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FUNCTIONAL SPACES AND FUNCTIONAL COMPLETION [1]

by N. ARONSZAJN and K. T. SMITH

INTRODUCTION

The incentive for the development of a general theory of functional completion has been the need for complete classes of admissible functions in differential problems. Traditionally the admissible functions have been assumed to be sufficiently regular, but during the evolution of existence proofs it became necessary to reconsider the hypotheses of regularity. In the final analysis, existence proofs use the completeness of the class of admissible functions with respect to a norm determined by the problem. On the other hand, the usual classes of sufficiently regular admissible functions are not complete.

In some instances it has proved feasible to adjoin to the usual class of admissible functions suitable ideal objects to obtain a class with the required properties of completeness, the « abstract completion », to extend the differential operator to such ideal objects, to prove the existence in the enlarged class of a solution to the problem in question, and finally to prove by using the special character of the problem that the solution is necessarily one of the original admissible functions [2]. Often the last step is unmanageable, however, and then the very questions of which the differential problem is composed, questions of differentiability of the solution, its boundary values, etc., are meaningless. Furthermore, comparison of the enlarged classes arising from two different problems is

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(2) See for example, K. O. Friedrichs [16].

not possible in any direct way, and there are questions in which such comparisons are necessary ([3]).

In some problems, especially those connected with the Laplace operator, there have been scattered attempts to complete the usual class of admissible functions by the adjunction of concrete functions determined in a definite way by the original class of functions and its norm ([4]). The success of these attempts was notable for the reason that the problem of completion by functions was not then well defined. They are the fore-runners of the general theory of functional completion.

The basic difficulty in the completion by functions of a functional class lies in the impossibility of using functions which have significant values at each point. It is in the nature of the problem that if there is a functional completion at all, then associated with it are certain exceptional sets of points. Any two functions which differ only on one of the exceptional sets must be considered equivalent.

Thus the problem of functional completion divides into two parts. The first of these is to find a suitable class of exceptional sets. The second is to find the functions, defined modulo these exceptional sets, which must be adjoined in order to obtain a complete functional class. It turns out that there may be an infinite number of suitable exceptional classes (of exceptional sets) in a given problem, but to any one of them corresponds essentially one functional completion. As to the infinite number of suitable exceptional classes, it is clear that the most suitable is the class whose exceptional sets are the smallest, for to it corresponds the completion whose functions are defined with the best possible precision. Whenever such a minimal exceptional class exists the corresponding completion is called the *perfect completion*. Use of the perfect completion is especially important in differential problems, for if the exceptional sets are too large, then it is impossible to discuss derivatives, boundary values, etc., in the normal way.

In the first sections of Chapter I of this paper we give the

([3]) Comparison of the enlarged classes for two different problems is an essential part of some recent approximation methods; see N. ARONSZAJN [3, 7].

([4]) e.g. O. NIKODYM [21]; J. W. CALKIN [11]; C. B. MORREY [20].

precise definitions and general theory of functional completion in an abstract setting [5].

We define exactly the classes of sets which will be called exceptional classes, then the functional classes, normed functional classes and functional spaces relative to a given exceptional class $\mathfrak{A}$. This leads finally to a precise definition of a functional completion relative to $\mathfrak{A}$ or relative to any larger exceptional class $\mathfrak{A}' \supset \mathfrak{A}$. We give a construction of the functional completion relative to $\mathfrak{A}'$, supposing that it exists.

The bulk of the chapter is devoted to the more difficult problem of determining the exceptional classes relative to which a functional completion does exist. We introduce set functions $\delta(A)$, $\tilde{\delta}(A)$, and $c_\varphi(A)$. The last, constructed from δ by means of functions $\varphi(t)$ of a variable $t \geq 0$, are called capacities. In certain classical cases they coincide with classical capacities. The classes of sets for which the functions δ, $\tilde{\delta}$, and c_φ vanish give bounds for the exceptional classes relative to which a completion can exist. We introduce the « majoration property », and under assumption that it holds (which is always true in cases met in applications) we prove that one of the above bounds is exactly the exceptional class for the perfect completion, if the perfect completion exists. Under the same assumption necessary and sufficient conditions for the existence of the perfect completion are obtained. We obtain also some properties of the functions constituting the complete class. These are of importance in applications.

The chapter is concluded by a discussion of proper functional completion, the case where it is actually possible to use functions defined everywhere.

Chapter II is given to examples. We do not show any of

[5] A general theory of functional completion was announced by N. Aronszajn in [2] and presentend in [8]. The new presentation given in this paper differs from its predecessor in several respects. The most important is the use of set functions to replace the classes of sets $\mathfrak{S}\{M_n\}$. The set functions are simpler conceptually and easier to handle. Another improvement is the introduction of the majoration property and the solution for spaces having this property of the problem of perfect completion. By using the majoration property it is possible to obtain the perfect completion in all the examples in which formely the theory of measurable spaces was used. Consequently it has been possible to defer discussion of the latter until the time when they will be used in the theory of pseudo-reproducing kernels. Finally the choice of examples is quite different in the two papers.

the applications of the theory to differential problems, for these will be treated fully in forthcoming papers. Rather, we have chosen the examples with the object of bringing out in concrete cases the significance of the notions introduced in Chapter I. In some of the examples, however, especially example 3, we are able to use the general theory to give new proofs of known results.

The first example treats a well known space of analytic functions.

The second example is the completion of a space of continuous functions in which the norm is the L^p norm with respect to a Borel measure μ in a locally compact topological space. The example is one which is thoroughly discussed in measure theory; here it serves exclusively as illustration. One point which might be unexpected is that the perfect completion is not always the space $L^p(\mu)$, though for the usual topological spaces — say metrizable spaces — it is.

The third example is the completion of classes of functions harmonic in a domain and continuous in the closed domain in which the norm is the L^p norm on the boundary. We obtain the extension to n-dimensional spheres, and more generally to n-dimensional domains of bounded curvature, of theorems which are classical in the case of the circle in the plane. In particular, by using the capacities as defined in the general theory we obtain the extension of Fatou's theorem to these domains [6].

The last example is the completion of the class of potentials of M. Riesz of order α, $0 < \alpha < n$, of finite energy [7]. We obtain the perfect completion on the basis of the general theory of Chapter I, and we prove that the exceptional sets for the perfect completion are the sets of outer capacity 0. We establish the following connection between the set functions and capacities of the general theory and the usual inner and outer capacities: $\underset{\smile}{\delta}(A)^2 = \overset{\smile}{\delta}(A)^2 = c_2(A) = \gamma_0(A)$ for any set A,

[6] The theorem in question is that concerning the convergence of a harmonic function to its boundary values. Its extension to domains of bounded curvature was obtained by C. de la VALLÉE POUSSIN [25]. A further extension to more general domains was obtained by I. I. PRIVALOFF and P. KOUZNETZOFF [22].

[7] The perfect completion for the case $\alpha = 2$ was conjectured by N. ARONSZAJN [2]. The perfect completion for arbitrary α was constructed first in J. DENY [15]. An independent construction for $\alpha = 2$ was announced in N. ARONSZAJN [6].

where c_2 is our capacity formed with the function $\varphi(t) = t^2$ and where γ_0 is the usual outer capacity of order α. Furthermore, $\gamma_i(A) = \gamma_0(A)$ for any analytic set A, where γ_i is the usual inner capacity of order α [8]. These results justify our terminology.

[8] We prove this result by applying the general theory of capacities of G. Choquet [14, 14a]. The result is new for $\alpha > 2$.

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CHAPTER I

GENERAL THEORY

§ 1. **Linear functional classes.** — If f and g are real or complex-valued functions defined on respective subsets A and B of an abstract set $\mathcal{E}$, then $f + g$ and αf, α real or complex, denote the following functions: $f + g$ is defined on the set $A \cap B$, and $(f + g)(x) = f(x) + g(x)$; αf is defined on the set A, and $(\alpha f)(x) = \alpha f(x)$. A *real linear functional class* is a class $\mathfrak{F}$ of real valued functions, each defined on a subset of a fixed abstract set $\mathcal{E}$, such that if f and g belong to $\mathfrak{F}$ and α is real, then $f + g$ and αf belong to $\mathfrak{F}$. A *complex linear functional class* is the obvious analogue. A *linear functional class*, or simply a *functional class*, is a real or a complex linear functional class.

The abstract set $\mathcal{E}$ in which the functions of a linear functional class $\mathfrak{F}$ are defined is called the basic set of $\mathfrak{F}$. A given function f in $\mathfrak{F}$ is not necessarily defined on the whole of the basic set $\mathcal{E}$; the subset on which f is not defined is called the exceptional set of f. Members f and g of $\mathfrak{F}$ are equal only if they are identical.

