Annales de l'institut Fourier

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*Annales de l'institut Fourier*, tome 28, nº 1 (1978), p. 193-206
<http://www.numdam.org/item?id=AIF_1978__28_1_193_0>

Ann. Inst. Fourier, Grenoble
28, 1 (1978), 193-206

# THE GENERAL COMPLEX CASE OF THE BERNSTEIN-NACHBIN APPROXIMATION PROBLEM

by S. MACHADO* and J.B. PROLLA

## 1. Introduction.

Throughout this paper X denotes a Hausdorff topological space, and A $\quad C(X;K)$, where $K = \mathbf{R}$ or $\mathbf{C}$, denotes a subalgebra. A *vector fibration over* X is a pair $(X, (F_x)_{x \in X})$, where each $F_x$ is a vector space over the field K. A *cross-section* is then any element $f$ of the vector space Cartesian product of the vector spaces $F_x$, i.e., $f = (f(x))_{x \in X}$. A *weight on* X is a function $v$ on X such that $v(x)$ is a seminorm over $F_x$ for each $x \in X$. A *Nachbin space* $LV_\infty$ is a vector space of cross-sections $f$ such that the mapping $x \in X \longmapsto v(x)[f(x)]$ is upper semi-continuous and null at infinity on X for each weight $v \in V$, equipped with the topology defined by the family of seminorms of the form

$$\|f\|_v = \sup \{v(x)[f(x)] ; x \in X\} .$$

For simplicity, and without loss of generality, the set V is assumed to be *directed,* i.e., given $u, v \in V$ there is $w \in W$ and $t > 0$ such that $u(x) \leqslant t \cdot w(x)$ and $v(x) \leqslant t \cdot w(x)$, for all $x \in X$.

Throughout this paper $W \subset LV_\infty$ denotes a vector subspace which is an A-module, i.e., if $a \in A$ and $g \in W$, then the cross-section $ag = (a(x)g(x))_{x \in X}$ belongs to W. In this context, the *Bernstein-Nachbin approximation problem* consists in asking for a description of the closure of W in $LV_\infty$. Let P be a closed, pairwise disjoint covering

(*) Research supported in part by a grant from FINEP (Financiadora de Estudos e Projetos).

of X. We say that W is P-*localizable in* $LV_\infty$ if its closure consists of those $f \in LV_\infty$ such that, given any $S \in P$, any $v \in V$ and any $\epsilon > 0$, there is some $g \in W$ such that $v(x)[f(x) - g(x)] < \epsilon$ for all $x \in S$. The *strict Bernstein-Nachbin approximation problem* consists in asking for necessary and sufficient conditions for an A-module W to be P-localizable, when P is the set $P_A$ of all equivalence classes $Y \subset X$ modulo X/A. We recall that the equivalence relation X/A is defined as follows. For any pair $x, y \in X$, $x$ is equivalent to $y$ modulo X/A if, and only if, $a(x) = a(y)$ for all $a \in A$.

In [7], the sufficient conditions for localizability established by Nachbin (see e.g. Nachbin [6]) were extended to the context of vector-fibrations, and a fortiori to vector-valued functions, in the case of modules over *real* or *self-adjoint* complex algebras. In this paper we extend the results of [7] to the general complex case, at the same time getting stronger results. We extend the results of [7] in the same way that Bishop's theorem generalizes the Stone-Weierstrass theorem.

## 2. Definitions and lemmas.

In this section we collect all pertinent definitions. As we said in the Introduction, $W \subset LV_\infty$ is always an A-module.

DEFINITION 1. – *Let $\Omega_n$ be the set of all fundamental weights on $\mathbf{R}^n$. (See* [7], *pg. 302.) We denote by $\Omega_n^s$ the subset of $\Omega_n$ consisting of those $\omega \in \Omega_n$ which are symmetric,* i.e., *$\omega(t) = \omega(|t|)$ for all $t \in \mathbf{R}^n$, where $|t| = (|t_1|, \ldots, |t_n|)$, if $t = (t_1, \ldots, t_n)$. Let $\Gamma_1$ be the subset of $\Omega_1$ consisting of those $\gamma \in \Omega_1$ such that $\gamma^k \in \Omega_1$ for all $k > 0$. We denote by $\Gamma_1^s$ the intersection $\Gamma_1 \cap \Omega_1^s$. Notice that $\Omega_n^d \subset \Omega_n^s$ and similarly $\Gamma_1^d \subset \Gamma_1^s$. Here $\Omega_n^d$ denotes the subset of all $\omega \in \Omega_n$ such that $|u| \leqslant |t|$ implies $\omega(u) \geqslant \omega(t)$ for all $u$, $t \in \mathbf{R}^n$ and then $\Gamma_1^d = \Gamma_1 \cap \Omega_1^d$.*

