i=1}^{n} \theta_i y^*(e_{i,n})}{1 - \sum_{i=1}^{n} \theta_i \Phi(e_{i,n})} \mid \theta_i = \pm 1; \ i = 1, \ldots, n \right\}
\]
\[
h_2(y^*) = \min \left\{ \frac{1 - y^*(e - e_{i,n})}{\Phi(e_{i,n})} \mid \Phi(e_{i,n}) > 0; \ i = 1, \ldots, n \right\}
\]
and consider \( g(y^*) = \min (h_1(y^*), h_2(y^*), 1 + y^*(e)) \).

Hence \( g: B(A(S)^*) \rightarrow \mathbb{R} \) is \( \omega^* \)-continuous, concave and nonnegative. In addition, \( f(y^*) \leq g(y^*) \) holds for all \( y^* \in H \).

By [3] Theorem 2.1. there is \( e_{n+1,n+1} \in A(S) \) with
\[
y^*(e_{n+1,n+1}) \leq g(y^*)
\]
for all \( y^* \in B(A(S)^*) \) and \( y^*(e_{n+1,n+1}) = f(y^*) \) for all \( y^* \in H \).

Hence, \( \| e - [e_{i,n} - \Phi(e_{i,n})e_{n+1,n+1}] \| \leq 1 \) and
\[
\| e - e_{n+1,n+1} \| \leq 1.
\]
Thus \( 0 \leq e_{i,n} - \Phi(e_{i,n})e_{n+1,n+1} \) and \( 0 \leq e_{n+1,n+1} \) for \( i = 1, \ldots, n \). Furthermore \( \Phi_0(e_{n+1,n+1}) = 0 \), hence \( e_{n+1,n+1} \in A_0(S) \). That means, \( e_{n+1,n+1} \) and \( e_{i,n} - \Phi(e_{i,n})e_{n+1,n+1} \) are the elements of a peaked partition in \( A_0(S) \). □

Lemma 4. — Let \( r_1, \ldots, r_n > 0 \) with \( \sum_{i=1}^{n} r_i < 1 \) and a peaked partition \( \{e_{1,n}, \ldots, e_{n,n}\} \subset A_0(S) \) be given. Then there is a positive element \( \Phi \in \text{ex } B(A_0(S)^*) \) with \( \Phi(e_{i,n}) = r_i \) for all \( i \leq n \).
Proof. — Let \( \{x_n \mid n \in \mathbb{N} \} \) be dense in \( A_0(S) \). Set linear span \( \{e_i, n \mid i \leq n \} = E \). Define \( \Phi|_E \) by \( \Phi(e_i, n) = r_i \) for all \( i \). Assume that we have defined \( \Phi \) already on a positively generated \( l_\infty \)-subspace \( \tilde{E} \supset E \) of \( A_0(S) \) so that \( \|\Phi|_{\tilde{E}}\| < 1 \). Then there is a basis \( \{e_{i, m} \mid i \leq m \} \) of \( \tilde{E} \) consisting of a peaked partition so that \( \Phi(e_{i, m}) > 0 \) for all \( i = 1, \ldots, m \).

Now, let \( 0 < \varepsilon < 1/2^{m+1} \left( 1 - \sum_{i=1}^{m} \Phi(e_{i, m}) \right) \). There is a positive linear extension \( \Psi \in \operatorname{ex} B(A_0(S)^*) \) of \( \Phi \) by Lemma 1 and Lemma 2. We derive from \( \operatorname{ex} S = S \) that \( \operatorname{ex} B(A_0(S)^*)_+ \) is \( \sigma^* \)-dense in \( B(A_0(S)^*)_+ \). It follows that there is \( \Omega \in \operatorname{ex} B(A_0(S)^*)_+ \) with \( \Phi(e_{i, m}) \geq \Omega(e_{i, m}) \) for all \( i = 1, \ldots, m \) and with \( \sum_{i=1}^{m} |\Omega(e_{i, m}) - \Phi(e_{i, m})| < \varepsilon \). We infer from Lemma 3 that there is peaked partition