In particular, f and g are different whenever their exceptional sets are different. For this reason a linear functional class is not necessarily a vector space in the ordinary sense. In fact, if f and g are any two functions with different exceptional sets, then $0.f \neq 0.g$, for the former has the exceptional set of f, and the latter has the exceptional set of g; $0.f \neq 0.g$ is impossible in a vector space. Similarly, the identity $(f + g) - g = f$ fails in a general linear functional class. These examples give already the main deviation from vector space behavior, however: addition is associative and commutative, the usual distributive laws hold, and $1.f = f$.

Let $\mathfrak{L}$ be the class of all exceptional sets of functions in $\mathcal{F}$. It is clear that the union of each pair of sets in $\mathfrak{L}$ is again in $\mathfrak{L}$, for the union of the exceptional sets of f and g is the exceptional set of $f+g$. An equivalence relation is defined on $\mathcal{F}$ as follows: $f \equiv f'$ if f and f' are defined and equal save on some subset of a set in $\mathfrak{L}$. It is immediately verified that if $f \equiv f'$ and $g \equiv g'$, then $\alpha f \equiv \alpha f'$ and

$$f + g \equiv f' + g'.$$

The equivalence classes in $\mathcal{F}$, under the usual definitions of addition and scalar multiplication of equivalence classes, form a vector space [9].

§ 2. Functional classes rel. $\mathfrak{A}$ and normed functional classes. — Let $\mathcal{F}$ be a linear functional class on a basic set $\mathcal{E}$, and let $\mathfrak{L}$ be the class of exceptional sets of functions in $\mathcal{F}$. In practice it often happens that more sets must be considered exceptional than those already in $\mathfrak{L}$. In order to treat examples of this kind we are compelled to introduce a general notion of exceptional class. An exceptional class will serve to define, as $\mathfrak{L}$ defined in the last section, an equivalence relation on the class $\mathcal{F}$. In this definition, subsets of exceptional sets play the same role as the exceptional sets themselves, so it is justifiable to insist that each subset of an exceptional set be exceptional. In order to ensure that the equivalence be compatible with the linear operations in $\mathcal{F}$, we require that a finite union of exceptional sets be exceptional. In order to ensure that it be compatible with limit processes, we require that even a countable union of exceptional sets be exceptional. The formal definition follows.

An *exceptional class* in the basic set $\mathcal{E}$ is a class $\mathfrak{A}$ of subsets of $\mathcal{E}$ which is

(2. 1) hereditary: if $A \in \mathfrak{A}$ and $B \subset A$, then $B \in \mathfrak{A}$.

(2. 2) σ-additive: if $A_n \in \mathfrak{A}$, $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{A}$ [10].

[9] This equivalence relation is not the only one which transforms $\mathcal{F}$ into a vector space. The relation with the smallest equivalence classes is given by: $\equiv f'$ if $f = f'$ wherever both are defined.

[10] We shall use the following standard notation: if $\mathfrak{A}$ is a class of subsets of a

A linear functional class $\mathfrak{F}$ is a *linear functional class relative to* $\mathfrak{A}$ if $\mathfrak{A}$ is an exceptional class which contains the exceptional set of each f in $\mathfrak{F}$. If $\mathfrak{F}$ is a functional class relative to $\mathfrak{A}$ (written rel. $\mathfrak{A}$), then $\mathfrak{A}$ is called an exceptional class for $\mathfrak{F}$, and the sets in $\mathfrak{A}$ are called exceptional sets. In order to avoid unnecessary repetition we make the following conventions: the letter $\mathfrak{F}$, with or whitout indices, will denote a linear functional class; $\mathfrak{E}$ will denote its basic set; $\mathfrak{A}$, with or without indices, will denote an exceptional class in $\mathfrak{E}$.

It is clear that for each linear functional class $\mathfrak{F}$ there exists an exceptional class, which in general is not unique. The largest exceptional class for $\mathfrak{F}$ is the class of all subsets of $\mathfrak{E}$; the smallest exceptional class for $\mathfrak{F}$ is the class $\mathfrak{L}_{\sigma h}$, where $\mathfrak{L}$ is the class of all exceptional sets of functions in $\mathfrak{F}$; the intersection of any family of exceptional classes for $\mathfrak{F}$ is again an exceptional class for $\mathfrak{F}$.

Any exceptional class $\mathfrak{A}$ for the functional class $\mathfrak{F}$ defines on $\mathfrak{F}$ a natural equivalence relation : $f \equiv f'$ if f and f' are defined and equal save on a set in $\mathfrak{A}$. As before, the equivalence classes form a vector space, but usually it is more convenient to work directly with the functional class and its functions than with the vector space and its equivalence classes. Consequently, the equivalence notation, $f \equiv f'$ will be used rarely. In its stead we shall write $f = f'$ exc. $\mathfrak{A}$. In fact, we shall say that any proposition is true exc. $\mathfrak{A}$ if the set of points at which it is not true belongs to the exceptional class $\mathfrak{A}$. Also, for two sets A and B we shall say $A \subset B$ exc. $\mathfrak{A}$ if $A - B \in \mathfrak{A}$. Similarly, $A = B$ exc. $\mathfrak{A}$ means $(A - B) \cup (B - A) \in \mathfrak{A}$.

If $\mathfrak{F}$ is a functional class rel. $\mathfrak{A}$, then so is the class $\mathfrak{F}'$ of all functions defined exc. $\mathfrak{A}$ and equal exc. $\mathfrak{A}$ to some function in $\mathfrak{F}$. $\mathfrak{F}'$ is called the *saturated extension* of $\mathfrak{F}$ rel. $\mathfrak{A}$. $\mathfrak{F}$ is saturated rel. $\mathfrak{A}$ if it coincides with its saturated extension. Let $\mathfrak{F}$ and $\mathfrak{F}_1$ be functional classes rel. $\mathfrak{A}$ and $\mathfrak{A}_1$ respectively. From the relation $\mathfrak{F} \subset \mathfrak{F}_1$ one obtains no relation in general between $\mathfrak{A}$ and $\mathfrak{A}_1$. If $\mathfrak{F}$ is saturated, however, then $\mathfrak{A} \subset \mathfrak{A}_1$.

set $\mathfrak{E}$, then $\mathfrak{A}_h$ is the class of all subsets of sets in $\mathfrak{A}$; $\mathfrak{A}_\sigma$ is the class of all countableunions of sets in $\mathfrak{A}$; $\mathfrak{A}_\delta$ is the class of all countable intersections of sets in $\mathfrak{A}$. With this notation the fact that $\mathfrak{A}$ is an exceptional class can be written $\mathfrak{A} = \mathfrak{A}_{\sigma h}$.

A pseudo-norm on a functional class $\mathfrak{F}$ is a real valued function $\|f\|$ on $\mathfrak{F}$ with the properties:

(2. 3) $\qquad \|f\| \geq 0,$

(2. 4) $\qquad \|\alpha f\| = |\alpha| \|f\|,$

(2. 5) $\qquad \|f + g\| \leq \|f\| + \|g\|.$

It can be proved by the homogeneity property (2. 4) that if a function f in $\mathfrak{F}$ is equal to 0 wherever it is defined, then $\|f\| = 0$. A *normed functional class rel.* $\mathfrak{A}$ is a functional class $\mathfrak{F}$ rel. $\mathfrak{A}$ together with a pseudo-norm on $\mathfrak{F}$ which has the property:

(2. 6) $\quad \|f\| = 0$ *if and only if* $f = 0$ exc. $\mathfrak{A}$.

A pseudo-norm with property (2. 6) will be called a norm.

The following statements can be proved without difficulty. In each of them $\mathfrak{F}$ is a functional class with a fixed pseudo-norm.

1) *If* $\mathfrak{F}$ *is a normed functional class rel.* $\mathfrak{A}$, *then so is its saturated extension* (*with the natural extension of the norm*).

2) *If* $\mathfrak{F}' \subset \mathfrak{F}$, *then* $\mathfrak{F}'$ (*with the pseudo-norm of* $\mathfrak{F}$) *is a normed functional class rel.* $\mathfrak{A}$ *whenever* $\mathfrak{F}$ *is.*

3) *If* $\mathfrak{F}$ *is a normed functional class rel.* $\mathfrak{A}'$ *and rel.* $\mathfrak{A}'' \supset \mathfrak{A}'$, *then it is also a normed functional class rel.* $\mathfrak{A}$ *whenever* $\mathfrak{A}' \subset \mathfrak{A} \subset \mathfrak{A}''$.

4) *If* $\mathfrak{F}$ *is a normed functional class relative to each of a family of exceptional classes, then it is also normed functional class relative to the intersection of the family.*

Condition (2. 6) comprises two implications. Taken separately they provide bounds above and below for the exceptional classes relative to which $\mathfrak{F}$ can be a normed functional class. Let $\mathfrak{L}'$ be the class of all subsets B of $\mathfrak{E}$ such that for some f in $\mathfrak{F}$ with $\|f\| = 0$, $B \subset \underset{x}{E}\,[f(x)$ is *undefined*, or $f(x) \neq 0]$.