DEFINITION 2. – *Let* P *be a closed, pairwise disjoint covering of* X. *We say that* W *is sharply* P-*localizable in* $LV_\infty$ *if, given $f \in LV_\infty$ and $v \in V$, there is some $S \in P$ such that*

$$\inf\{\|f - g\|_v \,;\, g \in W\} = \inf\{\|f | S - g | S\|_v \,;\, g \in W\}.$$

DEFINITION 3. – *For each* $v \in V$ *and each* $\delta > 0$, *we denote by* $L(W\,; v, \delta)$ *the set of all cross-sections* $f \in LV_\infty$ *such that, for each equivalence class* $Y \subset X$ modulo $X/A$, *there is* $g \in W$ *such that* $\| f \mid Y - g \mid Y \|_v < \delta$.

DEFINITION 4. – *Let* $\mathfrak{S}$ *be the class of all ordinal numbers whose cardinal numbers are less or equal than* $2^{|X|}$, *where* $|X|$ *is the cardinal number of* X. *For each* $\sigma \in \mathfrak{S}$, *we define a closed, pairwise disjoint covering* $P_\sigma$ *of* X. *For* $\sigma = 1$, $P_1 = \{X\}$. *Assume that* $P_\tau$ *has been defined for all* $\tau < \sigma$. *We consider two cases.*

a) $\sigma = \tau + 1$, *for some* $\tau \in \mathfrak{S}$ ; *let* $S \in P_\tau$. *Define* $A_S = \{a \in A\,;$ $a \mid S$ *is real valued*$\}$. *Consider the partition of* S *into equivalence classes modulo* $S/(A_S \mid S)$. *The partition* $P_\sigma$ *is then defined as the collection of all such equivalence classes, as* S *ranges over* $P_\tau$.

b) *If* $\sigma$ *has no predecessor, define* $x \equiv y$ *if, and only if,* $x$ *and* $y$ *belong to the same set* $S_\tau \in P_\tau$ *for all* $\tau < \sigma$. *The partition* $P_\sigma$ *is then defined as the collection of all equivalence classes modulo the above equivalence relation.*

This defines $P_\sigma$ for all $\sigma \in \mathfrak{S}$, and $P_\sigma$ is a refinement of $P_\tau$, for $\tau < \sigma$. This construction is taken from Bishop [1], who attributed it to Silov. In Bishop [1], it is shown that there exists an ordinal $\rho \in \mathfrak{S}$ such that $P_{\rho+1} = P_\rho$, and each $S \in P_\rho$ is *anti-symmetric for* A, i.e., for $a \in A$, $a \mid S$ being real-valued implies that $a \mid S$ is constant. In fact, given an anti-symmetric subset $K \subset X$, for each $\sigma \in \mathfrak{S}$ there is $S_\sigma \in P_\sigma$ such that $K \subset S_\sigma$. Hence, each $S \in P_\rho$ is a *maximal anti-symmetric set for* A. The collection of all maximal anti-symmetric sets for A is denoted by $\mathcal{K}_A$. So we have $P_\rho = \mathcal{K}_A$.

DEFINITION 5. – *We say that the* A*-module* W *is sharply localizable under* A *in* $LV_\infty$ *if, given* $f \in LV_\infty$ *and* $v \in V$, *for each* $\sigma \in \mathfrak{S}$ *there exists an element* $S_\sigma \in P_\sigma$ *such that :*

a) $S_\sigma \subset S_\tau$ *for all* $\tau < \sigma$,

b) $\inf \{\| f - g \|_v\,; g \in W\} = \inf \{\| f \mid S_\sigma - g \mid S_\sigma \|_v\,; g \in W\}$.

*Remark.* – Since $P_\rho = \mathcal{K}_A$, W is sharply localizable under A in $LV_\infty$ implies that W is sharply $\mathcal{K}_A$-localizable.

DEFINITION 6. – *We say that a subset* $G(A) \subset A$ *is a set of generators for* A, *if the subalgebra over* K *of* A *generated by* $G(A)$ *is dense in* A *for the compact-open topology of* $C(X;K)$ ; *and we say that a set of generators* $G(A) \subset A$ *is a strong set of generators if, for any* $\sigma \in \mathfrak{S}$ *and any* $S \in P_\sigma$, *the set* $(A_S \cap G(A))|S$ *is a set of generators for the algebra* $A_S|S$. (*Recall that* $A_S = \{a \in A \,;\, a|S$ *is real-valued*$\}$.) *For example, the whole algebra* A *is a strong set of generators for* A. *Also, if the algebra* A *has a set of generators* $G(A)$ *consisting only of real-valued functions, then* $G(A)$ *is a strong set of generators for the algebra* A.

