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CONICAL MEASURES AND VECTOR MEASURES

by Igor KLUVÁNEK

One of the most important objects related to a family $\{f_\iota : \iota \in \mathrm{I}\}$ of (real-valued) random variables is the joint distribution of this family. It is a mesure on the cylindrical σ-algebra in the space $\mathbf{R}^{\mathrm{I}}$. Indeed, all properties of this family (as a family of random variables) are reflected by its distribution. It can happen that neither the family of random variables itself nor the probability measure on the underlying space are available but we are given the indefinite integrals of all random variables f_ι, $\iota \in \mathrm{I}$ (assuming them integrable). In that case the joint distribution cannot be determined. However, the conical measure on $\mathbf{R}^{\mathrm{I}}$ determined by the joint distribution still can be reconstructed. It is clear that there remain many properties of the family of random variables which can be inferred from this conical measure.

Noticing the obvious fact that the family of indefinite integrals of the functions f_ι, $\iota \in \mathrm{I}$, can be interpreted as a measure with values in the space $\mathbf{R}^{\mathrm{I}}$, we propose to investigate the indicated situation in a more general setting. To every abstract measure with values in a locally convex topological vector space E, there corresponds a conical measure on E and also on the weak completion E'^* of E. There are many properties of the vector measure which can be determined from the corresponding conical measure. Also conversely, the vector measure can be made a tool for studying the conical measure.

In the first Section, it is proved that every conical measure on a weak complete space E is represented (not uniquely) as integration

with respect to a non-negative measure λ on the cylindrical σ-algebra in E. And λ can even be extended on a larger σ-algebra so as to become a direct sum of finite measures.

This result and the connection between measures with values in E and conical measures on E is used in Sections 2 and 3 to obtain information on the general structure of abstract E-valued measures. In the final Section are stated some results concerning conical measures. In particular, some conditions are given guaranteeing that a conical measure on E be expressible as integration with respect to a finite (σ-additive) measure on the cylindrical σ-algebra in E.

1. Representation of conical measures.

Let E be a real locally convex topological vector space with topological dual E′ and algebraic dual E*.

The vector lattice of functions on E generated by E′ is denoted by h (E). Every element f of h (E) can be written as

$$f(x) = \sup \{f_i(x) : 1 \leqslant i \leqslant l\} - \sup \{f_i(x) : l + 1 \leqslant i \leqslant k\}, \; x \in \mathrm{E},$$

where $f_i \in \mathrm{E}'$, for $1 \leqslant i \leqslant k$, $1 \leqslant l < k$.

A non-negative linear form on h (E) is termed a *conical measure* on E (see Choquet [3] or [4]).

The interest in this Section is focused on weak complete spaces. If E is a weak complete topological vector space then there exists an index set I such that, up to a topological vector space isomorphism,

$$\mathrm{E} = \mathbf{R}^{\mathrm{I}} ;$$

the structure of the topological vector space on $\mathbf{R}^{\mathrm{I}}$ being that of the product of one-dimensional ones.

Without a loss of generality it will be assumed that I is well-ordered and, indeed, that I is an interval of ordinal numbers with 0 as the least element.

For every $\kappa \in \mathrm{I}$, let p_κ be the projection of $\mathrm{E} = \mathbf{R}^{\mathrm{I}}$ onto

its κ-th co-ordinate ; that is, $p_\kappa(x) = x_\kappa$, for every element $x = (x_\iota)_{\iota \in I}$ of E.

It is known that $\{p_\iota : \iota \in I\}$ forms a Hamel basis for E'. Also h (E) is the vector lattice generated by $\{p_\iota : \iota \in I\}$.

For every $\kappa \in I$, define

$$T_\kappa = \{x \in E : p_\iota(x) = 0 \text{ for } \iota < \kappa \text{ and } | p_\kappa(x) | = 1\}.$$

Furthermore, let

$$T = \bigcup_{\kappa \in I} T_\kappa.$$

Given a function f on E, by $f \mid T$ is denoted its restriction to T and by $f \mid T_\kappa$ its restriction to T_κ; h (E) $\mid T_\kappa = \{f \mid T_\kappa : f \in h(E)\}$, $\kappa \in I$; h (E) $\mid T = \{f \mid T : f \in h(E)\}$.

Let S_κ be the σ-algebra of subsets of T_κ generated by h (E) $\mid T_\kappa$, i.e. the associated σ-algebra such that all functions in h (E) $\mid T_\kappa$ are S_κ-mesurable, $\kappa \in I$. Let S be the direct sum of σ-algebras S_κ, $\kappa \in I$. That is, S consists of all sets $X \subset T$ such that $X \cap T_\kappa \in S_\kappa$, for every $\kappa \in I$.

