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ON A THEOREM OF M. ITO (*)

by Gunnar FORST

Let G be a locally compact abelian group and let No be a convolution kernel on G (i.e. a positive measure on G) satisfying the principle of unicity of mass. A convolution kernel N on G is said to be a divisor of No if there exists a convolution kernel N' on G such that No = N * N'. If such an N' exists and is uniquely determined, it is denoted No/N; this is the case if N' exists and is a Hunt kernel on G. The set of divisors of No is denoted D(No).

The following result (1) was recently proved by M. Itô, cf. Théorème 1 in [1].

**Theorem.** — The set

\[ C(No) = \{ N \in D(No) | No/N \text{ is a Hunt kernel on } G \} \]

is a convex cone.

The proof in [1] of this result is somewhat complicated, and it is the purpose of this note to give a simpler proof, mainly using the ideas from [1].

**Lemma 1** ([1], p. 292-293). — Let N be a non-zero convolution kernel. In order that there exist a Hunt kernel N' and an N'-invariant convolution kernel η such that No = N * N' + η, it is necessary and sufficient that N + pNo ∈ D(No) for all p > 0. In the affirmative case N' and η are uniquely deter-

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(1) The statement of Théorème 1 in [1] is not true; the argument uses Corollaire 2 of [1] which is false. What the proof actually shows is what is formulated in the Theorem.
mined as the vague limits
\[ N' = \lim_{p \to 0} (N_0/(N + pN_0)) \quad \text{and} \quad \gamma_i = \lim_{t \to \infty} N_0 * \alpha_i, \]

where \((\alpha_i)_{i \geq 0}\) is the vaguely continuous convolution semigroup (vccs) on \(G\) associated with \(N'\).

**Lemma 2** (cf. [1], p. 295-299). — Let \(N_i\) for \(i = 1, 2\) be a Hunt kernel with associated vccs \((\alpha_i^i)_{i \geq 0}\). Suppose that the convolution \(N_1 * N_2\) exists. Then \((\alpha_1^1 * \alpha_2^2)_{i \geq 0}\) defines a vccs and \(N = \int_0^\infty \alpha_1^1 * \alpha_2^2 \, dt\),
defines a Hunt kernel satisfying
\[ N_1 * N_2 = N * (N_1 + N_2). \quad (*) \]

**Proof.** — For \(s, t \geq 0\) we have \(\alpha_1^1 * N_1 \leq N_1 \) and \(\alpha_2^2 * N_2 \leq N_2\) which implies that the convolution \(\alpha_1^1 * \alpha_2^2\) exists. By the convergence Lemma in [1] p. 295, it is rather easy to see that the mapping \((s,t) \mapsto \alpha_1^1 * \alpha_2^2\) is vaguely continuous on \([0, \infty[ \times [0, \infty[,\) and in particular \((\alpha_1^1 * \alpha_2^2)_{i \geq 0}\) is a vccs. For \(n > 0\) we have (vaguely)
\[
\left( \int_0^n \alpha_1^1 * \alpha_2^2 \, dt \right) * (N_1 + N_2) = \left( \int_0^n \alpha_1^1 * \alpha_2^2 \, dt \right) * \left( \int_0^\infty (\alpha_1^1 + \alpha_2^2) \, ds \right)
\]
\[
= \int_0^\infty \left( \int_0^n \alpha_1^1 * \alpha_2^2 \, ds \right) \left( \int_0^\infty \alpha_1^1 + \alpha_2^2 \, ds \right) \, dt
\]
\[
\leq \int_0^\infty \left( \int_t^\infty \alpha_1^1 * \alpha_2^2 \, du \right) \left( \int_t^\infty \alpha_1^1 + \alpha_2^2 \, du \right) \, dt
\]
\[
= \int_0^\infty \int_t^\infty \alpha_1^1 * \alpha_2^2 \, ds \, dt = N_1 * N_2.
\]
This shows that \(\int_0^\infty \alpha_1^1 * \alpha_2^2 \, dt\) converges vaguely, and equation (*) follows by a monotone convergence argument.\]

**Proof of the theorem.** — We shall see that \(C(N_0)\) is convex. Let \(N_1, N_2 \in C(N_0)\) and let \((\alpha_i^i)_{i \geq 0}\), \(i = 1, 2\), be the vccs associated with the Hunt kernel \(N_i' = N_0/N_i, i = 1, 2\). Let \((N_{i', p})_{p \geq 0}\), \(i = 1, 2\), be the resolvent for \(N_i'\),
\[ N_{i', p} = \int_0^\infty e^{-pt} \alpha_i \, dt \quad \text{for} \quad p > 0 \quad \text{and} \quad i = 1, 2. \]
For \(p > 0\) and \(i = 1, 2\) we have \(N_{i', p} * (N_i + pN_0) = N_0\) which shows that for \(p > 0\)
\[ N_{1', p} * N_{2', p} * N_0 \leq N_{1', p} * \frac{1}{p} N_0 \leq \left( \frac{1}{p} \right)^2 N_0. \]
It follows that the convolution $N_{1,p}^* N_{2,p}^*$ exists, and by Lemma 2 the convolution kernel

$$\tilde{N}_p = \int_0^\infty e^{-2\mu \alpha_1^2 + \alpha_1^2} dt$$

is well-defined and satisfies

$$N_{1,p}^* N_{2,p}^* = \tilde{N}_p^* (N_{1,p}^* + N_{2,p}^*).$$

Moreover we have

$$(N_1 + N_2 + 2pN_0)^* \tilde{N}_p = N_0. \quad (**)$$

In fact,

$$(N_1 + N_2 + 2pN_0)^* \tilde{N}_p = N_0^* N_{1,p}^* N_{2,p}^* + (N_1 + pN_0)^* N_{1,p}^* N_{2,p}^* \tilde{N}_p = N_0^* N_{2,p}^* \tilde{N}_p + N_0^* N_{1,p}^* \tilde{N}_p = N_0^* N_{1,p}^* N_{2,p}^*,$$

where all the convolutions exist, and this implies (**) because $N_{1,p}^* N_{2,p}^*$ satisfies the principle of unicity of mass.

By Lemma 1 the Hunt kernel

$$\tilde{N} = \lim_{p \to 0} N_p = \int_0^\infty \alpha_1^2 \alpha_1^2 dt$$

satisfies $\tilde{N} \ast (N_1 + N_2) + \eta = N_0$ for some $\tilde{N}$-invariant convolution kernel $\eta$, and since $N_0 = N_1 \ast N_1' = N_2 \ast N_2'$ we have (as in [1], p. 302)

$$\eta = \lim_{t \to -\infty} N_0 \ast \alpha_1^2 \alpha_1^2 \leq \lim_{t \to -\infty} N_0 \ast \alpha_1^2 = 0,$$

which shows that $\tilde{N} \ast (N_1 + N_2) = N_0$, i.e.

$$N_1 + N_2 \in C(N_0).$$

BIBLIOGRAPHY