Similarly, a subset $G(W) \subset W$ is a *set of generators for* W if the A-submodule of W generated by $G(W)$ is dense in W for the topology of the space $LV_\infty$. Let us call $G(W)^*$ the *real* linear span of $G(W)$.

LEMMA 7. – *Let* $A \subset C_b(X;\mathbf{R})$ *be a subalgebra containing the constants Given an equivalence class* $Y \subset X$ *modulo* $X/A$, *and a compact subset* $K \subset X$, *disjoint from* Y, *there is* $b \in A$ *such that* $0 < b \leqslant 1$, $b(y) = 1$ *for all* $y \in Y$, *and* $b(t) < 1$ *for all* $t \in K$.

*Proof.* – Choose $y_0 \in Y$. For each $t \in K$, there is $a_t \in A$ such that $a_t(y_0) \neq a_t(t)$. Define $b_t \in A$ by the following

$$b_t(x) = 1 - (a_t(x) - a_t(y_0))^2/2\ \|a_t - a_t(y_0)\|_X^2$$

for all $x \in X$, where for each $f \in C_b(X;\mathbf{R})$, $\|f\|_X = \sup\{|f(x)|\,;\, x \in X\}$. Then $0 < b_t \leqslant 1$ ; $b_t(y) = 1$ for all $y \in Y$ ; and $b_t(x) < 1$, if $a_t(x) \neq a_t(y_0)$ ; in particular, $b_t(t) < 1$. The collection of open sets $U_t = \{x \in X \,;\, b_t(x) < 1\}$, for $t \in K$, is an open covering of the compact set K. By compactness, there are $t_1, \ldots, t_n \in K$ such that $K \subset U_{t_1} \cup \ldots \cup U_{t_n}$. Now $b = (b_{t_1} + \ldots + b_{t_n})/n$ has all the desired properties.

LEMMA 8. – *Let* $A \subset C_b(X;\mathbf{R})$ *be a subalgebra containing the constants. For each equivalence class* $Y \subset X$ *modulo* $X/A$, *let there be given a compact set* $K_Y \subset X$, *disjoint from* Y. *Then there exist equivalence classes* $Y_1, \ldots, Y_n \subset X$ *modulo* $X/A$ *such that to each* $\delta > 0$, *there correspond* $a_1, \ldots, a_n \in A$ *with* $0 \leqslant a_i \leqslant 1$ ; $0 \leqslant a_i(t) < \delta$ *for all* $t \in K_{Y_i}$, $i = 1, \ldots, n$ ; *and* $a_1 + \ldots + a_n = 1$.

*Proof.* – Let $P_A$ be the set of all equivalence classes $Y \subset X$ modulo $X/A$. Select one element $Y_1$ in $P_A$ and let $P$ be the collection of all $Y \in P_A$ such that $Y \cap K_{Y_1} \neq \emptyset$. For each $Y \in P$, let $b_Y \in A$ be given by Lemma 7; choose real numbers $r_Y$ and $s_Y$ such that $0 \leqslant \sup \{b_Y(x) ; x \in K_Y\} \leqslant r_Y < s_Y < 1$. Put $B_Y = \{x \in X ; b_Y(x) > s_Y\}$, for each $Y \in P$. Clearly, $Y \subset B_Y$, so that the collection $\{B_Y ; Y \in P\}$ is an open cover of the compact set $K_{Y_1}$. By compactness, there are $Y_2, \ldots, Y_n$ in $P$ such that the finite collection $\{B_2, \ldots, B_n\}$ is a cover of $K_{Y_1}$, where $B_i = B_Y$ with $Y = Y_i$, $i = 2, \ldots, n$. For each index $i$, appeal to Jewett [3], Lemma 2, to get a real polynomial $p_i$ with $p_i(1) = 1$ ; $0 \leqslant p_i \leqslant 1$ ; $0 \leqslant p_i(t) < \delta$ for all $0 \leqslant t \leqslant r_i$; and $1 - \delta < p_i(t) \leqslant 1$ for all $s_i \leqslant t \leqslant 1$, where $r_i = r_Y$ and $s_i = s_Y$ with $Y = Y_i$. Consider $g_i = p_i(b_i)$, where $b_i = b_Y$ with $Y = Y_i$, for $i = 2, \ldots, n$. Define $a_2 = g_2$, $a_3 = (1 - g_2) g_3, \ldots, a_n = (1 - g_2)(1 - g_3) \ldots (1 - g_{n-1}) g_n$. This technique is from Rudin [9], item 2.13, For $i = 2, \ldots, n$, it is easily seen that $0 \leqslant a_i \leqslant 1$, and $a_i(x) \leqslant g_i(x) < \delta$ for all $x \in K_i$, where $K_i = K_Y$ with $Y = Y_i$. Moreover, we have

$$1 \geqslant a_2 + \ldots + a_n = 1 - (1 - g_2)(1 - g_3) \ldots (1 - g_n) \geqslant 0 .$$