Let $\mathfrak{L}''$ be the class of all subsets B of $\mathfrak{E}$ such that for every f in $\mathfrak{F}$ with $\|f\| > 0$, $B \not\supset \underset{x}{E}\,[f(x) \neq 0]$.

The classes $\mathfrak{L}'$ and $\mathfrak{L}''$ are both hereditary but they are not in general σ-additive or even additive.

5) *A necessary and sufficient condition that* $\mathfrak{F}$ *be a normed functional class rel.* $\mathfrak{A}$ *is that* $\mathfrak{L}' \subset \mathfrak{A} \subset \mathfrak{L}''$. *A necessary and sufficient condition that there be an exceptional class relative to which* $\mathfrak{F}$ *is a normed functional class is that* $\mathfrak{L}'_\sigma \subset \mathfrak{L}''$.

REMARK 1. — The inclusion $\mathfrak{L}' \subset \mathfrak{L}''$ does not hold for all $\mathfrak{F}$; even when it does, $\mathfrak{L}'_{\mathfrak{F}} \subset \mathfrak{L}''$ may not.

Example 1. — Take $\mathcal{E}$ to be the open interval $0 < x < 1$, and $\mathfrak{F}$ to be the class of functions on $\mathcal{E}$ with continuous bounded derivatives; define the pseudo-norm by $\|f\| = \int_0^1 |f'(x)|\,dx$. In this case the class $\mathfrak{L}'$ consists of all subsets of $\mathcal{E}$, the class $\mathfrak{L}''$ of all subsets with empty interior. There is no exceptional class relative to which $\mathfrak{F}$ is a normed functional class.

Example 2. — Take $\mathcal{E}$ to be the closed interval $0 \leqq x \leqq 1$, and $\mathfrak{F}$ to be the class of continuous functions on $\mathcal{E}$; define the norm by $\|f\| = \sup |f(x)|$. In this case $\mathfrak{L}'$ is (0), and $\mathfrak{L}''$ is again the class of subsets of $\mathcal{E}$ with empty interior. $\mathfrak{F}$ is a normed functional class relative to the class $\mathfrak{A}'$ of sets of Lebesgue measure 0, and also relative to the class $\mathfrak{A}''$ of sets of first category; but there is no $\mathfrak{A}$ larger than $\mathfrak{A}'$ and $\mathfrak{A}''$ relative to which $\mathfrak{F}$ is a normed functional class.

CONCLUSION. — If there is any exceptional class relative to which a given functional class with a pseudo-norm is a normed functional class, then there is a smallest such class, but there may not be a largest.

In any functional class $\mathfrak{F}$ with a pseudo-norm convergence (in norm) is defined as follows: a sequence $\{f_n\}$ of functions in $\mathfrak{F}$ converges to a function f in $\mathfrak{F}$ (written $f_n \to f$, or $f = \lim f_n$) if $\|f_n - f\| \to 0$. The sequence $\{f_n\}$ is Cauchy if $\|f_n - f_m\| \to 0$. $\mathfrak{F}$ is complete if each Cauchy sequence of functions in $\mathfrak{F}$ converges to some function in $\mathfrak{F}$.

REMARK 2. — A sequence in $\mathfrak{F}$ may have several limits. If $\mathfrak{F}$ is a normed functional class rel. $\mathfrak{A}$, any two are equal exc. $\mathfrak{A}$.

REMARK 3. — Suppose that $\mathfrak{F}$ is a normed functional class rel. $\mathfrak{A}$, and let V be the vector space associated with $\mathfrak{F}$ by means of the equivalence relation defined by $\mathfrak{A}$. It is clear that the pseudo-norm has a constant value on each equivalence class. If this constant value is taken as the norm of the class, then V becomes a normed linear space in the usual sense. A convergent sequence in $\mathfrak{F}$ corresponds to a convergent sequence in V, a Cauchy sequence in $\mathfrak{F}$ to a Cauchy sequence in V. $\mathfrak{F}$ is complete if and only if V is complete.

§ 3. **Functional spaces.** — In a general normed functional class norm convergence of a sequence of functions f_n has no bearing upon the convergence of the functions pointwise. The object of the rest of this paper is to study functional classes in which the two kinds of convergence are linked.

A *functional space rel.* $\mathfrak{A}$ is a normed functional class rel. $\mathfrak{A}$ in which the following condition holds:

(3. 1) If $f_n \to f$, *then there is a subsequence* $\{f_{n_k}\}$ *such that* $f_{n_k}(x) \to f(x)$ exc. $\mathfrak{A}$.

In the statements below, $\mathfrak{F}$ is a functional class with a fixed pseudo-norm.

1) *If* $\mathfrak{F}$ *is a functional space rel.* $\mathfrak{A}$, *then so is its saturated extension.*

2) *If* $\mathfrak{F}' \subset \mathfrak{F}$, *then* $\mathfrak{F}'$ (*with the pseudo-norm of* $\mathfrak{F}$) *is a functional space rel.* $\mathfrak{A}$ *whenever* $\mathfrak{F}$ *is.*

3) *If* $\mathfrak{F}$ *is a functional space rel.* $\mathfrak{A}'$ *and rel.* $\mathfrak{A}'' \supset \mathfrak{A}'$, *then* $\mathfrak{F}$ *is a functional space rel.* $\mathfrak{A}$ *whenever* $\mathfrak{A}' \subset \mathfrak{A} \subset \mathfrak{A}''$.

4) *If* $\mathfrak{F}$ *is a functional space relative to each of a sequence of exceptional classes, then* $\mathfrak{F}$ *is a functional space relative to their intersection.*

PROOFS. — Statements 1), 2), and 3) can be obtained easily from their counterparts in the preceding section. Statement 4) is obtained as follows. Let $\mathfrak{A}$ be the intersection of the sequence $\{\mathfrak{A}_n\}$. If $\mathfrak{F}$ is a functional space relative to each $\mathfrak{A}_n$, then by 4), section 2, $\mathfrak{F}$ is a normed functional class rel. $\mathfrak{A}$. If $f_n \to f$, then there is a subsequence $\{f_{1,n}\}$ such that $f_{1,n}(x) \to f(x)$ exc. $\mathfrak{A}_1$; then a subsequence $\{f_{2,n}\}$ of $\{f_{1,n}\}$ such that $f_{2,n}(x) \to f(x)$ exc. $\mathfrak{A}_2$ — hence also exc. $\mathfrak{A}_1 \cap \mathfrak{A}_2$. The standard diagonal process yields a subsequence of the original $\{f_n\}$, which converges at every point exc. $\mathfrak{A}$; thus $\mathfrak{F}$ is a normed functional class rel. $\mathfrak{A}$ in which (3. 1) holds.

REMARK. — Even if $\mathfrak{F}$ is a functional space relative to some exceptional class $\mathfrak{A}$, 4) cannot be used to obtain the existence of a minimal exceptional class relative to which it is a functional space; for 4) provides only for countable intersection of exceptional classes. As yet there is neither a general proof nor a counter-example for the existence of such a minimal class. It is certain that there need not be a largest exceptional class relative to which $\mathfrak{F}$ is a functional space. This is shown by Example 2 of the last section.

Examples. — Example 2 of the last section provides two functional spaces. Other common functional spaces are the spaces L^p, $p \geqq 1$. To be specific, let $\mathcal{E}$ be the interval $0 \leqq x \leqq 1$, and let $\mathfrak{A}$ be the class of subsets of $\mathcal{E}$ of Lebesgue measure 0; then L^p, $p \geqq 1$, is the class of all functions f defined exc. $\mathfrak{A}$ which are measurable and such that

$$\|f\|_p = \left\{ \int^1 |f(x)|^p \, dx \right\}^{1/p} < \infty.$$

With the indicated norm, L^p is a functional space.

Proper functional spaces. A proper functional class is a functional class rel. $\mathfrak{A} = (0)$, the class consisting of the empty set. A *proper normed functional class* is a normed functional class rel. (0). A *proper functional space* is a functional space rel. (0).

5) *Either of the following statements is a necessary and sufficient condition that a proper normed functional class $\mathfrak{F}$ be a proper functional space.*

a) If $f_n \to f$, then $f_n(x) \to f(x)$ for each x in $\mathcal{E}$.

b) For each x in $\mathcal{E}$, the expression $f(x)$ is a continuous linear functional on $\mathfrak{F}$.

