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BIHOLOMORPHIC MAPS DETERMINED ON THE BOUNDARY

by Nozomu MOCHIZUKI

Let X, Y be complex manifolds of pure dimension n where the holomorphic functions on X separate points ; let D be a relatively compact open subset of X, and \widetilde{D} a neighborhood of \overline{D} . Let $f: \widetilde{D} \longrightarrow Y$ be a holomorphic map. The object of the present note is to show under certain conditions that if f is one-to-one when restricted to the boundary bD of D, then $f: D \longrightarrow f(D)$ is biholomorphic. In case X = Y =the complex plane, if bD is a rectifiable Jordan curve, then f(D) is the domain surrounded by the curve f(bD) and $f: D \longrightarrow f(D)$ is conformal. A corollary is deduced to extend this theorem to the case of higher dimensions. We begin with a lemma which will be stated in a form a little more general than actually needed.

LEMMA. — Let $f: \widetilde{D} \longrightarrow Y$ be a holomorphic map. If f has finite fibres on bD, then so does f on D.

Proof. – Let $F = \{p \in \widetilde{D} | f(p) = q_0\}, q_0 \in f(D)$, and suppose that $F \cap D$ is noncompact. Then $F \cap bD \neq \phi$; this constitutes a finite set of points $\{p_1, p_2, \ldots, p_s\}$. There exists a point p_i such that $F \cap D \cap U \neq \phi$ for every neighborhood U of p_i . We take mutually disjoint open neighborhoods U_i of p_i in \widetilde{D} , $i = 1, 2, \ldots, s$, for which $F \cap U_i = V_i^1 \cup V_i^2 \cup \ldots \cup V_i^{m_i}$ is the decomposition of F into irreductible branches at p_i , and the sets $R(V_i^m)$ of regular points of V_i^m are connected manifolds which are dense in V_i^m . There are a point p_j and a branch V_j^m such that $V_j^m \cap D \neq \phi$ and $V_j^m - \overline{D} \neq \phi$, because, if this is not the case, then $F \cap D$ and all the branches contained

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in \overline{D} constitute a compact subvariety of $D \cup \bigcup_{i=1}^{s} U_i$, so that $F \cap D$ becomes a finite set of points. The dimension of such a variety V_i^m at p_j is positive. We choose $p'_1 \in \mathbb{R}(V_j^m) \cap D$ and $p'_2 \in \mathbb{R}(V_j^m) - \overline{D}$.

Then there is a curve in the connected manifold $R(V_j^m) - \{p_j\}$ which joins p'_1 to p'_2 , and this must intersect *bD*. But this is impossible, and the proof is completed.

In what follows, differentiability will mean that of C^{∞} . We denote by ∂D the totality of regular points of bD; that is, $p_0 \in \partial D$ if and only if $p_0 \in bD$ and there exist a neighborhood U of p_0 and a differentiable coordinate system $\phi = (x_1, x_2, \dots, x_{2n}) : U \longrightarrow \Delta(0; \epsilon)$, the ϵ -cube in \mathbb{R}^{2n} centered at the origin 0, such that

$$\phi(p_0) = 0, \ D \cap U = \{p \in U \mid x_{2n}(p) \ge 0\}.$$

THEOREM. – Let D be a relatively compact open subset of X such that $\partial D \neq \phi$. If f is one-to-one on bD and f(D) - f(bD) is connected, then $f: D \longrightarrow f(D)$ is biholomorphic.

Proof. – We may assume that X and Y have countable bases for open sets. Note that $f: D \longrightarrow Y$ is an open map by the above lemma. Let G = f(D), $G_0 = G - f(bD)$, and $D_0 = D - f^{-1}(f(bD))$. G_0 is dense in G, since $f: bD \longrightarrow f(bD)$ is a homeomorphism. Let

$$S = \{ p \in D \mid \operatorname{rank}_n f < n \}.$$

By Sard's theorem, $D \cap S$ is a nowhere dense analytic subvariety of D, so it can be assumed, by shrinking \widetilde{D} if necessary, that S is nowhere dense in \widetilde{D} . The restricted map $f : D_0 \longrightarrow G_0$ is proper, and