Let $a_1 = 1 - (a_2 + \ldots + a_n)$. Then $0 \leqslant a_1 \leqslant 1$ and $a_1 + \ldots + a_n = 1$. Let $x \in B_2 \cup \ldots \cup B_n$ be given. There is some index $j \in \{2, \ldots, n\}$ such that $x \in B_j$. Then $1 \geqslant g_j(x) > 1 - \delta$, so that we have

$$a_2(x) + \ldots + a_n(x) = 1 - (1 - g_j(x)) \prod_{\substack{i=2 \\ i \neq j}}^{n} (1 - g_i(x)) > 1 - \delta .$$

That is, $a_1(x) < \delta$ for $x \in B_2 \cup \ldots \cup B_n \supset K_{Y_1}$, *QED.*

*Remark.* – The above two lemmas embody techniques of peak sets and peaking functions in the present context.

Notice the occurence of the unavoidable basic real analysis detail in the proof of Lemma 8 above : it is the very simple Lemma 2 of Jewett [3].

## 3. A solution of the Bernstein-Nachbin approximation problem.

THEOREM 9. – *Suppose that there exist sets of generators* G(A) *and* G(W), *for* A *and* W *respectively, such that :*

1) G(A) *consists only of real valued functions ;*

2) *given any* $v \in V$, $a_1, \ldots, a_n \in G(A)$, *and* $g \in G(W)$, *there are* $a_{n+1}, \ldots, a_N \in G(A)$, *with* $N \geqslant n$, *and* $\omega \in \Omega_N$ *such that*

$$v(x)\,[g(x)] \leqslant \omega(a_1(x), \ldots, a_n(x), \ldots, a_N(x))$$

*for all* $x \in X$.

*Then* W *is sharply localizable under* A *in* $LV_\infty$.

We first remark that, since G(A) consists only of real valued functions, $\rho = 2$ and $P_2 = P_A$, where $P_A$ is the closed, pairwise disjoint covering of X into equivalence classes modulo X/A. Hence, all that we have to prove is that W is sharply $P_A$-localizable in $LV_\infty$. The proof will be partitioned into several lemmas, and to state them we need a preliminary definition.

DEFINITION 10. – *Let us call* B *the subalgebra of* $C_b(X; \mathbf{R})$ *of all functions of the form* $q(a_1, \ldots, a_n)$, *where* $n \geqslant 1$, $a_1, \ldots, a_n \in G(A)$, *and* $q \in C_b(\mathbf{R}^n; \mathbf{R})$ *are arbitrary.*

LEMMA 11. – *Assume that* G(A) *consists only of real valued functions. Let* $f \in L(W; v, \lambda)$. *Then, for each* $\epsilon > 0$, *there exist* $b_1, \ldots, b_m \in B$, *and* $g_1, \ldots, g_m \in G(W)$ *such that*

$$\left\| f - \sum_{i=1}^{m} b_i g_i \right\|_v < \lambda + \epsilon .$$

*Proof.* – For each $Y \in P_A$, there exists $w_Y \in G(W)^*$ such that $v(x)\,[f(x) - w_Y(x)] < \lambda + \epsilon/2$, for all $x \in Y$. Let us define $K_Y = \{t \in X\,;\, v(t)[f(t) - w_Y(t)] \geqslant \lambda + \epsilon/2\}$. Then $K_Y$ is compact and disjoint from Y. Since the equivalence relations X/A and X/B are the same, we may apply Lemma 8 for the algebra B. Hence, there exist equivalence classes $Y_1, \ldots, Y_n \in P_A$ such that to each $\delta > 0$, there correspond $h_1, \ldots, h_n \in B$ with $0 \leqslant h_i \leqslant 1$ ; $0 \leqslant h_i(x) < \delta$

for $x \in K_i$, where $K_i = K_{Y_i}$ for $i = 1, \ldots, n$. Moreover, $h_1 + \ldots + h_n = 1$ on X. Let us choose $\delta > 0$ such that $nM\delta < \epsilon/2$, where $M = \max\{\|f - w_i\|_v ; i = 1, \ldots, n\}$, and $w_i = w_Y$ with $Y = Y_i$ for $i = 1, \ldots, n$. Let $w = h_1 w_1 + \ldots + h_n w_n$. We claim that $v(x)[f(x) - w(x)] < \lambda + \epsilon$, for all $x \in X$. Indeed,