\[
\{e_{i, m+1} \in A_0(S) \mid i = 1, \ldots, m + 1\}
\]

with \( e_{i, m} = e_{i, m+1} + \Omega(e_{i, m})e_{m+1, m+1}; i = 1, \ldots, m \). Set \( E_{m+1} = \text{span} \{e_{i, m+1} \mid i \leq m + 1\} \) and extend \( \Phi \) linearly by defining \( \Phi(e_{m+1, m+1}) = (1 + 2^{-m})^{-1} \). Hence \( \|\Phi|_{E_{m+1}}\| < 1 \).

Find a positively generated \( l_\infty \)-space \( F \subset A_0(S) \) with \( E_{m+1} \subset F \) and \( \inf \{\|x_k - y\| \mid y \in F\} \leq (m + 1)^{-1}\|x_k\| \) for all \( k \leq m \). Continue this process with \( F \) instead of \( E \). Finally we obtain an increasing sequence \( E_m \subseteq A_0(S) \) of positively generated \( l_\infty \)-spaces so that \( A_0(S) = \bigcup_{m=1}^{\infty} E_m \) where \( m \) runs through a subsequence of \( \mathbb{N} \). Furthermore there are peaked partitions \( \{e_{i, m} \in E_m \mid i \leq m\} \) so that \( \lim_{m \to \infty} \Phi(e_{m, m}) = 1 \). The latter condition implies that \( \Phi \) is a positive extreme point of \( B(A_0(S)^*) \). \( \square \)

Corollary. — Let \( e_{i, n} \in A_0(S) \) be a peaked partition and let \( 0 < r_i; i = 1, \ldots, n; \) be real numbers with \( \sum_{i=1}^{n} r_i < 1 \). Then there is a peaked partition \( \{e_{j, n+1} \in A_0(S) \mid j = 1, \ldots, n + 1\} \) with \( e_{i, n} = e_{i, n+1} + r_i e_{n+1, n+1}; i = 1, \ldots, n \).

Remark. — If we omit "\( \sum_{i=1}^{n} r_i < 1 \)" then the above corollary is no longer true (see [7], remark after the corollary.
of Lemma 2). The previous corollary does not hold either if we drop « 0 < r_i for all i ». This follows from the next lemma.

**Lemma 5.** — Let \( s_0 \in \text{ex } S \) be fixed. Then the set

\[
\Lambda(S, s_0) = \{ f \in B(A_0(S, s_0)) \mid f
\]
and \( 1 - f \) are smooth points of \( A(S) \} \) is dense in \( \partial B(A_0(S, s_0))_+ \).

**Proof.** — Let \( g \in \partial B(A_0(S, s_0))_+ \) and \( s_1 \in \text{ex } S \) so that \( g(s_1) = 1 \). Set \( F = \text{conv} \{ s_0, s_1 \} \). Let \( \{ x_n \mid n \in \mathbb{N} \} \) be dense in \( \{ x \in A_0(S, s_0) \mid \| x \| \leq 1 ; x|_F = 0 \} \). Define the affine continuous function \( h : F \to \mathbb{R} \) by \( h(s_0) = 0 \), \( h(s_1) = 1 \).

Furthermore let \( f_1(s) = 1 - 1/2 \sum_{n=1}^{\infty} 2^{-n}(x_n(s))^2 \) and

\[
f_2(s) = 1/2 \sum_{n=1}^{\infty} 2^{-n}(x_n(s))^2
\]
for all \( s \in S \). Then \( f_1 \) and \( f_2 \) are continuous; \( f_1 \) is concave, \( f_2 \) is convex. Furthermore \( f_2(s) \leq h(s) \leq f_1(s) \) for all \( s \in F \).

Hence there is an affine, continuous extension \( \tilde{h} : S \to \mathbb{R} \) of \( h \) with \( f_2(s) \leq \tilde{h}(s) \leq f_1(s) \) for all \( s \in S \) ([1], [2]).