$$f_0 : \mathrm{D}_0 - f^{-1} (f(\mathrm{D}_0 \cap \mathrm{S})) \longrightarrow \mathrm{G}_0 - f(\mathrm{D}_0 \cap \mathrm{S})$$

is a finitely sheeted covering map. $G_0 - f(D_0 \cap S)$ is dense in G; it follows that if f_0 is one-to-one, then so is $f: D \longrightarrow G$. For the differentiable map $f: D_0 \longrightarrow G_0$, the connectedness of G_0 guarantees the existence of a constant δ , the degree of f, such that if ω is a 2nform of compact support in G_0 then

$$\int_{D_0} f^* \omega = \delta \int_{G_0} \omega ;$$

this δ coincides with the number of sheets of the covering map $f_0([1])$. Thus, we have only to show that $\delta = 1$.

We shall show that $f(\partial D - S) \subset \partial G$, where it should be noted that $\partial D \notin S$ since ∂D is a real (2n - 1)-dimensional manifold. Let $p_0 \in \partial D - S$, $q_0 = f(p_0)$. We take an open neighborhood U' of p_0 in \widetilde{D} such that $f' = f | U' : U' \longrightarrow V'$ is biholomorphic where V' is a neighborhood of q_0 . We assume that

$$\phi = (x_1, x_2, \dots, x_{2n}) : \mathbf{U}' \longrightarrow \Delta(0; \epsilon)$$

is a coordinate system for which

$$\phi(p_0) = 0, \quad \overline{D} \cap U' = \{p \in U' \mid x_{2n}(p) \ge 0\}.$$

Let $y_i = x_i \circ f'^{-1}$, i = 1, 2, ..., 2n, then

$$\psi = (y_1, y_2, \dots, y_{2n}) : \mathbf{V}' \longrightarrow \Delta(0; \epsilon)$$

is a coordinate system for V'. Suppose that $q_0 \in G$ and $V' \subset G$. Since $q_0 \notin f(bD - U')$, we can find $V = \psi^{-1} (\Delta(0; \rho)), 0 < \rho < \epsilon$, so that $V \cap f(bD - U') = \phi$. Put $U = f'^{-1} (V)$. Let ω be a 2*n*-form : $\omega = g \, dy_1 \wedge dy_2 \wedge \ldots \wedge dy_{2n}$ where g is a differentiable function of compact support in V. Let $\{\rho_k\}$, $\{\rho'_k\}$ be sequences of positive numbers such that

$$\rho_1 < \rho_2 < \ldots < \rho$$
, $\rho_k \longrightarrow \rho$; $\rho_1 > \rho'_1 > \rho'_2 > \ldots$, $\rho'_k \longrightarrow 0$,

and let

$$\mathbf{Q}_{k} = \{ q \in \mathbf{V} \mid |y_{i}(q)| < \rho_{k}, 1 \leq i \leq 2n - 1; \rho_{k}' < |y_{2n}(q)| < \rho_{k} \},\$$

 $k = 1, 2, \ldots$. Note that $Q_k \subset G_0$. We choose differentiable functions g_k with the property that

$$g_{k}(q) = \begin{cases} g(q) &, q \in Q_{k} \\ 0 &, q \in Y - Q_{k+1} \end{cases}$$

and $|g_k(q)| \leq \text{const.}$ for all $q \in Y$ and k. Putting $\omega_k = g_k dy_1 \wedge dy_2 \wedge \ldots \wedge dy_{2n}$, we have

$$\int_{\mathcal{D}} f^* \omega_k = \delta \int_{\mathcal{G}} \omega_k \quad , \quad k = 1, 2, \ldots$$

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Let $H = D - \overline{U}$, then (supp $f^* \omega$) $\cap \overline{D} \subset H \cup (\overline{D} \cap U)$. The set $E = \{q \in V | y_{2n}(q) = 0\}$ is of measure zero in Y and, since f is locally biholomorphic on $\widetilde{D} - S$,

$$f^{-1}$$
 (E) $\cap \overline{\mathbb{D}} = (f^{-1}$ (E) $\cap \overline{\mathbb{D}} \cap \mathbb{S}) \cup (f^{-1}$ (E) $\cap (\overline{\mathbb{D}} - \mathbb{S}))$

is also of measure zero. Therefore, $g_k \longrightarrow g$, a.e., on \overline{G} and

$$g_k \circ f \longrightarrow g \circ f,$$

a.e., on $H \cup (\overline{D} \cap U)$. Thus, we obtain

$$I = \lim_{k \to \infty} \int_{D} f^* \omega_k = \int_{H} f^* \omega + \int_{D \cap U} f^* \omega, \quad I = \delta \int_{G} \omega.$$