$$v(x)[f(x) - w(x)] \leqslant \sum_{i=1}^{n} h_i(x)\, v(x)[f(x) - w_i(x)]$$

for all $x \in X$. Now, if $x \in K_i$ then $h_i(x) < \delta$, and therefore

$$h_i(x)\, v(x)[f(x) - w_i(x)] \leqslant \delta \, \|f - w_i\|_v \leqslant \delta M ;$$

on the other hand, if $x \notin K_i$, then the following estimate is true :

$$h_i(x)\, v(x)[f(x) - w_i(x)] \leqslant h_i(x)(\lambda + \epsilon/2) .$$

Combining both estimates, we get

$$\begin{aligned} v(x)[f(x) - w(x)] &\leqslant nM\delta + (\lambda + \epsilon/2)(h_1(x) + \cdots + h_n(x)) \\ &< \lambda + \epsilon . \end{aligned}$$

Since each $w_i \in G(W)^*$, there exist $b_1, \ldots, b_m \in B$ and $g_1, \ldots, g_m \in G(W)$ such that $w = b_1 g_1 + \cdots + b_m g_m$.

LEMMA 12. – *Suppose that the hypothesis of Theorem 9 are satisfied. Given* $v \in V$, $b \in B$, $g \in G(W)$ *and* $\delta > 0$, *there is* $w \in W$ *such that* $\|w - bg\|_v < \delta$.

*Proof.* – Suppose that $b = q(a_1, \ldots, a_n)$. Given $v \in V$ and $g \in G(W)$ there are $a_{n+1}, \ldots, a_N \in G(A)$, where $N \geqslant n$, and $\omega \in \Omega_N$ such that $v(x)[g(x)] \leqslant \omega(a_1(x), \ldots, a_n(x), \ldots, a_N(x))$ for all $x \in X$. Define $r \in C_b(\mathbf{R}^N ; \mathbf{R})$ by setting $r(t) = q(t_1, \ldots, t_n)$ for all $t = (t_1, \ldots, t_n, \ldots, t_N) \in \mathbf{R}^N$. By hypothesis $\omega \in \Omega_N$; hence $C_b(\mathbf{R}^N ; \mathbf{R})$ is contained in $C\omega_\infty(\mathbf{R}^N ; \mathbf{R})$ and $\mathcal{P}(\mathbf{R}^N)$ is dense in $C\omega_\infty(\mathbf{R}^N ; \mathbf{R})$. Given $\delta > 0$, we can find a real polynomial $p \in \mathcal{P}(\mathbf{R}^N)$ such that $\|p - r\|_\omega < \delta$. From this it follows that $\|w - bg\|_v < \delta$, where $w = p(a_1, \ldots, a_n, \ldots, a_N)\, g \in AW \subset W$.

LEMMA 13. – *Suppose that the hypothesis of Theorem 9 are satisfied. Then, for each* $f \in \mathrm{LV}_\infty$ *and* $v \in \mathrm{V}$ *we have*

$$d = \inf \{\| f - g \|_v \ ; g \in \mathrm{W}\}$$
$$= \sup \{\inf \{\| f \mid \mathrm{Y} - g \mid \mathrm{Y} \|_v \ ; g \in \mathrm{W}\} ; \mathrm{Y} \in \mathrm{P_A}\}.$$

*Proof.* – Clearly, $c \leqslant d$, where we have defined

$$c = \sup \{\inf \{\|f \mid \mathrm{Y} - g \mid \mathrm{Y}\|_v \ ; g \in \mathrm{W}\} ; \mathrm{Y} \in \mathrm{P_A}\}.$$

To prove the reverse inequality, let $\epsilon > 0$. For each $\mathrm{Y} \in \mathrm{P_A}$, there exists $g_\mathrm{Y} \in \mathrm{W}$ such that $v(x)[f(x) - g_\mathrm{Y}(x)] < c + \epsilon/3$ for all $x \in \mathrm{Y}$. Therefore, $f \in \mathrm{L}(\mathrm{W} ; v, c + \epsilon/3)$. By Lemma 11 applied with $\lambda = c + \epsilon/3$ and $\epsilon/3$, there exist $b_1, \ldots, b_m \in \mathrm{B}$ and $g_1, \ldots, g_m \in \mathrm{G(W)}$ such that:

$$\left\| f - \sum_{i=1}^{m} b_i g_i \right\|_v < (c + \epsilon/3) + \epsilon/3 .$$

By Lemma 12 applied with $\delta = \epsilon/3m$, there are cross-sections $w_1, \ldots, w_m \in \mathrm{W}$ such that $\| w_i - b_i g_i \|_v < \epsilon/3m$. From this it follows that $\|f - g\|_v < c + \epsilon$, where $g = w_1 + \cdots + w_m$. Since $g \in \mathrm{W}$, $d < c + \epsilon$. Since $\epsilon > 0$ was arbitrary, $d \leqslant c$, as desired.