PROOF. — The sufficiency of a) and the equivalence of a) and b) are evident. We prove the necessity of b). It is clear that the expression $f(x)$ is a linear functional on $\mathfrak{F}$. If it is not continuous, then it is unbounded on each sphere $\|f\| \leqq \varepsilon$, so for each n there is an f_n satisfying $\|f\| \leqq 1/n$ and $|f_n(x)| \geqq n$. Obviously $f_n \to 0$, but no subsequence of $f_n(x)$ does. This requires that x belong to an exceptional set, and contradicts the fact that there is no exceptional set but 0.

§ 4. **Functional completion.** — It is well known that the functional space L^p described in the example in the last section is obtained by completing a simpler functional class. Let $\mathcal{E}$ be the interval $0 \leqq x \leqq 1$, and let C_p denote the functional class of all continuous functions defined everywhere on $\mathcal{E}$ with the norm.

$$\|f\|_p = \left\{ \int_0^1 |f(x)|^p \, dx \right\}^{1/p}.$$

C_p is a proper normed functional class. It is not complete, nor is it a proper functional space. The exceptional class consisting of sets of Lebesgue measure 0 and the functional space L^p provide the solution to the following problem: to find an exceptional class $\mathfrak{A}$ relative to which C_p is a functional space, and to find a complete functional space $\mathfrak{F}$ rel. $\mathfrak{A}$ which contains C_p as a dense subset.

A normed functional class $\mathfrak{F}$ rel. $\mathfrak{A}$ is *embedded* in a normed functional class $\mathfrak{F}'$ rel. $\mathfrak{A}'$ if $\mathfrak{F}\subset\mathfrak{F}'$, $\mathfrak{A}\subset\mathfrak{A}'$, and the norm of each function in $\mathfrak{F}$ is the same as its norm as a function in $\mathfrak{F}'$. A subset $\mathfrak{D}$ of a normed functional class $\mathfrak{F}$ (or of any functional class with a pseudo-norm) is *dense* in $\mathfrak{F}$ if each f in $\mathfrak{F}$ is a limit of a sequence $\{f_n\}$ in $\mathfrak{D}$. A *functional completion* of a normed functional class $\mathfrak{F}$ rel. $\mathfrak{A}$ is a functional *space* $\mathfrak{F}'$ rel. $\mathfrak{A}'$ such that $\mathfrak{F}$ is embedded in $\mathfrak{F}'$ and is a dense subset of $\mathfrak{F}'$.

In the statements which follow $\mathfrak{F}$ and $\mathfrak{F}'$ denote normed functional classes rel. $\mathfrak{A}$ and $\mathfrak{A}'$, respectively.

1) *$\mathfrak{F}$ is embedded and dense in its saturated extension.*

2) *$\mathfrak{F}$ is complete if and only if its saturated extension is complete.*

3) *If $\mathfrak{F}'$ is a functional completion of $\mathfrak{F}$, then the saturated extension of $\mathfrak{F}'$ is also a functional completion of $\mathfrak{F}$, and it is the only saturated functional completion rel. $\mathfrak{A}'$.*

Proofs. — 1), 2) and the first part of 3) are obvious. Suppose that $\mathfrak{F}'$ and $\mathfrak{F}''$ are two saturated functional completions rel. $\mathfrak{A}'$ of $\mathfrak{F}$. We shall show that $\mathfrak{F}'\subset\mathfrak{F}''$, from which it will follow by symmetry that $\mathfrak{F}'$ and $\mathfrak{F}''$ are identical. Let f belong to $\mathfrak{F}'$. Then there is a sequence $\{f_n\}$ of functions in $\mathfrak{F}$ such that as elements of $\mathfrak{F}'$, $f_n\to f$, and such that $f_n(x)\to f(x)$ exc. $\mathfrak{A}'$. The sequence $\{f_n\}$ is necessarily Cauchy in $\mathfrak{F}'$, and since $\|g\|'=\|g\|=\|g\|''$ for all g in $\mathfrak{F}$, it is Cauchy in $\mathfrak{F}''$ too. As $\mathfrak{F}''$ is complete, there is an f'' in $\mathfrak{F}''$ such that $f_n\to f''$ in $\mathfrak{F}''$. For a suitable subsequence, therefore,

$$f''(x)=\lim f_{n_k}(x)=f(x)\text{ exc. }\mathfrak{A}'.$$

Since $\mathfrak{F}''$ is saturated, f belongs to $\mathfrak{F}''$. Thus $\mathfrak{F}'$ and $\mathfrak{F}''$ are identical functional classes; their norms agree as they agree on the dense subclass $\mathfrak{F}$.

In view of the first part of 3) there is never a loss of generality in restricting our discussion to saturated completions. This is sometimes convenient because of the uniqueness property described in the second part of 3).

4) *If* $\mathfrak{F}$ *has a functional completion rel.* $\mathfrak{A}'$, *then the saturated completion* $\tilde{\mathfrak{F}}'$ *rel.* $\mathfrak{A}'$ *is described as follows:*

(4. 1) *A function* f *defined in* $\mathcal{E}$ *belongs to* $\tilde{\mathfrak{F}}'$ *if and only if there is a Cauchy sequence* $\{f_n\}$ *in* $\mathfrak{F}$ *such that* $f_n(x) \to f(x)$ exc. $\mathfrak{A}'$. *If* f *belongs to* $\tilde{\mathfrak{F}}'$, *then* $\|f\| = \lim \|f_n\|$ *for any such Cauchy sequence.*

Proof. — From the definitions of functional completion it is clear that for each f in the completion there is a sequence with the properties listed. On the other hand, suppose that f is a function for which there exists such a sequence $\{f_n\}$. As $\{f_n\}$ is Cauchy, it has a limit f' in $\tilde{\mathfrak{F}}'$, and for a suitable subsequence $\{f_{n_k}\}$, $f'(x) = \lim f_{n_k}(x) = f(x)$ exc. $\mathfrak{A}'$. Since $\tilde{\mathfrak{F}}'$ is saturated, it must contain f.

5) *If* $\mathfrak{F}$ *has a functional completion rel.* $\mathfrak{A}'$ *and rel.* $\mathfrak{A}'' \supset \mathfrak{A}'$, *then it also has a functional completion rel.* $\mathfrak{A}'''$ *whenever* $\mathfrak{A}' \subset \mathfrak{A}''' \subset \mathfrak{A}''$.

Proof. — Under these circumstances a functional completion $\mathfrak{F}'$ rel. $\mathfrak{A}'$ is in fact also one rel. $\mathfrak{A}'''$. It is sufficient to show that $\mathfrak{F}'$ is a functional space rel. $\mathfrak{A}'''$; for this it is sufficient (see 3), section 3) to show that $\mathfrak{F}'$ is a functional space rel. $\mathfrak{A}''$. The only point which requires verification is that if $f = 0$ exc. $\mathfrak{A}''$, then $\|f\| = 0$. It is easy to see, however, from the description (4. 1) that $\mathfrak{F}'$ is embedded in the saturated completion rel. $\mathfrak{A}''$, so that $f = 0$ exc. $\mathfrak{A}''$ and $\|f\| \neq 0$ are incompatible.

6) *If* $\mathfrak{F}$ *has a functional completion relative to each of a sequence* $\{\mathfrak{A}_n\}$ *of exceptional classes, then it has a functional completion relative to their intersection.*

Proof. — Let $\mathfrak{A}'$ be the intersection, and let $\tilde{\mathfrak{F}}'$ be the class of functions described in (4. 1). Define $\|f\|$ for these functions as it is defined there. It is evident that $\tilde{\mathfrak{F}}'$ is a functional class rel. $\mathfrak{A}'$. Let $\mathfrak{F}_n$ be the saturated completion rel. $\mathfrak{A}_n$. From 4) it follows that for each n $\tilde{\mathfrak{F}}'$ is embedded in $\mathfrak{F}_n$. From this it follows directly that the norm on $\tilde{\mathfrak{F}}'$ is well defined, and that $\tilde{\mathfrak{F}}'$ is a functional space rel. $\mathfrak{A}_n$.

Consider a function f in $\mathfrak{F}_n$. There is a Cauchy sequence $\{f_k\}$ in $\mathfrak{F}$ converging exc. $\mathfrak{A}_n$ to f. Now, $\{f_k\}$ is Cauchy in every $\mathfrak{F}_j$, therefore convergent in every $\mathfrak{F}_j$. Hence for each j it contains a subsequence which converges pointwise exc. $\mathfrak{A}_j$. By the diagonal process it is possible to obtain a subsequence which converges exc. $\mathfrak{A}'$, converges therefore exc. $\mathfrak{A}'$ to a function f' in $\tilde{\mathfrak{F}}'$. We have proved that $\tilde{\mathfrak{F}}' \subset \mathfrak{F}_n$, and that each f in $\mathfrak{F}_n$ is equal exc. $\mathfrak{A}_n$ to an f' in $\tilde{\mathfrak{F}}'$. This means that $\mathfrak{F}_n$ is the saturated extension of $\tilde{\mathfrak{F}}'$ rel. $\mathfrak{A}_n$, so that $\tilde{\mathfrak{F}}'$, like $\mathfrak{F}_n$ is complete and is a functional space rel. $\mathfrak{A}_n$. By 4), section 3, $\tilde{\mathfrak{F}}'$ is a complete functional space rel. $\mathfrak{A}'$. That $\mathfrak{F}$ is embedded and dense in $\tilde{\mathfrak{F}}'$ does not require proof.