If λ_κ is a measure on S_κ, for each $\kappa \in I$, a measure λ on S is termed the direct sum of the measures λ_κ, $\kappa \in I$, if

$$\lambda(X) = \sum_{\kappa \in I} \lambda_\kappa (X \cap T_\kappa), \; X \in S.$$

We say also, in this case, that the measure space (T, S, λ) is the direct sum of the measure spaces $(T_\kappa, S_\kappa, \lambda_\kappa)$, $\kappa \in I$.

THEOREM 1.– *Let u be a conical measure on the weak complete space* $E = \mathbf{R}^I$. *Then, for every $\kappa \in I$, there exists a unique finite measure λ_κ on S_κ such that, if (T, S, λ) is the direct sum of the measure spaces $(T_\kappa, S_\kappa, \lambda_\kappa)$, $\kappa \in I$, then*

$$u(f) = \int_T (f \mid T)\, d\lambda \tag{1}$$

for every $f \in h$ (E). The vector lattice h (E) $\mid$ T is dense in $L^1(\lambda)$.

The main part of this Section is devoted to the proof of Theorem 1.

According to Choquet [3, Théorème 17] or [4, Theorem 38.3], u is a Daniell integral on h (E). Let L be a linear lattice of functions on E containing h (E) and such that

i) there is a Daniell integral on L which is an extension of u and which will be denoted by u, again ;

ii) the Dominated convergence theorem (Lebesgue theorem) holds for this extension of u on L ;

iii) for every $f \in L$ and $\epsilon > 0$, there exists $g \in h$ (E) such that $u(|f - g|) < \epsilon$;

iv) all function in L are real-valued (the values $\pm \infty$ are excluded) and belong to the least class of functions containing h (E) which is closed with respect to the point-wise convergence of sequences.

The general theory of Daniell integrals guarantees that such L and the corresponding extension of u exist.

We put, as customary, $L^+ = \{f \in L : f \geqslant 0\}$.

The property (iv) implies that every function f in L is positive-homogeneous ($\alpha > 0$ and $x \in$ E imply that $f(\alpha x) = \alpha f(x)$). The positive homogeneity in turn implies that u can be considered a Daniell integral on the vector lattice $L \mid \mathrm{T} = \{f \mid \mathrm{T} : f \in L\}$. Specifically, if $f_n \in L$, $n = 1, 2, \ldots$, and if $f_n(t) \downarrow 0$ for every $t \in$ T, then $f_n(x) \downarrow 0$ for every $x \in$ E and $u(f_n) \to 0$, $n \to \infty$.

Let u_ι and v_ι be functionals defined on L inductively for every $\iota \in$ I as follows.

Let $\kappa \in$ I. Assume that u_ι is already defined for each $\iota < \kappa$. Denote

$$v_\kappa(f) = u(f) - \sum_{\iota < \kappa} u_\iota(f), \tag{2}$$

for every $f \in L$, and put

$$u_\kappa(f) = \sup \{v_\kappa(f \wedge \alpha \mid p_\kappa \mid) : \alpha > 0\} \tag{3}$$

for every $f \in L^+$; furthermore $u_\kappa(f) = u_\kappa(f^+) - u_\kappa(f^-)$ for every $f \in L$.

LEMMA 2. – *The functionals u_ι, v_ι are well-defined Daniell integrals on and L and $0 \leqslant u_\iota \leqslant v_\iota \leqslant u$, for every $\iota \in$ I.*

Proof. – Let $\kappa \in$ I. Assume that u_ι, v_ι are Daniell integrals and $0 \leqslant u_\iota \leqslant v_\iota \leqslant u$, for every $\iota < \kappa$.

Given $f \in L^+$, we have

$$0 \leqslant v_\nu(f) = u(f) - \sum_{\iota<\nu} u_\iota(f)$$

for every $\nu < \kappa$, Hence

$$0 < \sum_{\iota<\kappa} u_\iota(f) = \sup \{\sum_{\iota<\nu} u_\iota(f) : \nu < \kappa\} \leqslant u(f).$$

i.e. $0 \leqslant v_\kappa(f) \leqslant u(f)$. In particular, v_κ is well-defined by (2). It is then clear also that v_κ is a linear form on L. The inequality

$$0 \leqslant v_\kappa(f) \leqslant u(f).$$

for $f \in L^+$, implies that v_κ is a Daniell integral.

Now, u_κ is well-defined by (3) and $0 \leqslant u_\kappa(f) \leqslant v_\kappa(f)$ for every $f \in \mathrm{L}^+$. The definition then extends on the whole of L. We show that u_κ is a linear form.