*Proof of Theorem 9.* – Let $f \in \mathrm{LV}_\infty$ and $v \in \mathrm{V}$ be given. Let Z be the quotient space of X by the equivalence relation X/A, and let $\pi : \mathrm{X} \longrightarrow \mathrm{Z}$ be the quotient map. By Lemma 1 of [7], the map

$$z \in \mathrm{Z} \longmapsto \| f \mid \pi^{-1}(z) - g \mid \pi^{-1}(z) \|_v$$

is upper semicontinuous and null at infinity on Z, for each $g \in \mathrm{W}$. Hence the map defined by

$$h(z) = \inf \{\| f \mid \pi^{-1}(z) - g \mid \pi^{-1}(z) \|_v \ ; g \in \mathrm{W}\}$$

for all $z \in \mathrm{Z}$, is upper semi-continuous and null at infinity on Z too. Therefore $h$ attains its supremum on Z at some point $z$. Consider the equivalence class $\mathrm{Y} = \pi^{-1}(z)$ modulo X/A. On the other hand, the supremum of the map $h$ is by Lemma 13 equal to $d$. Thus, we have found an equivalence class $\mathrm{Y} \subset \mathrm{X}$ modulo X/A such that

$$\inf \{\| f - g \|_v \ ; g \in \mathrm{W}\} = \inf \{\| f \mid \mathrm{Y} - g \mid \mathrm{Y} \|_v \ ; g \in \mathrm{W}\} .$$

By Definitions 2 and 5 and the remark made before Definition 10, the module W is sharply localizable under A in $\mathrm{LV}_\infty$.

*Remark.* – Theorem 9 above is a strengthened form of Theorem 2 of [7]. It reduces the search of sufficient conditions for sharp localizability to the search of fundamental weights in the sense of Bernstein in $\mathbf{R}^n$, i.e., to the Finite Dimensional Bernstein Approximation Problem.

THEOREM 14. – *Suppose that there exist sets of generators* $G(A)$ *and* $G(W)$, *for* $A$ *and* $W$ *respectively, such that :*

1) $G(A)$ *is a strong set of generators for* $A$ ;

2) *given any* $v \in V$, $a_1, \ldots, a_n \in G(A)$ *and* $g \in G(W)$, *there exists* $\omega \in \Omega_n^s$ *such that* $v(x)\,[g(x)] \leqslant \omega(|a_1(x)|, \ldots, |a_n(x)|)$ *for all* $x \in X$.

*Then* $W$ *is sharply localizable under* $A$ *in* $LV_\infty$.

*Proof.* – Let $\sigma \in \mathfrak{S}$. Assume that for each $\tau < \sigma$ we have found an element $S_\tau \in P_\tau$ such that

a) $S_\tau \subset S_\mu$ for all $\mu < \tau$ ;

b) $\inf\{\|f - g\|_v \,;\, g \in W\} = \inf\{\|f\,|\,S_\tau - g\,|\,S_\tau\|_v \,;\, g \in W\}$.

*First case.* $\sigma = \tau + 1$ for some $\tau \in \mathfrak{S}$. By the induction hypothesis there is $S_\tau \in P_\tau$ such that a) and b) are true. Let $A_\tau$ be the subalgebra of all $a \in A$ such that $a\,|\,S_\tau$ is real-valued. By Theorem 9 applied to the algebra $A_\tau\,|\,S_\tau$ and the module $W\,|\,S_\tau$ there is a set $S_\sigma \in P_\sigma = P_{\tau+1}$ such that

$$\inf_{g\in W} \|f\,|\,S - g\,|\,S\|_v = \inf_{g\in W} \|f\,|\,S_\sigma - g|\,S_\sigma\|_v$$

On the other hand, $S_\sigma \subset S_\tau$, by construction.