This proof shows the possibility of using (4. 1) not only in describing a functional completion known a priori to exist, but also in making an existence proof. Whenever $\mathfrak{F}$ is a normed functional class rel. $\mathfrak{A} \subset \mathfrak{A}'$, (4. 1) defines a class of functions $\tilde{\mathfrak{F}}'$ which is a functional class rel. $\mathfrak{A}'$. It also gives a procedure to define a norm in $\tilde{\mathfrak{F}}'$; this norm is well defined if and only if it does not depend on the choice of the Cauchy sequence $\{f_n\}$ converging to f pointwise exc. $\mathfrak{A}'$.

7) (*a*) *$\tilde{\mathfrak{F}}'$ is a normed functional class rel. $\mathfrak{A}'$ if and only if for each Cauchy sequence $\{f_n\}$ in $\mathfrak{F}$ which converges pointwise exc. $\mathfrak{A}'$ the conditions $f_n(x) \to 0$ exc. $\mathfrak{A}'$ and $\|f_n\| \to 0$ are equivalent. If $\tilde{\mathfrak{F}}'$ is a normed functional class rel. $\mathfrak{A}'$, then $\mathfrak{F}$ is embedded and dense in $\tilde{\mathfrak{F}}'$.*

(*b*) *If $\tilde{\mathfrak{F}}'$ is a normed functional class, and if each Cauchy sequence in $\mathfrak{F}$ contains a subsequence which converges exc. $\mathfrak{A}'$, then $\tilde{\mathfrak{F}}'$ is complete. $\tilde{\mathfrak{F}}'$ is a functional completion of $\mathfrak{F}$ if and only if it satisfies this condition on Cauchy sequences and is a functional space rel. $\mathfrak{A}'$.*

PROOF. — (*a*) If $\tilde{\mathfrak{F}}'$ is a normed functional class rel. $\mathfrak{A}'$ (which implies in particular that the norm in $\tilde{\mathfrak{F}}'$ is well defined), then a sequence $\{f_n\}$ of the type indicated has a limit f in $\tilde{\mathfrak{F}}'$ to which it converges pointwise exc. $\mathfrak{A}'$. Each condition which follows is obviously equivalent to the conditions adjacent to it: (i) $||f_n|| \to 0$; (ii) $||f|| = 0$; (iii) $f(x) = 0$ exc. $\mathfrak{A}'$; (iv) $f_n(x) \to 0$ exc. $\mathfrak{A}'$.

Suppose that $\mathcal{F}$ has the property described in (a). If for an f in $\tilde{\mathcal{F}}'$ there are Cauchy sequences $\{f_n\}$ and $\{g_n\}$ in $\mathcal{F}$ which converge exc. $\mathfrak{A}'$ to f, then $|\, ||f_n|| - ||g_n||\, | \leqslant ||f_n - g_n|| \to 0$, for $\{f_n - g_n\}$ is a Cauchy sequence which converges to 0 exc. $\mathfrak{A}'$. Therefore the procedure of (4. 1) for norming $\tilde{\mathcal{F}}'$ is well defined; the norm of an f does not depend on the particular approximating sequence. The proof that $\tilde{\mathcal{F}}'$ is a normed functional class rel. $\mathfrak{A}'$ in which $\mathcal{F}$ is embedded offers no difficulty. In order to show that $\mathcal{F}$ is dense in $\tilde{\mathcal{F}}'$ we verify the fact that if f is a pointwise limit exc. $\mathfrak{A}'$ of a Cauchy sequence $\{f_n\}$ in $\mathcal{F}$, then $||f_n - f|| \to 0$. For each n, $f_n - f$ is a pointwise limit exc. $\mathfrak{A}'$ of the Cauchy sequence $\{f_n - f_m\}$ in $\mathcal{F}$, so that by definition $||f_n - f|| = \lim\limits_{m \to \infty} ||f_n - f_m||$, and this can be made arbitrarily small by proper choice of n because the sequence $\{f_n\}$ is Cauchy.

(b) If $\tilde{\mathcal{F}}'$ is a functional completion, then by definition it is a complete class and a functional space rel. $\mathfrak{A}'$; hence each Cauchy sequence has a subsequence which converges exc. $\mathfrak{A}'$.

To prove completeness under the hypothesis in (b) it is sufficient, since we have already established that $\mathcal{F}$ is embedded and dense in $\tilde{\mathcal{F}}'$, to prove that each Cauchy sequence in $\mathcal{F}$ has a limit in $\tilde{\mathcal{F}}'$. By hypothesis each Cauchy sequence in $\mathcal{F}$ has a subsequence which converges exc. $\mathfrak{A}'$. The pointwise limit of this subsequence belongs necessarily to $\tilde{\mathcal{F}}'$, and it is the limit in norm of the subsequence, therefore also of the sequence.

Remark 1. — It is particularly important in applications to make use of completions for which the exceptional sets are as small as possible, for in these the functions are determined most accurately. If there is a smallest exceptional class $\mathfrak{A}$ relative to which a given $\mathcal{F}$ has a functional completion, then the saturated completion rel. $\mathfrak{A}$ is called the *perfect* completion of $\mathcal{F}$. Proposition 4) is relevant here, but it cannot be used, even with the hypothesis that there exists some completion, to deduce that there exists a perfect completion. It provides only for countable intersection of exceptional classes. This general question of existence is open. We were able, however, to settle it under special assumptions general enough to have a wide range of applications (see the end of section 6).

§ 5. **The functions δ and $\tilde{\delta}$ and the classes which they define.** — In this section and the next we introduce certain functions and classes of sets which lead toward solutions, partial or complete, to the following problems : (*i*) to decide when a given normed functional class admits a functional completion; (*ii*) to decide when it admits a perfect completion; (*iii*) to describe the exceptional sets for a perfect completion. The classes introduced will provide explicit bounds for the exceptional class of a perfect completion; in every example where a perfect completion has been found, its exceptional class coincides with the bounds given. Throughout the two sections $\mathfrak{A}$ is a fixed exceptional class, $\mathcal{F}$ is a fixed normed functional class rel. $\mathfrak{A}$. The initial definitions follow.

DEFINITION *a*). — $\mathfrak{L}$ is the class of all sets $\mathrm{B} \subset \mathcal{E}$ for which there is an f in $\mathcal{F}$ satisfying $|f(x)| \geqq 1$ on B exc. $\mathfrak{A}$; for each B in $\mathfrak{L}$, $\delta(\mathrm{B})$ is the infimum, over all f in $\mathcal{F}$ satisfying $|f(x)| \geqq 1$ on B exc. $\mathfrak{A}$, of the numbers $\|f\|$.

DEFINITION *b*). — $\tilde{\mathfrak{L}}$ is the class of all sets $\mathrm{B} \subset \mathcal{E}$ for which there is a Cauchy sequence $\{f_n\}$ in $\mathcal{F}$ satisfying $\liminf |f_n(x)| \geqq 1$ on B exc. $\mathfrak{A}$; for each B in $\tilde{\mathfrak{L}}$, $\tilde{\delta}(\mathrm{B})$ is the infimum, over all Cauchy sequences $\{f_n\}$ in $\mathcal{F}$ satisfying $\liminf |f_n(x)| \geqq 1$ on B exc. $\mathfrak{A}$, of the numbers $\lim \|f_n\|$.

DEFINITION *c*). — $\mathfrak{L}^0$ is the class of all sets B in $\mathfrak{L}$ with $\delta(\mathrm{B}) = 0$; $\tilde{\mathfrak{L}}^0$ is the class of all sets B in $\tilde{\mathfrak{L}}$ with $\tilde{\delta}(\mathrm{B}) = 0$.

The first two statements below follow directly from these definitions.

1) *If* $\mathrm{A} \in \mathfrak{A}$, *then* $\mathrm{A} \in \mathfrak{L}$ *and* $\delta(\mathrm{A}) = 0$; *if* $\mathrm{B} \in \mathfrak{L}$ *and* $\mathrm{B}' = \mathrm{B}$ exc. $\mathfrak{A}$, *then* $\mathrm{B}' \in \mathfrak{L}$, *and* $\delta(\mathrm{B}) = \delta(\mathrm{B}')$; *if* $\mathrm{B} \in \mathfrak{L}$ *and* $\mathrm{B}' \subset \mathrm{B}$, *then*

$\mathrm{B}' \in \mathfrak{L}$, *and* $\delta(\mathrm{B}') \leqq \delta(\mathrm{B})$. *The same statements hold for* $\tilde{\mathfrak{L}}$ *and* $\tilde{\delta}$. *Consequently,* $\mathfrak{L}$, $\mathfrak{L}^0$, $\tilde{\mathfrak{L}}$, *and* $\tilde{\mathfrak{L}}^0$ *are all hereditary and contain* $\mathfrak{A}$.