Let $f, g \in L^+$. Since $(f+g) \wedge \alpha \mid p_\kappa \mid \leqslant (f \wedge \alpha \mid p_\kappa \mid + (g \wedge \alpha \mid p_\kappa \mid)$ for every $\alpha > 0$, the definition (3) gives that $u_\kappa(f+g) \leqslant u_\kappa(f) + u_\kappa(g)$. On the other hand, for every $\epsilon > 0$ there is $\alpha > 0$ and $\beta > 0$ such that

$$u_\kappa(f) - \epsilon < v_\kappa(f \wedge \alpha \mid p_\kappa \mid), \; u_\kappa(g) - \epsilon < v_\kappa(g \wedge \beta \mid p_\kappa \mid).$$

Consequently,

$$u_\kappa(f) + u_\kappa(g) - 2\epsilon < v_\kappa(f \wedge \alpha \mid p_\kappa \mid) + v_\kappa(g \wedge \beta \mid p_\kappa \mid)$$

$$\leqslant v_\kappa((f+g) \wedge (\alpha+\beta) \mid p_\kappa \mid) \leqslant u_\kappa(f+g).$$

So, $u_\kappa(f+g) = u_\kappa(f) + u_\kappa(g)$.

For every $f \in L^+$ and $c > 0$,

$$u_\kappa(cf) = \sup \{v_\kappa(cf \wedge \alpha \mid p_\kappa \mid) : \alpha > 0\} =$$

$$= c \sup \{v_\kappa(f \wedge (\alpha/c) \mid p_\kappa \mid) : \alpha > 0\} = c\, u_\kappa(f).$$

The linearity of u_κ follows.

The inequality $0 \leqslant u_\kappa(f) \leqslant v_\kappa(f)$, for each $f \in L^+$, implies that u_κ is a Daniell integral.

LEMMA 3. – *Let $f \in L^+$. Then*

$$\sum_{\iota \leqslant \kappa} u_\iota(f) = \sup \{u(f \wedge \varphi) : \varphi \in L, \varphi(t) = 0 \text{ for } t \in \bigcup_{\kappa < \iota} T_\iota\} \quad (4)$$

and

$$\sum_{\iota < \kappa} u_\iota(f) = \sup \{u(f \wedge \varphi) : \varphi \in L, \varphi(t) = 0 \text{ for } t \in \bigcup_{\kappa \leqslant \iota} T_\iota\} \quad (5)$$

for every $\kappa \in I$.

Proof. – If $\kappa = 0$, the statement (4) is true by the definition of u_0 and (5) is true vacuously.

Let $\kappa \in I$, $\kappa > 0$. Assume that (4) holds if κ is replaced by any $\nu \in I$, $\nu < \kappa$. Then (5) follows. An inductive proof will be proving (4).

Let φ_n be functions vanishing on every T_ι with $\kappa \leqslant \iota$ such that $\varphi_n \leqslant \varphi_{n+1}$, $n = 1,2,\ldots$, and $u(f \wedge \varphi_n) \to \sum_{\iota < \kappa} u_\iota(f)$. Let

$$g = \lim f \wedge \varphi_n = \sup f \wedge \varphi_n.$$

Then

$$u(g) = u(f \wedge g) = \sum_{\iota < \kappa} u_\iota(f). \quad (6)$$

Moreover, if $\theta \in L^+$ is a function vanishing on every T_ι with $\kappa \leqslant \iota$, then

$$u((f - g) \wedge \theta) = 0. \quad (7)$$

Indeed, if (7) were not true then

$$u(f \wedge (g + \theta)) = u(f \wedge g) + u((f - g) \wedge \theta) > u(g)$$

would follow, contradicting (6) and (5).

Also, if $f' \leqslant f, f' \in L^+$, then $\sum_{\iota < \kappa} u_\iota(f') = u(f' \wedge g)$. This because, for any $\varphi \in L^+$ vanishing on every T_ι with $\kappa \leqslant \iota$, we have

$$u(f' \wedge \varphi) \leqslant u(f' \wedge g) + u((f' - f' \wedge g) \wedge (\varphi - \varphi \wedge g)) \leqslant u(f' \wedge g) +$$
$$+ u((f - g) \wedge (\varphi - \varphi \wedge g)) = u(f' \wedge g),$$

choosing $0 = \varphi - \varphi \wedge g$ in (7).

We have, in fact, proved that $v_\kappa(f') = u(f') - u(f' \wedge g)$ for every $f' \in L^+, f' \leqslant f$.

Let $h = \lim_{n\to\infty} f \wedge n \mid p_\kappa \mid$. Hence $u_\kappa(f) = v_\kappa(h) = v_\kappa(f \wedge h)$. Now

$$\sum_{\iota \leqslant \kappa} u_\iota(f) = \sum_{\iota < \kappa} u_\iota(f) + u_\kappa(f) = u(f \wedge g) + v_\kappa(f \wedge h) = u(f \wedge g) +$$

$$+ \; u(f \wedge h) - u(f \wedge h \wedge g) = u(f \wedge (g \vee h)).$$

Suppose that $\varphi \in L^+$ and that φ vanishes on every T_ι such that $\kappa < \iota$. Then

$$u(f \wedge \varphi) \leqslant u(f \wedge (g \vee h) \wedge \varphi) + u((f - g \vee h) \wedge \varphi). \tag{8}$$

The function $\theta = (f - g \vee h) \wedge \varphi$ vanishes on every T_ι with $\kappa \leqslant \iota$ and $\theta = (f - g) \wedge \theta$, hence, by (7), $u((f - g \vee h) \wedge \varphi) = 0$. So (8) gives that $u(f \wedge \varphi) \leqslant u(f \wedge (g \vee h))$, proving (4).