Thus \( \tilde{h}(s_0) = 0 \), \( \tilde{h}(s_1) = 1 \), \( 0 < \tilde{h}(s) < 1 \) for \( s \neq s_0 \), \( s_1 \).

Then \( \lim_{\varepsilon \to 0} \frac{(1 - \varepsilon)g + \varepsilon \tilde{h}}{\| (1 - \varepsilon)g + \varepsilon \tilde{h} \|} = g \).

Now, if we take \( e_{1,1} \in \Lambda(S, s_0) \) and suppose that there is \( \Phi \in \text{ex } B(A_0(S, s_0)^*) \) with \( \Phi(e_{1,1}) = 0 \) then there must be \( s_1 \in \text{ex } S \) with \( s_1 \neq s_0 \) so that \( e_{1,1}(s_1) = 0 \), which is a contradiction. This concludes our above remark.

**Proposition 6.** — Let \( S \) be the Poulsen simplex and \( s \), \( \bar{s} \in \text{ex } S \). Consider \( x \in \Lambda(S, s) \) and \( y \in \Lambda(S, \bar{s}) \). Then there is an isometric (linear and order-) isomorphism \( T : \)

\[
A_0(S, s) \to A_0(S, \bar{s}) \text{ (onto) with } T(x) = y.
\]

**Proof.** — In the following we set \( X = A_0(S, s) \) and \( Y = A_0(S, \bar{s}) \). We claim that there are peaked partitions

\[
\{ e_i, n \mid i \leq n \} \subset X, \quad \{ f_i, n \mid i \leq n \} \subset Y, \quad n \in \mathbb{N};
\]
and real numbers \( a_{i,n} \); \( i \leq n \); \( n \in \mathbb{N} \); with

\[
\begin{align*}
e_{i,n} & = e_{i,n+1} + a_{i,n} e_{n+1,n+1} \\
f_{i,n} & = f_{i,n+1} + a_{i,n} f_{n+1,n+1} \\
0 & < a_{i,n}; \quad i \leq n; \quad \sum_{i=1}^{n} a_{i,n} < 1; \quad n \in \mathbb{N}; \\
e_{1,1} & = x; \quad f_{1,1} = y.
\end{align*}
\]

For this purpose we construct peaked partitions

\[
\begin{align*}
\{e_{i,n} \mid i \leq n\} & \subseteq X \\
\{f_{i,n} \mid i \leq n\} & \subseteq Y; \quad n \in \mathbb{N}; \quad j \leq n; \quad \text{such that}
\end{align*}
\]

\[
\begin{align*}
e_{i,n} & = e_{i,n+1} + a_{i,n} e_{n+1,n+1} \\
(2') & f_{i,n} = f_{i,n+1} + a_{i,n} f_{n+1,n+1} \\
(3) & \|e_{i,n} - e_{i+1,n}\| \leq 2^{-j} \\
(3') & \|f_{i,n} - f_{i+1,n}\| \leq 2^{-j}.
\end{align*}
\]

We proceed by induction:

Let \( \{x_n \mid n \in \mathbb{N}\} \) be dense in \( X \) and let \( \{y_n \mid n \in \mathbb{N}\} \) be dense in \( Y \). Assume that

\[
\begin{align*}
\{e_{i,k}^{(p)} \mid i \leq k\}, \quad \{f_{i,k}^{(p)} \mid i \leq k\}
\end{align*}
\]

and \( 0 < a_{i,j}; \quad j = 1, \ldots, n - 1; \quad k \leq p; \quad k, \ p = 1, \ldots, n; \) have been introduced already such that \( e_{1,1}^{(p)} = x \) and \( f_{1,1}^{(p)} = y \).