2) $\mathfrak{L} \subset \tilde{\mathfrak{L}} \subset \mathfrak{L}_\sigma$; *if* $\mathrm{B} \in \mathfrak{L}$, *then* $\delta(\mathrm{B}) \geqq \tilde{\delta}(\mathrm{B})$. *Hence* $\mathfrak{L}^0 \subset \tilde{\mathfrak{L}}^0$ ([11]).

3) (*a*) *If* $\mathcal{F}$ *is a functional space rel.* $\mathfrak{A}$ *then* $\mathfrak{A} = \mathfrak{L}^0$.

(*b*) *If* $\mathcal{F}$ *is complete and a functional space rel.* $\mathfrak{A}$, *then* $\mathfrak{L} = \tilde{\mathfrak{L}}$, $\delta(\mathrm{B}) = \tilde{\delta}(\mathrm{B})$ *and* $\mathfrak{A} = \mathfrak{L}^0 = \tilde{\mathfrak{L}}^0$.

4) (*a*) *For each* $\mathrm{B} \in \mathfrak{L}^0$ *there is a sequence* $\{f_n\}$ *in* $\mathcal{F}$ *such that* $\|f_n\| \to 0$ *and* $|f_n(x)| \to \infty$ *on* B exc. $\mathfrak{A}$.

(*b*) *If* $\mathrm{B} \subset \mathcal{E}$ *is such that for some sequence* $\{f_n\}$ *in* $\mathcal{F}$, $\|f_n\| \to 0$ *and* $\lim |f_n(x)| > 0$ *on* B *exc.* $\mathfrak{A}$, *then* $\mathrm{B} \in \mathfrak{L}^0_\sigma$.

5) *If* $\mathrm{B} \subset \mathcal{E}$ *is such that for some Cauchy sequence* $\{f_n\}$ *in* $\mathcal{F}$, $|f_n(x)| \to \infty$ *on* B exc. $\mathfrak{A}$, *then* $\mathrm{B} \in \tilde{\mathfrak{L}}^0$.

Proofs. — 3) (*a*). By 1) $\mathfrak{A} \subset \mathfrak{L}^0$. On the other hand if $\mathrm{B} \in \mathfrak{L}^0$, there exists $\{f_n\}$ such that $\|f_n\| \to 0$ and $|f_n(x)| \geqq 1$ for $x \in \mathrm{B}$ exc. $\mathfrak{A}$. By definition of functional spaces it follows that $\mathrm{B} \in \mathfrak{A}$.

3) (*b*). In view of statements 1) and 2) and 3) (*a*), we have only to prove that $\mathfrak{L} \supset \tilde{\mathfrak{L}}$ and $\delta(\mathrm{B}) \leqq \tilde{\delta}(\mathrm{B})$. Let $\mathrm{B} \in \tilde{\mathfrak{L}}$ and let $\{f_n\}$ be a Cauchy sequence such that $\liminf |f_n(x)| \geqq 1$ for $x \in \mathrm{B}$ exc. $\mathfrak{A}$. Since $\mathcal{F}$ is a complete functional space we can find a subsequence $\{f'_n\}$ and a function $f \in \mathcal{F}$ such that $\{f'_n\} \to f$ and $f'_n(x) \to f(x)$ exc. $\mathfrak{A}$. It follows, $|f(x)| \geqq 1$ for $x \in \mathrm{B}$ exc. $\mathfrak{A}$ and $\lim \|f'_n\| = \|f\|$ and thus both our assertions are proved.

4) (*a*) If $\delta(\mathrm{B}) = 0$, then for each n there is a function g_n in $\mathcal{F}$ such that $\|g_n\| \leqq 1/n^2$ and $|g_n(x)| \geqq 1$ on B exc. $\mathfrak{A}$. Take $f_n = n g_n$.

(*b*) If $\mathrm{B}_{k,n} = \underset{x}{\mathrm{E}}\left[|f_n(x)| \geqq \frac{1}{k}\right]$, then $\delta(\mathrm{B}_{k,n}) \leqq k\|f_n\|$ and $\mathrm{B} \subset \bigcup_{k=1}^{\infty} \bigcup_{l=1}^{\infty} \bigcap_{n=l}^{\infty} \mathrm{B}_{k,n}$ exc. $\mathfrak{A}$. On the other hand, $\bigcap_{n=l}^{\infty} \mathrm{B}_{k,n} \in \mathfrak{L}^0$, for $\delta\left(\bigcap_{n=l}^{\infty} \mathrm{B}_{k,n}\right) \leqq \inf_{n \geqq l} \delta(\mathrm{B}_{k,n}) \leqq \inf_{n \geqq l} k\|f_n\| = 0$. Hence the result.

5) Let $\mathrm{M} = \lim \|f_n\|$. For each $\varepsilon > 0$, the sequence $\{\varepsilon f_n\}$ is a Cauchy sequence in $\mathcal{F}$ satisfying $\liminf |\varepsilon f_n(x)| \geqq 1$ on B exc. $\mathfrak{A}$. Therefore $\tilde{\delta}(\mathrm{B}) \leqq \varepsilon \mathrm{M}$.

([11]) In general the equality $\mathfrak{L}^0 = \tilde{\mathfrak{L}}^0$ and even $\mathfrak{L}^0_\sigma = \tilde{\mathfrak{L}}^0_\sigma$ is not true, as will be shown at the end of the example in section 9.

The rest of the section is given to the statement and proof of its main theorem. The theorem displays necessary and sufficient conditions that $\mathfrak{F}$ be a functional space, or that it admit a functional completion, relative to a given exceptional class $\mathfrak{A}' \supset \mathfrak{A}$. The conditions for the existence of a functional completion rel. $\mathfrak{A}'$, unlike those given in section 4, are expressible within $\mathfrak{F}$ and $\mathfrak{A}'$, without recourse to the auxiliary class $\tilde{\mathfrak{F}}'$ (which is always the functional class defined by (4. 1)). However, new information about $\tilde{\mathfrak{F}}'$ is required for the proof of the theorem. Since this has independent interest, we state it as a lemma distinct from the main line of argument. When $\tilde{\mathfrak{F}}'$ is a normed functional class rel. $\mathfrak{A}'$, the classes $\mathfrak{L}$ and $\mathfrak{L}^0$ and the function δ formed for $\tilde{\mathfrak{F}}'$ are denoted by $\overline{\mathfrak{L}}$, $\overline{\mathfrak{L}^0}$, and $\bar{\delta}$.

THEOREM. — *Let $\mathfrak{F}$ be a normed functional class rel. $\mathfrak{A}$, and let $\mathfrak{A}' \supset \mathfrak{A}$.*

(*a*) *In order that $\mathfrak{F}$ be a functional space rel. $\mathfrak{A}'$ it is necessary and sufficient that conditions 1_a and 2_a be satisfied.*

1_a *If $f(x) = 0$ exc. $\mathfrak{A}'$, then $\|f\| = 0$.*

2_a *Each sequence of sets B_n, such that $\delta(B_n) \to 0$, contains a subsequence whose limit superior belongs to $\mathfrak{A}'$* ([12]).

(*b*) *In order that $\mathfrak{F}$ have a functional completion rel. $\mathfrak{A}'$ it is necessary and sufficient that conditions 1_b, 2_b, and 3_b be satisfied.*

1_b *For each Cauchy sequence $\{f_n\}$ in $\mathfrak{F}$ which converges pointwise exc. $\mathfrak{A}'$ the conditions $f_n(x) \to 0$ exc. $\mathfrak{A}'$ and $\|f_n\| \to 0$ are equivalent.*

2_b *Each Cauchy sequence $\{f_n\}$ in $\mathfrak{F}$ contains a subsequence which converges pointwise exc. $\mathfrak{A}'$.*

3_b *Each sequence of sets B_n such that $\tilde{\delta}(B_n) \to 0$ contains a subsequence whose limit superior belongs to $\mathfrak{A}'$.*

LEMMA. — *Let $\mathfrak{F}$ be a normed functional class rel. $\mathfrak{A}$, let $\mathfrak{A}' \supset \mathfrak{A}$, and suppose that conditions 1_b and 2_b are satisfied. Then $\tilde{\mathfrak{F}}'$*

([12]) The standard definitions of the limits superior and inferior of a sequence $\{B_n\}$ of sets are as follows: $\limsup B_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n$; $\liminf B_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} B_n$. The limit superior consists of those points which belong to infinetely many B_n, the limit inferior of those points which belong to all but finitely many B_n.

is a complete normed functional class rel. $\mathfrak{A}'$. *If* $\mathrm{B}' = \mathrm{B}$ *exc.* $\mathfrak{A}'$ *for some set* $\mathrm{B} \in \tilde{\mathfrak{L}}$, *then* $\mathrm{B}' \in \overline{\mathfrak{L}}$ *and* $\bar{\delta}(\mathrm{B}') \leqq \tilde{\delta}(\mathrm{B})$. *If* $\mathrm{B}' \in \overline{\mathfrak{L}}$, *then there is a set* $\mathrm{B} \in \tilde{\mathfrak{L}}$ *such that* $\mathrm{B}' = \mathrm{B}$ *exc.* $\mathfrak{A}'$ *and* $\bar{\delta}(\mathrm{B}') = \tilde{\delta}(\mathrm{B})$.