LEMMA 4. – *For every $f \in L$ vanishing on every T_ι such that $\kappa \leqslant \iota$, the relation*

$$\sum_{\iota < \kappa} u_\iota(f) = u(f) \tag{9}$$

holds. The equality

$$\sum_{\iota \in I} u_\iota(f) = u(f) \tag{10}$$

holds for every $f \in L$.

Proof. – The first statement (9) follows immediately form (4). This implies (10) for every $f \in h(E)$ since each $f \in h(E)$ vanishes on every T_ι, $\kappa < \iota$, for some $\kappa \in I$.

Let now $f \in L$. For every $\epsilon > 0$ there exists $g \in h(E)$ such that $u(|f - g|) < \epsilon$. Consequently,

$$|u(f) - \sum_{\iota \in I} u_\iota(f)| \leqslant |u(f) - u(g)| + |u(g) - \sum_{\iota \in I} u_\iota(g)| +$$

$$+ |\sum_{\iota \in I} u_\iota(g) - \sum_{\iota \in I} u_\iota(f)| \leqslant u(|f - g|) + |u(g) - \sum_{\iota \in I} u_\iota(g)| +$$

$$+ \sum_{\iota \in I} u_\iota(|f - g|) \leqslant 2\, u(|f - g|) < 2 \;\; \epsilon.$$

LEMMA 5. – *Let* $\kappa \in \mathrm{I}$. *Let* $f_n \in L$, $n = 1,2,\ldots$, *and let* $f_n(t) \downarrow 0$, *for every* $t \in \mathrm{T}_\kappa$. *Then* $u_\kappa(f_n) \to 0$.

Proof. – If $g \in L$ and $g(t) = 0$ for every $t \in \mathrm{T}_\kappa$, then

$$g(t) \wedge \alpha \mid p_\kappa(t) \mid = 0$$

for every $t \in \bigcup_{\kappa \leqslant \iota} \mathrm{T}_\iota$. Hence, by Lemma 3, $v_\kappa(g \wedge \alpha \mid p_\kappa \mid) = 0$ for each $\alpha > 0$. So, $u_\kappa(g) = 0$.

Now put $g_1 = f_1$ and $g_n = g_{n-1} \wedge f_n$, $n = 2,3,\ldots$, Then

$$g_n(t) \downarrow g(t),$$

for every $t \in \mathrm{T}$, where g is a function in L such that $g(t) = 0$ for all $t \in \mathrm{T}_\kappa$. Hence $u_\kappa(f_n) = u_\kappa(g_n) \to u_\kappa(g) = 0$.

Proof of theorem 1. – By Lemma 5, the functional

$$f \mid \mathrm{T}_\kappa \to u_\kappa(f), f \in L$$

is a Daniell integral on the vector lattice $\{f \mid \mathrm{T}_\kappa : f \in L\}$, for every $\kappa \in \mathrm{I}$. Since this vector lattice, or even $h(\mathrm{E}) \mid \mathrm{T}_\kappa$, contains non-zero constants, there exists a unique measure λ_κ on S_κ such that

$$u_\kappa(f) = \int_{\mathrm{T}_\kappa} (f \mid \mathrm{T}_\kappa)\, d\lambda_\kappa$$

for every $f \in L$. The measure λ_κ is finite, $\lambda_\kappa(\mathrm{T}_\kappa) < \infty$, and $h(\mathrm{E}) \mid \mathrm{T}_\kappa$ is dense in $L^1(\lambda_k)$.

If λ is the direct sum of measures λ_κ, $\kappa \in \mathrm{I}$, then (1) holds for every $f \in L$ and $h(\mathrm{E}) \mid \mathrm{T}$ becomes dense in $L^1(\lambda)$.

The cylindrical σ-algebra $C = C(\mathrm{E})$ in the space E is by definition the smallest σ-algebra of subsets of E such that every continuous linear form on E is C-measurable ; it is the smallest σ-algebra such that every function in $h(\mathrm{E})$ is C-measurable.