Set \( E_n = \text{Span} \{e_{i,n}^{(n)} \mid i \leq n\} \); \( F_n = \text{Span} \{f_{i,n}^{(n)} \mid i \leq n\} \)

(*) There are positively generated \( l_\infty \)-subspaces \( E_k \subseteq X \) with \( E_{k-1} \subseteq E_k; \quad k = n + 1, \ldots, m; \) so that

\[
\begin{align*}
\text{(4)} & \inf \{\|x_j - x\| \mid x \in E_n\} \leq 2^{-n}\|x_j\|; \quad j = 1, \ldots, n.
\end{align*}
\]

Consider a system of peaked partitions \( \{e_{i,k}^{(k)} \mid i \leq k\} \) spanning \( E_k \) and real numbers \( 0 \leq b_{i,k} \) with

\[
\begin{align*}
e_{i,k-1}^{(k-1)} & = e_{i,k}^{(k)} + b_{i,k-1} e_{k,k}^{(k)}; \quad \sum_{i=1}^{k-1} b_{i,k-1} \leq 1; \\
k & = n + 1, \ldots, m.
\end{align*}
\]

Notice that (6) \( 0 < \sum_{i=1}^{k-1} b_{i,k-1} \) for all \( k \).

Since otherwise there is \( \Phi \in \text{ex} B(X^*) \) with \( \Phi|_{E_{k-1}} = 0 \) and \( \Phi(e_{k,k}) = 1 \). As \( x \in E_{k-1} \), there are two different \( s, s_1 \in \text{ex} S \) with \( x(s) = x(s_1) = 0 \), a contradiction.
We first perturb \( \{e_i^{(n)} \mid i \leq n\} \):

**Step (n + 1):**

Consider

\[
(7) \quad x = e^{(n)}_{1,n} = e^{(n)}_1 + \sum_{j=2}^n k_{j} e^{(n)}_{j,n} = e^{(n+1)}_{1,n+1} + \sum_{j=2}^n k_{j} e^{(n+1)}_{j,n+1}
\]

where \( 0 \leq k_j \leq 1; 2 \leq j \leq n \). Even \( k_j < 1 \) holds properly

for all \( j = 2, \ldots, n \); since otherwise there would be two
different \( s_1, s_2 \in \text{ex} \, S \) with \( x(s_1) = x(s_2) = 1 \); which can be
inferred from (7) similarly as the proof of (6). Using the same
kind of argument shows \( 0 < k_j \) for all \( j = 2, \ldots, n \).

In view of (6) there is some \( b_{i,n} \neq 0 \).

(a) Let \( \sum_{i=1}^n b_{i,n} < 1 \):

Let \( i_0 \) be an index with \( b_{i_0,n} \neq 0 \). Set \( k_1 = 1 \) and

\[
\rho = \min \left( \left( 1 - \sum_{i=1}^n b_{i,n} \right) k_{i_0}(n - 1) - \sum_{j=1}^n k_{j}^{-1}; 1/n \right).
\]

Define

\[
a_{i,n} = \left( 1 - 2^{-n} \rho \sum_{j \neq i_0}^n k_{j} \right) b_{i,n}; \quad a_{i,n} = b_{i,n} + 2^{-n} \rho k_{i_0} b_{i_0,n}; \quad i \neq i_0.
\]

(b) Assume now \( \sum_{i=1}^n b_{i,n} = 1 \).

From our assumption \( x \in \Lambda(S,s) \) together with (7) it
follows similarly as above that there is \( i \geq 2 \) with \( b_{i,n} > 0 \).
Assume without loss of generality that \( b_{n,n} > 0 \).

Let \( \rho = \min \left( \frac{1}{2} (1 - k_{n})(n - 1) - \sum_{j=1}^{n-1} k_{j}^{-1}; 1/n \right) \).