*Second case.* The ordinal $\sigma \in \mathfrak{S}$ has no predecessor. Define $S_\sigma = \cap\{S_\tau \,;\, \tau < \sigma\}$. Then $S_\sigma \in P_\sigma$ and $S_\sigma \subset S_\tau$ for all $\tau < \sigma$. Assume that $\inf\{\|f\,|\,S_\sigma - g\,|\,S_\sigma\|_v \,;\, g \in W\} < d$, where we have defined $d = \inf\{\|f - g\|_v \,;\, g \in W\}$. (The case $d = 0$ is trivial.) There exists $g \in W$ such that $\|f\,|\,S_\sigma - g\,|S_\sigma\|_v < d$. Let $U \subset X$ be the open set $\{t \in X \,;\, v(t)\,[f(t) - g(t)] < d\}$. Then the complement of $U$ in $X$ is compact, and $S_\sigma \subset U$. By compactness, there exist $\tau_1 < \cdots < \tau_n < \sigma$ such that $X\backslash U \subset (X\backslash S_1) \cup \ldots \cup (X\backslash S_n)$, where $S_i = S_\tau$ with $\tau_i = \tau$. However, since $S_n \subset \ldots \subset S_1$, it follows that $S_n \subset U$, a contradiction to b), because $\tau_n < \sigma$.

*Remark.* – Theorem 14 above implies the case $N = n$ of Theorem 3, [7]. Indeed, if A is self-adjoint, and G(A) is a set of generators for A satisfying the hypothesis of Theorem 3 of [7], with $N = n$, then the set ReG(A) $\cup$ *Im*G(A) is a strong set of generators for A satisfying the hypothesis of Theorem 14 above, since $\Omega_n^d \subset \Omega_n^s$, and $|\,\mathrm{Re}\, a\,| \leqslant |a|$, $|\,Im\, a\,| \leqslant |\,a\,|$, for any complex number $a = \mathrm{Re}\; a + i\, Im\; a$.

Our next theorem reduces the search of sufficient conditions for sharp localizability to the One-Dimensional Bernstein Approximation Problem.

THEOREM 15. – *Suppose that there exist sets of generators* G(A) *and* G(W), *for* A *and* W *respectively, such that :*

1) G(A) *is a strong set of generators for* A ;

2) *given any* $v \in V$, $a \in G(A)$, *and* $g \in G(W)$, *there exists* $\gamma \in \Gamma_1^s$ *such that* $v(x)\,[g(x)] \leqslant \gamma(|a(x)|)$ *for all* $x \in X$.

*Then* W *is sharply localizable under* A *in* $LV_\infty$.

*Proof.* – Given any $v \in V$, $a_1, \ldots, a_n \in G(A)$, and $g \in G(W)$, there are $\gamma_i \in \Gamma_1^s$ such that $v(x)\,[g(x)] \leqslant \gamma_i(|a_i(x)|)$ for all $x \in X$, $i = 1, \ldots, n$. Define $\omega$ on $\mathbf{R}^n$ by $\omega(t) = [\gamma_1(t_1) \ldots \gamma_n(t_n)]^{1/n}$ for all $t = (t_1, \ldots, t_n)$. Then $\omega \in \Omega_n$ by Lemma 1, § 27, [6]. Obviously, $\omega(t) = \omega(|t|)$ for all $t \in \mathbf{R}^n$. Hence, $\omega \in \Omega_n^s$. By Theorem 14, W is sharply localizable under A in $LV_\infty$.

*Remark.* – Theorem 15 above implies Theorem 6 of [7]. Indeed, the same argument used in the previous remark applies here.

## 4. Sufficient conditions for sharp localizability.

THEOREM 16. – (Analytic criterion). – *Suppose that there exist sets of generators* G(A) *and* G(W) *such that :*

1) G(A) *is a strong set of generators for* A ;

2) *given any* $v \in V$, $a \in G(A)$, *and* $g \in G(W)$, *there are constants* $M > 0$ *and* $m > 0$ *such that* $v(x)\,[g(x)] \leqslant M e^{-m|a(x)|}$ *for all* $x \in X$.

*Then* W *is sharply localizable under* A *in* $LV_\infty$.

*Proof.* – The function $\gamma(t) = \mathrm{M}\, e^{-m|t|}$ defined for all $t \in \mathrm{R}$, belongs to $\Gamma_1^s$ by Lemma 2, § 28 of [6]. It remains to apply Theorem 15 above.

THEOREM 17. – (Quasi-analytic criterion). – *Suppose that there exist sets of generators* G (A) *and* G (W) *such that :*

1) G (A) *is a strong set of generators for* A ;

2) *given any* $v \in \mathrm{V}$, $a \in \mathrm{G}(\mathrm{A})$, *and* $g \in \mathrm{G}(\mathrm{W})$, *we have*

$$\sum_{m=1}^{\infty} (\mathrm{M}_m)^{-1/m} = +\infty$$

*where* $\mathrm{M}_m = \|a^m g\|_v$ *for* $m = 0, 1, 2, \ldots.$

*Then* W *is sharply localizable under* A *in* $\mathrm{LV}_\infty$.