PROOF. OF THE LEMMA. — The truth of the first part of the lemma, which states that $\tilde{\mathcal{F}}'$ is a complete normed functional class rel. $\mathfrak{A}'$ can be seen from proposition 7) section 4.

Suppose that $\mathrm{B}' \subset \mathcal{E}$ is equal exc. $\mathfrak{A}'$ to some $\mathrm{B} \in \tilde{\mathfrak{L}}$. For each $\varepsilon > 0$ there is a Cauchy sequence $\{f_n\}$ in $\mathcal{F}$ satisfying $\liminf |f_n(x)| \geqq 1$ on B exc. $\mathfrak{A}$ and $\lim \|f_n\| \leqq \tilde{\delta}(\mathrm{B}) + \varepsilon$. Because of 2_b it can be assumed that $\{f_n\}$ converges pointwise exc. $\mathfrak{A}'$. Then its pointwise limit $\tilde{f}$ belongs to $\tilde{\mathcal{F}}'$ and satisfies $|\tilde{f}(x)| \geqq 1$ on B' exc. $\mathfrak{A}'$. Therefore $\mathrm{B}' \in \overline{\mathfrak{L}}$, and

$$\bar{\delta}(\mathrm{B}') \leqq \|\tilde{f}\| = \lim \|f_n\| \leqq \tilde{\delta}(\mathrm{B}) + \varepsilon,$$

so that $\bar{\delta}(\mathrm{B}') \leqq \tilde{\delta}(\mathrm{B})$.

Suppose that B' belongs to $\overline{\mathfrak{L}}$. For each $\varepsilon > 0$ there is a function $\tilde{f}$ in $\tilde{\mathcal{F}}'$ satisfying $|\tilde{f}(x)| \geqq 1$ on B' exc. $\mathfrak{A}'$ and $\|\tilde{f}\| \leqq \bar{\delta}(\mathrm{B}') + \varepsilon$. There is also a Cauchy sequence $\{f_n\}$ in $\mathcal{F}$ which converges pointwise to $\tilde{f}$ exc. $\mathfrak{A}'$. Let B_ε be the set of points x in B' such that $\liminf |f_n(x)| \geqq 1$. Then $\mathrm{B}_\varepsilon \in \tilde{\mathfrak{L}}$, $\mathrm{B}_\varepsilon = \mathrm{B}'$ exc. $\mathfrak{A}'$, and

$$\tilde{\delta}(\mathrm{B}_\varepsilon) \leqq \lim \|f_n\| = \|\tilde{f}\| \leqq \bar{\delta}(\mathrm{B}') + \varepsilon.$$

If $\mathrm{B} = \bigcap_{n=1}^{\infty} \mathrm{B}_{\varepsilon_n}$, where $\varepsilon_n \to 0$, then $\mathrm{B} \in \tilde{\mathfrak{L}}$, $\mathrm{B} = \mathrm{B}'$ exc. $\mathfrak{A}'$, and $\tilde{\delta}(\mathrm{B}) \leqq \bar{\delta}(\mathrm{B}')$. The inequality $\tilde{\delta}(\mathrm{B}) \geqq \bar{\delta}(\mathrm{B}')$ was established in the last paragraph.

PROOF. OF THE THEOREM. — First we shall use the lemma and results from section 4 to show that (b) is implied by (a). Then we shall prove (a).

Because of 4), section 4, $\mathcal{F}$ has a functional completion rel. $\mathfrak{A}'$ if and only if $\tilde{\mathcal{F}}'$ itself is a functional completion rel. $\mathfrak{A}'$. Because of 7), section 4, $\tilde{\mathcal{F}}'$ is a functional completion rel. $\mathfrak{A}'$ if and only if 1_b and 2_b hold, and in addition $\tilde{\mathcal{F}}'$ is a functional space. If

1_b and 2_b are assumed, then, by virtue of the lemma, 3_b (as it stands) is equivalent to 2_a (as applied to $\tilde{\mathcal{F}}'$). Therefore (b) is implied by (a).

Suppose that $\mathcal{F}$ is a functional space rel. $\mathfrak{A}'$. Obviously 1_a holds. If $\{B_n\}$ is a sequence of sets in $\mathfrak{L}$, then for each n there is a function f_n in $\mathcal{F}$ satisfying $|f_n(x)| \geqq 1$ on B_n exc. $\mathfrak{A}$ and $\|f_n\| \leqq \delta(B_n) + 1/n$. If $\delta(B_n) \to 0$, then, as $\mathcal{F}$ is a functional space rel. $\mathfrak{A}'$, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges pointwise to 0 exc. $\mathfrak{A}'$. Since $\limsup |f_{n_k}(x)| \geqq 1$ on $\limsup B_{n_k}$ exc. $\mathfrak{A}$, $\limsup B_{n_k}$ must belong to $\mathfrak{A}'$; and 2_a holds.

Suppose that 1_a and 2_a hold. It is clear (from 1_a alone) that $\mathcal{F}$ is a normed functional class rel. $\mathfrak{A}'$. Given a sequence $\{f_n\}$ in $\mathcal{F}$ with $\|f_n\| \to 0$, set $B_n = \underset{x}{\mathrm{E}}\,[|f_n(x)| \geqq 1/M_n]$, where $\{M_n\}$ is any sequence of positive numbers converging to infinity and such that $M_n\|f_n\| \to 0$. Then $B_n \in \mathfrak{L}$, and $\delta(B_n) \to 0$. By hypothesis, there is a subsequence $\{B_{n_k}\}$ of $\{B_n\}$ with $\limsup B_{n_k} \in \mathfrak{A}'$. On the complement of $\limsup B_{n_k}$, $f_{n_k}(x) \to 0$ exc. $\mathfrak{A}'$. Therefore $f_{n_k}(x) \to 0$ exc. $\mathfrak{A}'$; hence the defining property of a functional space is true.

Corollary. — *If $\mathcal{F}$ is a functional space rel. $\mathfrak{A}'$, then $\mathfrak{A}' \supset \mathfrak{L}^0_\sigma$. If $\mathcal{F}$ has a functional completion rel. $\mathfrak{A}'$, then $\mathfrak{A}' \supset \tilde{\mathfrak{L}}^0_\sigma$.*

This follows from the fact that for $B \in \mathfrak{L}^0$ $\left(\text{or } B \in \tilde{\mathfrak{L}}^0\right)$ we can put $B_n = B$ in condition 2_a (or 3_b).

Remark 1. — The second part of the theorem and the lemma show that $\tilde{\mathfrak{L}}$ and $\tilde{\delta}$ play the same role for completion of $\mathcal{F}$ as $\mathfrak{L}$ and δ for $\mathcal{F}$. A simple consequence of the lemma is that $\overline{\mathfrak{L}}$ is the class of all sets equal to some set in $\tilde{\mathfrak{L}}$ exc. $\mathfrak{A}'$, and that $\bar{\delta}(B') = \min \tilde{\delta}(B)$ for all $B \in \tilde{\mathfrak{L}}$ such that $B = B'$ exc. $\mathfrak{A}'$.

§ 6. **Capacities.** — In section 5 a lower bound for the exceptional class of a perfect functional completion was given. In this section an upper bound is given, and additional conditions for the existence of functional completions are obtained. The description of the upper bound resembles that of the lower bound: certain set functions on the basic set are introduced, and the upper bound is determined as the class of null sets for these functions. In some of the differential problems which have had a decisive effect on the development of the thoery of functional spaces the set functions in question prove to include among them the classical capacities. For this reason they will be called capacities in the general case also. Throughout the section, $\mathfrak{A}$ is an exceptional class; $\mathcal{F}$ is a normed functional class rel. $\mathfrak{A}$; and δ, $\mathfrak{L}$, etc. are the functions and classes defined in section 5.