Define

\[
a_{1,n} = b_{1,n} + 2^{-n} k_{n}(1 + \rho) b_{n,n};
\]

\[
a_{i,n} = b_{i,n} + 2^{-n} k_{n} \rho b_{n,n}; \quad 2 \leq i \leq n - 1 \; \text{(if } n > 2)\]

\[
a_{n,n} = \left( 1 - 2^{-n} - 2^{-n} \rho \sum_{j=1}^{n-1} k_{j} \right) b_{n,n}.
\]
Hence in either case \( 0 < a_{i,n} \) for all \( i = 1, \ldots, n \) and 
\[ \sum_{i=1}^{n} a_{i,n} < 1. \]
Furthermore

\[ |a_{i,n} - b_{i,n}| \leq 2^{-2n} \quad \text{for all} \quad i \leq n. \]

Define

\[
\begin{align*}
e^{(n+1)}_{i,n} &= e^{(n+1)}_{i,n+1} + a_{i,n} e^{(n+1)}_{n+1,n+1} \quad i \leq n + 1 \\
e^{(n+1)}_{i,n} &= e^{(n+1)}_{i,n+1} + a_{i,n-1} e^{(n+1)}_{n,n} \quad i \leq n \\
&
\end{align*}
\]

From (8) and (9) we derive easily \( \|e^{(n+1)}_{i,k} - e^{(n)}_{i,k}\| \leq 2^{-n}; \)
\( k = 1, \ldots, n + 1; i \leq n. \) Hence \((2)_{n+1}\) and \((3)_{n+1}\) are 
established.

Furthermore, because the elements \( k_j \) of (7) depend only on \( a_{i,k}; i \leq k \leq n - 1; \) we obtain

\[
\begin{align*}
e^{(n+1)}_{1,1} &= e^{(n+1)}_{1,n} + \sum_{j=2}^{n} k_j e^{(n+1)}_{j,n} \\
&= e^{(n+1)}_{1,n+1} + \sum_{j=2}^{n} k_j e^{(n+1)}_{j,n+1} + \left( a_{1,n} + \sum_{j=2}^{n} k_j a_{j,n} \right) e^{(n+1)}_{n+1,n+1} \\
&= e^{(n+1)}_{1,n+1} + \sum_{j=2}^{n} k_j e^{(n+1)}_{j,n+1} + \left( b_{1,n} + \sum_{j=3}^{n} k_j b_{j,n} \right) e^{(n+1)}_{n+1,n+1} \\
&= e^{(n)}_{1,1} = x.
\end{align*}
\]

Now, in step \((n + 2)\), repeat the procedure of step 
\((n + 1)\) but replace \( E_{n+1} \) by \( E_{n+2} \) and \( n + 1 \) by \( n + 2. \)

Then turn to step \((n + 3), \ldots, \) step \((m)\). We obtain 
\((2)_{n+1}, \ldots, (2)_m\) and \((3)_{n+1}, \ldots, (3)_m\).

Consider now \( F_n \). Find positively generated \( l^*_\infty \) subspaces
\( F_n \subset F_{n+1} \subset \ldots \subset F_m \subset Y \) and peaked partitions spanning \( F_k, \{f^{(m)}_{i,k} \in F_k | i \leq k\} \) with

\[
\begin{align*}
f^{(m)}_{i,k} &= f^{(m)}_{i,k+1} + a_{i,k} f^{(m)}_{k+1,k+1}; \quad k = n, \ldots, m - 1
\end{align*}
\]

where we have set \( f^{(n)}_{i,n} = f^{(n)}_{i}; i = 1, \ldots, n. \) This is possible 
by the Corollary after Lemma 4. Define

\[
\begin{align*}
f^{(j)}_{i,k} &= f^{(m)}_{i,k}; \quad i \leq k; \quad n + 1 \leq k \leq m; \quad n + 1 \leq j \leq m \\
f^{(j)}_{i,k} &= f^{(n)}_{i,k}; \quad i \leq k; \quad 1 \leq k \leq n; \quad n + 1 \leq j \leq m.
\end{align*}
\]

Find positively generated \( l^*_\infty \)-subspaces \( F_k \) of \( Y \) with
\( F_{k-1} \subseteq F_k; \; k = m + 1, \ldots, r \); such that

\[
\inf \{ \| y_j - x \| \mid x \in F_r \} \leq 2^{-m} \| y_j \|; \; j = 1, \ldots, m.
\]

Repeat (*) with \( r \) instead of \( m \) and \( F_r \) instead of \( E_m \). This yields \((2')_{m+1}, \ldots, (2')_r \) and \((3')_{m+1}, \ldots, (3')_r \).