COROLLARY 6. – *If* E *is a weak complete space then, for every conical measure* u *on* E, *there exists a* σ*-additive measure* λ *on the cylindrical* σ*-algebra* C *such that*

$$u(f) = \int_{\mathrm{E}} f\, d\lambda \tag{11}$$

for every $f \in h(\mathrm{E})$. *Moreover there exists a* σ*-algebra* D *of subsets of* E *such that* $C \subset D$ *and for every conical measure* u *on* F *there is a* σ*-additive measure* λ *on* D *which is a direct sum of finite measures such that* (10) *holds.*

Proof. – Define D to be the family of all sets $\mathrm{X} \subset \mathrm{E}$ such that $\mathrm{X} \cap \mathrm{T}_\kappa \in S_\kappa$, for each $\kappa \in \mathrm{I}$. Given a conical measure u, define the measure λ on D by $\lambda(\mathrm{X}) = \sum_{\kappa \in \mathrm{I}} \lambda(\mathrm{X} \cap \mathrm{T}_\kappa)$, $\mathrm{X} \in D$. Then, clearly, (11) holds and λ is a direct sum of finite measures.

It is clear that $C \subset D$ and that if λ is restricted to C then (11) remains valid.

2. Families of scalar measures.

In this Section a theorem is proved establishing an isometry and vector lattice isomorphism of any vector lattice of finitely additive real measures onto a dense linear sub-lattice of a space $L^1(\lambda)$ with a suitable measure λ which is direct sum of finite measures. This result can be, of course, obtained using Kakutani's concrete representation theorem of abstract L-spaces. The aim is, however, to base the proof on Theorem 1 so as to show its implications and to achieve a unity of method. This could be of interest by offering a new and, possibly, more direct approach to Kakutani's theorem. The proved theorem will be used as a lemma in the subsequent Section concerning the structure of vector measures.

Let T be an abstract space and let R be an algebra of subsets of T. By $\mathrm{ba}(R)$ is denoted the set of all bounded finitely additive real-valued functions (measures) on R. The set $\mathrm{ba}(R)$ carries several structures. It is a vector lattice with respect to the set-wise linear operations and order. If $\mu, \nu \in \mathrm{ba}(R)$, then $\mu \vee \nu$ is the least element in $\mathrm{ba}(R)$ majoring both μ and ν and $\mu \wedge \nu$ is the greatest element majored by μ and ν ; $|\mu| = \mu \vee (-\mu)$ for every $\mu \in \mathrm{ba}(R)$. The norm $\mu \mapsto \|\mu\| = |\mu|(\mathrm{T})$, $\mu \in \mathrm{ba}(R)$, makes of $\mathrm{ba}(R)$ a Banach space.

The set of all σ-additive elements of $\mathrm{ba}(R)$ is denoted by $\mathrm{ca}(R)$. It is a closed normed subspace of $\mathrm{ba}(R)$, hence also a Banach space ; also it is a vector sub-lattice of $\mathrm{ba}(R)$.

Let $H \subset ba(R)$. Two sets $X_1, X_2 \in R$ are declared H-equivalent if $|\mu|(X_1 \Delta X_2) = 0$, for every $\mu \in H$. The H-equivalence class of a set X is denoted $[X]_H$ whenever the distinction is material ; otherwise we do not distinguish between a set in R and its H-equivalence class. The set of all H-equivalence classes is denoted by $R(H)$. It is a Boolean algebra with respect to the induced operations ; i.e. $R(H)$ equals to R modulo the ideal of sets H-equivalent to ϕ.

Every measure $\mu \in H$ defines the Nikodym pseudo-metric d_μ on R and on $R(H)$; namely $d_\mu(X, Y) = |\mu|(X \Delta Y)$, for $X, Y \in R$. By $\tau(H)$ is denoted the topology and also the uniform structure on R and on $R(H)$ determined by the system $\{d_\mu : \mu \in H\}$ of pseudo-metrics.

In the situation when R is a σ-algebra and λ is a σ-additive measure on R having possibly infinite values, $R(\lambda)$ and $\tau(\lambda)$ are interpreted as $R(L^1(\lambda))$ and $\tau(L^1(\lambda))$, respectively, where $L^1(\lambda)$ in its turn is interpreted as the family of indefinite integrals of λ-integrable functions, i.e. as a subset of $ca(R) \subset ba(R)$. In the case when λ is localizable in the sense of Segal [10], and this is the case of our main interest, we can say that $\tau(\lambda)$ stands for the topology $\tau(H)$ where H is the set of all finite measures absolutely continuous with respect to λ. In this situation, a net X_i of sets in R tends to $X \in R$ in the topology $\tau(\lambda)$ if and only if $\lambda(X_i \Delta X) \to 0$.

Let $H \subset ba(R)$ be a vector sub-lattice. Let $\hat{E} = H^*$ be the algebraic dual of the vector space H.