*Proof.* – Define $\gamma$ on $\mathbf{R}$ as the proof of Theorem 9, [7], and then apply Theorem 15 above.

THEOREM 18. – (Bounded case). – *Suppose that there exist sets of generators* G (A) *and* G (W), *for* A *and* W *respectively, such that :*

1) G (A) *is a strong set of generators for* A ;

2) *given any* $v \in \mathrm{V}$, $a \in \mathrm{G}(\mathrm{A})$, *and* $g \in \mathrm{G}(\mathrm{W})$, *the function* $a$ *is bounded on the support of* $v\,[g]$.

*Then* W *is sharply localizable under* A *in* $\mathrm{LV}_\infty$.

*Proof.* – Let $v \in \mathrm{V}$, $a \in \mathrm{G}(\mathrm{A})$, and $g \in \mathrm{G}(\mathrm{W})$ be given. Let $m > \sup\{|a(x)| \,;\, x \in \mathrm{S}\}$, where S is the support of the function $v\,[g]$ ; and let $\mathrm{M} > \|g\|_v$. If $\gamma$ is the characteristic function of the interval $[-m, m] \subset \mathbf{R}$ times the constant M, then $\gamma \in \Gamma_1^s$ and $v(x)\,[g(x)] \leqslant \gamma(|a(x)|)$ for all $x \in \mathrm{X}$. It remains to apply Theorem 15.

## 5. Vector-valued functions.

The above Theorem 18 generalizes Theorem 4, § 2 of Kleinŝtück [4], which in turn was a generalization of Theorem 4.5 of Prolla [8], and the result of Summers [10].

Indeed, consider the case in which $F_x = E$, for all $x \in X$, where E is a locally convex space ; V is a directed set of upper semicontinuous positive functions on X ; and we take the Nachbin space $LV_\infty$ to be $CV_\infty(X ; E)$, i.e., the vector space of all continuous functions $f \in C(X ; E)$ such that $vf$ vanishes at infinity on X, equipped with the topology given by the seminorms

$$\|f\|_{v,p} = \sup \{v(x) p(f(x)) ; x \in X\}$$

when $v \in V$ and $p \in cs(E)$, the set of all continuous seminorms on E. In this case Theorem 18 reads as follows :

THEOREM 19. – *Let* $W \subset CV_\infty(X ; E)$ *be an* A*-module. Suppose that there exist sets of generators* G(A) *and* G(W), *for* A *and* W *respectively, such that :*

1) G(A) *is a strong set of generators for* A ;

2) *given any* $v \in V$, $p \in cs(E)$, $a \in G(A)$, *and* $g \in G(W)$, *the function a is bounded on the support of the function* $vp(g)$.

*Then* W *is sharply localizable under* A *in* $CV_\infty(X ; E)$.

COROLLARY 20. – *Let* $W \subset CV_\infty(X ; E)$ *be an* A*-module. Suppose that every* $a \in A$ *is bounded on the support of every* $v \in V$. *Then* W *is sharply localizable under* A *in* $CV_\infty(X ; E)$.

*Proof.* – The set A is a strong set of generators for A. Since the support of $v$ contains the supports of $x \longmapsto v(x)p(g(x))$ for any continuous seminorm $p \in cs(E)$ and any $g \in W$, we may apply Theorem 19 with G(A) = A and G(W) = W.

COROLLARY 21. – (Kleinstück [4]). – *Assume the hypothesis of Corollary 20. Then for every* $f \in CV_\infty(X ; E)$, $f$ *belongs to the closure of* W *in* $CV_\infty(X ; E)$ *if, and only if, given any* $v \in V$, $p \in cs(E)$, $\epsilon > 0$, *and* $K \in \mathcal{K}_A$ *there exists* $g \in W$ *such that* $v(x)\, p(f(x) - g(x)) < \epsilon$ *for all* $x \in K$.

*Proof.* – By Corollary 20, W is sharply localizable under A in $CV_\infty(X ; E)$. Therefore, W is sharply $\mathcal{K}_A$–localizable, i.e., given $f \in CV_\infty(X ; E)$, $v \in V$, $p \in cs(E)$, there is some maximal antisymmetric set $K \in \mathcal{K}_A$ such that

$$\inf \{\|f - g\|_{v,p} ; g \in W\} = \inf \{\|f | K - g | K\|_{v,p} ; g \in W\} .$$

This formula generalizes that obtained by Glicksberg in the case of Bishop's Theorem (see [2]), and from it there follows the desired conclusion.

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Manuscrit reçu le 4 octobre 1976
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