Let *h* be a nonnegative differentiable function of compact support in V such that $h(q_0) > 0$ and let $\theta = h \, dy_1 \wedge \ldots \wedge dy_{2n-1}$. The support of $f^*\theta$ in H is compact, so we get from the preceding formula applied to $\omega = d\theta$

$$I = \int_{D \cap U} d(f^*\theta) = \int_{\partial D \cap U} f^*\theta = \int_E \theta > 0, \quad I = \delta \int_G d\theta = 0,$$

a contradiction. Thus, we have proved $f(\partial D - S) \subset bG$. Now take U', V' as in the above. Since $f(\partial D \cap U') \subset bG \subset f(bD)$ where $f(\partial D \cap U')$ is open in f(bD), we can find a neighborhood W of q_0 in V' so that $bG \cap W \subset f(\partial D \cap U')$. Take $V = \psi^{-1} (\Delta (0; \rho))$ in W such that $V \cap f(bD - U') = \phi$, and let $U = f'^{-1}$ (V). We see that $bG \cap V = f(\partial D \cap U)$. V is decomposed as follows :

$$V = f (D \cap U) \cup f (\partial D \cap U) \cup f (U - \overline{D})$$
$$= (G \cap V) \cup (bG \cap V) \cup (V - \overline{G}),$$

where $f(D \cap U) \subseteq G \cap V, V = \overline{G} \subseteq f(U = \overline{D})$. Suppose that

$$V - \overline{G} = \phi$$
.

Then, from $V - f(\partial D \cap U) \subset G$, we can deduce a contradiction just as in the above. Thus, $V - \overline{G} \neq \phi$ and, from the connectedness of $f(U - \overline{D})$, we see that $f(U - \overline{D}) \cap G \cap V = \phi$, which implies that $f(D \cap U) = G \cap V$. It follows that

$$G \cap V = \{q \in V \mid y_{2n}(q) > 0\}$$

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and
$$b \in G \cap V = \partial G \cap V$$
. In the present situation, let
 $Q_k = \{q \in V | |y_i(q)| < \rho_k, \rho'_k < y_{2n}(q) < \rho_k\}, k = 1, 2, ...,$

and choose g_k as before for $\omega = g \, dy_1 \wedge dy_2 \wedge \ldots \wedge dy_{2n}$. For $\omega = d\theta$, we have

$$\mathbf{I} = \int_{\partial \mathbf{D} \cap \mathbf{U}} f^* \theta = \int_{\partial \mathbf{G} \cap \mathbf{V}} \theta, \quad \mathbf{I} = \delta \int_{\mathbf{G}} d\theta = \delta \int_{\partial \mathbf{G} \cap \mathbf{V}} \theta ;$$

these yield $\delta = 1$. This completes the proof.

As a typical example in which the condition of Theorem is satisfied, we deal with the following case.

COROLLARY. – Let D be a bounded open subset of the complex n-space \mathbb{C}^n such that b D is topologically a (2n-1)-dimensional sphere in \mathbb{R}^{2n} with $\partial D \neq \phi$, and let $f : \widetilde{D} \longrightarrow \mathbb{C}^n$ be holomorphic. If f is oneto-one on bD, then $f : D \longrightarrow f(D)$ is biholomorphic where f(D) is the domain surrounded by the sphere f(b D).

Proof. -f(bD) is a (2n-1)-sphere in \mathbb{C}^n , so that $\mathbb{C}^n - f(bD)$ is decomposed into two components G and G' with f(bD) = bG = bG'. Let G be the bounded component. Let $f(D) \cap G' \neq \phi$. If $G' \notin f(D)$, then $bf(D) \cap G' \neq \phi$, which contradicts $bf(D) \subset f(bD)$; hence we have $G' \subset f(D)$, which contradicts the boundedness of f(D). Thus, $f(D) \subset G$. It follows from the same reasoning that f(D) = G. We have f(bD) = bf(D), and the proof is completed.

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