For every $\mu \in H$ and $x \in \hat{E}$, denote $\mu^*(x) = x(\mu)$. If we give $\hat{E}$ its $\sigma(E, H)$-topology, then $\hat{E}'$ will be identified with H by the star-mapping. This is to say, μ^* belongs to $\hat{E}'$, for every $\mu \in \hat{H}'$. and, conversely, every element of $\hat{E}'$ is equal to μ^* for some $\mu \in H$. It follows that every function $f \in h(\hat{E})$ can be expressed as

$$f(x) = \sup\{x(\mu_i) : 1 \leqslant i \leqslant l\} - \sup\{x(\mu_i) : l+1 \leqslant i \leqslant k\} =$$

$$= \sup\{\mu_i^*(x) : 1 \leqslant i \leqslant l\} - \sup\{\mu_i^*(x) : l+1 \leqslant i \leqslant k\}, x \in \hat{E}, \qquad (12)$$

for some $\mu_i \in H$, $1 \leqslant i \leqslant k$, $1 \leqslant l < k$.

LEMMA 7. – *There exists a unique vector lattice homomorphism* $\Phi : h(\hat{E}) \to H$ *such that* $\Phi(\mu^*) = \mu$, *for* $\mu \in H$. *For every* $f \in h(\hat{E})$, *written in the form* (12), *the value* $\Phi(f)$ *is given by*

$$\Phi(f) = \bigvee_{i=1}^{l} \mu_i - \bigvee_{i=l+1}^{k} \mu_i, \tag{13}$$

where the lattice operations are those of H *or* ba (R).

Proof. – If such a homomorphism exists then, clearly, (13) holds. Hence it suffices to show that (13) defines unambiguously a homomorphism $\Phi : h(\hat{E}) \to H$.

Suppose that $f \in h(\hat{E})$ is given by (12) and that

$g(x) = \sup \{x(\nu_i) : 1 \leqslant i \leqslant l'\} - \sup \{x(\nu_i) : l' + 1 \leqslant i \leqslant k'\}$, $x \in \hat{E}$, with $\nu_i \in H$, $1 \leqslant i \leqslant k'$, $1 \leqslant l' < k'$.

If $f = g$, i.e. $f(x) = g(x)$, for every $x \in \hat{E}$, then, in particular,

$$\sup \{\mu_i(X) : 1 \leqslant i \leqslant l\} - \sup \{\mu_i(X) : l + 1 \leqslant i \leqslant k\} =$$

$$= \sup \{\nu_i(X) : 1 \leqslant i \leqslant l'\} - \sup \{\nu_i(X) : l' + 1 \leqslant i < k'\},$$

for every $X \in R$. It then follows that

$$\bigvee_{i=1}^{l} \mu_i - \bigvee_{i=l+1}^{k} \mu_i = \bigvee_{i=1}^{l'} \nu_i - \bigvee_{i=l'+1}^{k'} \nu_i$$

Consequently, $\Phi(f) = \Phi(g)$.

Hence the mapping $\Phi : h(\hat{E}) \to H$ is defined unambigously by (13). The formula (13) now implies easily that Φ is a vector lattice homomorphism.

Let $\hat{T}$ be the subset of $\hat{E}$ and $\hat{S}$ the σ-algebra of subsets of $\hat{T}$ as in Theorem 1 ; this theorem is applicable since $\hat{E}$ is a weak complete space. It means that every conical measure u on $\hat{E}$ is represented as

$$u(f) = \int_{\hat{T}} f(t)\, d\lambda(t), f \in h(\hat{E}), \tag{14}$$

where λ is a measure on $\hat{S}$, a direct sum of finite measures. We say that λ represents u.

THEOREM 8. – *Let* $u(f) = \Phi(f)$ (T), *for every* $f \in h(\hat{E})$. *Then* u *is a conical measure on* $\hat{E}$; *let* λ *be the measure on* $\hat{S}$ *representing* u.

The mapping $f \mid \hat{T} \to \Phi(f), f \in h(\hat{E})$, *is a vector lattice isomorphism and an isometry of the dense subset* $h(\hat{E}) \mid \hat{T}$ *of* $L^1(\lambda)$ *onto* H.

There exists an injective mapping $\gamma : R(H) \to \hat{S}(\lambda)$ which is a *Boolean algebra isomorphism of* $R(H)$ *onto* $\gamma(R(H))$ *and also a homeomorphism of* $R(H)$ *in its* $\tau(H)$*-topology onto* $\gamma(R(H))$ *in the relative* $\tau(\lambda)$*-topology and* $\gamma(R(H))$ *is* $\tau(\lambda)$*-dense in* $\hat{S}(\lambda)$.

Proof. – Clearly, u is a conical measure.