Then go back to \( E_m \) and find positively generated \( L^k \)-subspaces \( E_{m+1} \subseteq \ldots \subseteq E_r \) of \( X \) with \( E_m \subseteq E_{m+1} \) and peaked partitions \( \{ e_{i,k}^{(r)} \mid i \leq k \} \) of \( E_k \) with

\[
e_{i,k}^{(r)} = e_{i,k+1}^{(r)} + a_{i,k} e_{k+1,k+1}^{(r)}; \; k = m, \ldots, r - 1.
\]

(We have set \( e_{i,m}^{(r)} = e_{i,m}^{(m)} \)).

Define

\[
e_{i,k}^{(r)} = e_{i,k}^{(r)}; \; i \leq k; \; m + 1 \leq k \leq r, \; m + 1 \leq j \leq r;
\]

\[
e_{i,k}^{(r)} = e_{i,k}^{(m)}; \; i \leq k; \; 1 \leq k \leq m, \; m + 1 \leq j \leq r.
\]

Finally go back to (*) and repeat everything with \( E_r \) and \( F_r \) instead of \( E_n \) and \( F_n \), respectively. By \((3)\) and \((3')\) we obtain

\[
e_{i,n} = \lim_{j \to \infty} e_{i,n}^{(j)}; \; f_{i,n} = \lim_{j \to \infty} f_{i,n}^{(j)}; \; i \leq n, \; n \in \mathbb{N};
\]

which are elements of peaked partitions with

\[
e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1,n+1}; \; f_{i,n} = f_{i,n+1} + a_{i,n} f_{n+1,n+1}
\]

\( i \leq n; \; n \in \mathbb{N}; \; f_{1,1} = y; \; e_{1,1} = x \) \((2)\) and \((2')\). From \((4), (10)\) and \((3), (3')\) we infer that

closed span \( \{ f_{i,n} \mid i \leq n; \; n \in \mathbb{N} \} = Y \)

and

closed span \( \{ e_{i,n} \mid i \leq n; \; n \in \mathbb{N} \} = X \).

We define an isometric isomorphism \( T: A_0(S;\delta) \to A_0(S;\tilde{\delta}) \) by

\[
T(e_{i,n}) = f_{i,n}; \; i \leq n; \; n \in \mathbb{N}.
\]

Proposition 6 establishes the assertion (a) of the Theorem if we extend \( T \) isometrically on \( A(S) \) by defining \( T(1) = 1 \).

Proof of \((b)\):

Let \( u, \; \nu \in \text{ex } S \) so that \( g(u) > 0 \) and \( g(\nu) < 0 \). If there were an isometric isomorphism \( (\text{onto}) \) then in view of Lemma 5 there would be \( \tilde{g} \in \partial B(A_0(S;\delta_1)) \) with \( \tilde{g}(u) > 0 \) and \( \tilde{g}(\nu) < 0 \) so that \( \tilde{g}(s) \neq 0 \) for all \( s \in S; \; s \neq s_1 \). But
then \( s_1 = \lambda u + (1 - \lambda)v \) for suitable \( \lambda; 0 < \lambda < 1 \). Hence \( u = v = s_1 \), a contradiction.

(c) has been proved already by Lemma 5.

**Concluding remarks.** — The assertion (a) of the Theorem cannot be extended on any dense subset of \( \partial B(A(S))^+ \) since otherwise any element of \( \partial B(A(S))^+ \) would be extreme point of \( B(A(S)) \) which is certainly not true. This follows from the fact that for any \( e \in \text{ex} B(A(S)) \),

\[
\max (\|x + e\|, \|x - e\|) = 1 + \|x\|
\]

holds for all \( x \in A(S) \). (cf. [4] Theorem 4.7. and 4.8.).

**BIBLIOGRAPHY**


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