Given $X \in R$, let

$$v_X(f) = \Phi(f)(X), f \in h(\hat{E}).$$

Then v_X is a conical measure on $\hat{E}$ and $v_X \leqslant u = v_T$. Let λ_X be the measure on $\hat{S}$ which represents v_X. Then $\lambda_X \leqslant \lambda = \lambda_T$. Since λ is a direct sum of finite measures, the Radon-Nikodym theorem is applicable [10, Theorem 5.1]. Hence there exists an $\hat{S}$-measurable function g_X on $\hat{T}$ such that $0 \leqslant g_X \leqslant 1$,

$$\lambda_X(\hat{X}) = \int_X g_X \, d\lambda$$

for every $\hat{X} \in \hat{S}$, and

$$v_X(f) = \int_{\hat{T}} f(t) \, g_X(t) \, d\lambda(t).$$

for every $f \in h(\hat{E})$. The relations $v_{T-X} + v_X = u$ and $v_X \wedge v_{T-X} = 0$ imply that $g_{T-X}(t) + g_X(t) = 1$ and $\min\{g_{T-X}(t), g_X(t)\} = 0$, λ-almost everywhere on $\hat{T}$. It follows that g_X is λ-equivalent to the characteristic function of a set in $\hat{S}$; denote it and also its λ-equivalence class by $\gamma(X)$. It is clear that $\gamma(X)$ depends on the H-equivalence class $[X]_H$ only and not on its individual representative X. Hence γ is well-defined on $R(H)$.

Accepting these conventions it is clear that $\gamma : R(H) \to \hat{S}(\lambda)$ is injective and that it is a Boolean algebra isomorphism of $R(H)$ onto $\gamma(R(H))$.

These definitions also give that

$$\mu(X) = \Phi(\mu^*)(X) = v_X(\mu^*) = \int_T \mu^*(t) \, g_X(t) \, d\lambda(t) =$$

$$= \int_{\gamma(X)} \mu^*(t) \, d\lambda(t),$$

for every $\mu \in \mathrm{H}$ and $\mathrm{X} \in R$. More generally,

$$\Phi(f)(\mathrm{X}) = \int_{\gamma(\mathrm{X})} f(t)\, d\lambda(t),$$

for every $f \in h(\hat{\mathrm{E}})$ and $\mathrm{X} \in R$. Since Φ is a vector lattice isomorphism, the variation $|\Phi(f)|$ of $\Phi(f)$ is equal to $\Phi(|f|)$, where $|f|$ is the absolute value of f. It then follows that

$$\|\Phi(f)\| = |\Phi(f)|(\mathrm{T}) = \Phi(|f|)(\mathrm{T}) = \int_{\gamma(\mathrm{T})} |f(t)|\, d\lambda(t) =$$
$$= \int_{\hat{\mathrm{T}}} |f|\,\hat{\mathrm{T}}\, d\lambda = \|f\,|\,\hat{\mathrm{T}}\|_1.$$

That is, the norm of $\Phi(f)$ as an element of $\mathrm{ba}(R)$ is equal to the L^1-norm of $f\,|\,\hat{\mathrm{T}}$. This establishes the isometry of $h(\hat{\mathrm{E}})\,|\,\hat{\mathrm{T}}$ and H.

Moreover, since

$$\|\mu\| = |\mu|(\mathrm{T}) = \sup\{|\mu(\mathrm{X})| + |\mu(\mathrm{Y})| : \mathrm{X}, \mathrm{Y} \in R,\ \mathrm{X} \cap \mathrm{Y} = \phi\},$$

we have also

$$\|\varphi\|_1 = \sup\{|\int_{\gamma(\mathrm{X})} \varphi\, d\lambda| + |\int_{\gamma(\mathrm{Y})} \varphi\, d\lambda| : \mathrm{X}, \mathrm{Y} \in R, \mathrm{X} \cap \mathrm{Y} = \phi\}, \quad (15)$$

for every $\varphi \in h(\hat{\mathrm{E}})\,|\,\hat{\mathrm{T}}$. Now, (15) holds also for every $\varphi \in L^1(\lambda)$, since $h(\hat{\mathrm{E}})\,|\,\hat{\mathrm{T}}$ is dense in $L^1(\lambda)$. It follows that, if $\varphi \in L^1(\lambda)$ and if the integral of φ vanishes on every set $\gamma(\mathrm{X})$, with $\mathrm{X} \in R$, then φ is λ-equivalent to 0. This implies that $\gamma(R)$ is $\tau(\lambda)$-dense in $\hat{S}(\hat{\lambda})$.

We say that the algebra R separates the points of the space T if, for every $t_1\ t_2$ in T, $t_1 \neq t_2$, there exists $\mathrm{X} \in R$ such that $t_1 \in \mathrm{X}$ and $t_2 \notin \mathrm{X}$.

There is no loss of generality in assuming that R indeed separates points of T. If R does not separate points of T we introduce the equivalence relation in T by declaring $t_1 \equiv t_2$ if and only if $t_1 \in \mathrm{X}$, implies $t_2 \in \mathrm{X}$, for every $\mathrm{X} \in R$. Then, if we replace T by so obtained set $\widetilde{\mathrm{T}}$ of equivalence classes and R by its image $\widetilde{R}$ under the natural mapping of T into $\widetilde{\mathrm{T}}$, we obtain a situation where the algebra separates the points of the space $\widetilde{\mathrm{T}}$ and $\mathrm{ba}(\widetilde{R})$ coincides with $\mathrm{ba}(R)$ both as a vector lattice and as a Banach space.

We say that the vector lattice $\mathrm{H} \subset \mathrm{ba}(R)$ separates the algebra R, if the equivalence classes $[\mathrm{X}]_{\mathrm{H}}$, $\mathrm{X} \in R$, are reduced to individual

sets ; i.e. there is not another element in R but X which is equivalent to X, for every $X \in R$. In this case $R(H) = R$ as sets and also as Boolean algebras.

In particular, H separates R if H contains all Dirac measures carried by points of T.

COROLLARY 9. – *If the algebra R separates the points of the space* T *and if the vector lattice* H *separates R, then there exists an injection* $\delta : T \to \hat{T}$ *such that* $\delta(X) = \gamma(X) \cap \delta(T)$, *for every* $X \in R$.

Proof. – Since λ is a direct sum of finite measures, $\hat{S}(\lambda)$ is a complete Boolean algebra. For every $t \in T$, let

$$[\hat{X}_t]_\lambda = \bigcup_{t \in X \in R} \gamma(X)$$

The element $[\hat{X}_t]_\lambda$ is not equal to $[\phi]_\lambda$. To prove that it suffice to notice that, since λ is a direct sum of finite measures, the dual of the Banach space $L^1(\lambda)$ is $L^\infty(\lambda)$; further each $\gamma(X)$ represents an element of the unit ball in $L^\infty(\lambda)$, so it suffices to use the weak-star compactness of the unit ball and notice that the finite intersections are all of norm 1.

Since H separates R and R separates T, the elements $[\hat{X}_{t_1}]_\lambda$ and $[X_{t_2}]_\lambda$ are disjoint in $S(\lambda)$ if $t_1 \neq t_2$. Now we use again the fact that λ is a direct sum of finite measure to deduce the existence of lifting in $L^\infty(\lambda)$ or in $\hat{S}(\lambda)$ and, hence, the possibility to choose the representations $\hat{X}_t$ of $[\hat{X}_t]_\lambda$ in such a way that they are disjoint as sets (have no points in common) if they correspond to different elements of T.

Let $\delta(t)$ be an arbitrary element of $\hat{X}_t$, for each $t \in T$. Then the mapping $\delta : T \to \hat{T}$ has the claimed properties.

COROLLARY 10. – *If R (H) is τ (H)-complete then* $\gamma : R(H) \to \hat{S}(\lambda)$ *is a Boolean algebra isomorphism of R (H) onto $\hat{S}(\lambda)$ and, hence, R (H) is a complete Boolean algebra.*

3. Representation of vector measures.

Let E be a locally convex topological vector space.

Let T be an abstract space and let R be an algebra of subsets of T.

An additive and bounded mapping $m : R \to E$ is termed a *pre-measure.* The boundedness means that the range

$$m(R) = \{m(X) : X \in R\}$$

is a bounded subset of E.

If R is σ-algebra and m is σ-additive then it is termed a vector measure (shortly a measure). In this case the boundedness is a consequence of other assumptions.

Given a pre-measure $m : R \to E$, declare two elements X_1 and X_2 of R to be m-equivalent if $m(X) = 0$ for every $X \subset X_1 \Delta X_2$, $X \in R$. The m-equivalence class of an element $X \in R$ is denoted by $[X]_m$ whenever the distinction between a set and its equivalence class is essential. The family of all m-equivalence classes is denoted by $R(m)$. Again, $R(m)$ is considered a Boolean algebra equal to R modulo the ideal of sets m-equivalent to ϕ.

Let U be a convex symmetric neighborhood of 0 in E and let U° be its polar. Define the pseudo-metric d_U on R and on $R(m)$ by putting

$$d_U(X, Y) = \sup \{|x' \circ m|(X \Delta Y) : x' \in U^\circ\}.$$

for every X, Y in R.

The system of pseudo-metrics d_U, for all symmetric convex neighbourhoods U, defines a topology and a uniform structure on R and on $R(m)$; this topology and the uniformity will be denoted by $\tau(m)$. On $R(m)$, $\tau(m)$ is separated (Hausdorff).

If $R(m)$ is a complete space in the uniform structure $\tau(m)$, the pre-measure m is termed *closed* [5].

Let S be σ-algebra of subsets of T and let λ be a non-negative σ-additive mesure on S (admitting also value ∞). A function $f : T \to E$ is said to be integrable if, for every $x' \in E'$, the real-valued function $x' \circ f$ is integrable and if, for every $X \in S$, there exists an element $x_X \in E$ such that