If $\varphi(t)$ is a non-negative real-valued function satisfying (i) $\varphi(t)$ is defined for all $t \geqslant 0$, $\varphi(t) > 0$ for $t > 0$; (ii) $\varphi(t)$ is non-decreasing; (iii) $\varphi(0) = \lim_{t \to 0} \varphi(t) = 0$; then φ determines a set function c_φ on $\mathfrak{L}_\sigma$ as follows:

(6. 1) *For each* $B \in \mathfrak{L}_\sigma$, $c_\varphi(B) = \inf \sum_{n=1}^{\infty} \varphi[\delta(B_n)]$, *where the infimum is taken over all sequences* $\{B_n\} \subset \mathfrak{L}$ *such that* $B \subset \bigcup_{n=1}^{\infty} B_n$.

The set function c_φ is called the φ-capacity. Only routine calculation with the definition is needed to establish the following properties of $c = c_\varphi$.

(6. 2) (a) *For each* $B \in \mathfrak{L}_\sigma$, $c(B)$ *is a non-negative real number or* $+\infty$.
(b) *If* $B \subset B'$ *then* $c(B) \leqq c(B')$; $c(0) = 0$.
(c) *If* $B = \bigcup_{n=1}^{\infty} B_n$, *then* $c(B) \leqq \sum_{n=1}^{\infty} c(B_n)$.
(d) *For each* $B \in \mathfrak{L}$, $c(B)$ *is finite.*
(e) *To each* $\varepsilon > 0$ *corresponds a* $\delta > 0$ *such that if* $\delta(B) \leqslant \delta$, *then* $c(B) \leqslant \varepsilon$.

In order to shorten notations and make proofs easier to read we will operate directly with the properties (6. 2), rather than with the functions φ explicitly. Accordingly we make two definitions: a *capacity* is a set function c on $\mathfrak{L}_\sigma$ with the properties (*a*)-(*c*) in (6. 2); a capacity is *admissible* if it has also properties (*d*) and (*e*).

The class of admissible capacities will be called Ω. The class of sets which are of capacity 0 for a given admissible capacity c will be called $\mathfrak{A}_c$; the class of sets which are of capacity 0 for all admissible capacities will be called $\mathfrak{A}_\Omega$.

REMARK 1. — One of the chief objects of the section is to show that $\mathfrak{A}_\Omega$ is an upper bound for the exceptional class of a perfect completion, if a perfect completion exists. It might seem that acceptance of abstract capacities makes the bound better than it would be if only φ-capacities were accepted. This is not true. Given any admissible capacity c, it is easy to construct a φ-capacity c_φ such that if $c(\mathrm{B}) \neq 0$ then $c_\varphi(\mathrm{B}) \neq 0$. A similar comment is to the point with regard to weakening (d) and (e) by deleting (d) and replacing (e) by a condition of the following nature: (e') *there is a number* $\delta_0 > 0$ *such that whenever* B_0 *is fixed and satisfies* $\delta(\mathrm{B}_0) < \delta_0$, *then* (e) *holds with respect to the subsets of* B_0.

It will be observed that the conditions (a) — (c) are exactly the defining conditions for an outer measure on $\mathfrak{L}_\sigma$. Thus every capacity is an outer measure on the hereditary σ-ring $\mathfrak{L}_\sigma$. In spite of this, it would be deceptive to use the term outer measure instead of the term capacity. The problems with which we are concerned are of an entirely different kind from those in measure theory. Measurability, for instance, is irrelevant; and in fact it may happen that the only measurable sets in $\mathfrak{L}_\sigma$ are the sets of measure 0.

1) *A capacity* c *on* $\mathfrak{L}_\sigma$ *is admissible if and only if* $c(\mathrm{B})$ *is finite for each* B *in* $\mathfrak{L}$, $c(\mathrm{A}) = 0$ *for each* A *in* $\mathfrak{A}$, *and either of the two equivalent conditions* (a) *or* (b) *below holds.*

(a) *To each pair of numbers* $\varepsilon > 0$ *and* $\eta > 0$ *corresponds a* $\delta > 0$ *such that if* $\|f\| < \delta$, *then* $c\left(\underset{x}{\mathrm{E}}\,[|f(x)| \geqq \varepsilon]\right) < \eta$.

(b) *If* $\|f_n\| \rightarrow 0$, *then* $\{f_n\}$ *converges to* 0 *in capacity (with respect to* c); *that is, for each* $\varepsilon > 0$, $\lim c(\mathrm{B}_{n,\varepsilon}) = 0$, *where* $\mathrm{B}_{n,\varepsilon} = \underset{x}{\mathrm{E}}\,[|f_n(x)| \geqq \varepsilon]$.

PROOF. — It is obvious that conditions (a) and (b) are equivalent. Let c be admissible; choose $\delta_0 > 0$ such that $\delta(\mathrm{B}) < \delta_0$ implies $c(\mathrm{B}) \leqslant \eta/2$ and put $\delta = \varepsilon\delta_0$. Then $\|f\| < \delta$ gives. $\delta\left(\underset{x}{\mathrm{E}}\,[|f(x)| \geqq \varepsilon]\right) \leqq \left\|\frac{f}{\varepsilon}\right\| < \frac{\delta}{\varepsilon} = \delta_0$, hence condition (a). That (a) implies admissibility of c follows by a similar argument in reverse.

2) *Let c be an admissible capacity on $\mathfrak{L}_\sigma$. To each $B' \in \tilde{\mathfrak{L}}$ and $\varepsilon > 0$ correspond sets B and D such that $B' \subset B \cup D$, $B \in \mathfrak{L}$, $\delta(B) \leqq \tilde{\delta}(B')$, and $c(D) < \varepsilon$.*

Proof. — Choose, as 1)-(a) permits, a sequence of numbers δ_n such that if $\|f\| \leqq \delta_n$, then $c\left(\underset{x}{\mathrm{E}}\,[|f(x)| \geqq 1/2^n]\right) \leqq 1/2^n$. Then choose Cauchy sequences $\{f_n^{(k)}\}$ such that: $\liminf_{n\to\infty} |f_n^{(k)}(x)| \geqq 1$ on B' exc. $\mathfrak{A}$, $\sup_n \|f_n^{(k)}\| < \tilde{\delta}(B') + \frac{1}{2^k}$ and $\|f_n^{(k)} - f_{n-1}^{(k)}\| \leqq \delta_n$, for $k = 1, 2, \ldots$. Let $B_n^{(k)} = \underset{x}{\mathrm{E}}\left[|f_n^{(k)}(x)| \geqq 1 - \frac{1}{2^n}\right]$ and let

$$A_n^{(k)} = \underset{x}{\mathrm{E}}\left[|f_n^{(k)}(x) - f_{n-1}^{(k)}(x)| \geqq \frac{1}{2^n}\right].$$

For all n and k, we have clearly, $B' \subset B_n^{(k)} + \bigcup_{l=n+1}^{\infty} A_l^{(k)}$ exc. $\mathfrak{A}$. Hence for every $i = 1, 2, \ldots$,

$$B' \subset \bigcap_{n=i}^{\infty}\left(B_n^{(n)} + \bigcup_{l=n+1}^{\infty} A_l^{(n)}\right) \subset \bigcap_{n=i}^{\infty} B_n^{(n)} + \bigcup_{n=i}^{\infty}\bigcup_{l=n+1}^{\infty} A_l^{(n)} \text{ exc. } \mathfrak{A}.$$

Since $\delta(B_n^{(n)}) \leqq \frac{1}{1-2^{-n}}\|f_n^{(n)}\| < \frac{1}{(1-2^{-n})}\left(\tilde{\delta}(B') + \frac{1}{2^n}\right)$ we get for $B_i = \bigcap_{n=i}^{\infty} B_n^{(n)}$, $\delta(B_i) \leqq \tilde{\delta}(B')$. For $D_i = \bigcup_{n=i}^{\infty}\bigcup_{l=n+1}^{\infty} A_l^{(n)}$ we have $c(D_i) \leqq \sum_{n=i}^{\infty}\sum_{l=n+1}^{\infty} c(A_l^{(n)}) \leqq \sum_{n=i}^{\infty}\sum_{l=n+1}^{\infty} \frac{1}{2^l} = \frac{1}{2^{i-1}}$. For i large enough $c(D_i) < \varepsilon$ and the inclusion $B' \subset B_i + D_i$ exc. $\mathfrak{A}$ proves our statement.

3) *If c is an admissible capacity on $\mathfrak{L}_\sigma$, then to each $\varepsilon > 0$ corresponds a $\delta > 0$, namely, the δ of (6. 2) (e), such that if $B \in \tilde{\mathfrak{L}}$ and $\tilde{\delta}(B) \leqq \delta$ then $c(B) \leqq \varepsilon$. In particular, if $\tilde{\delta}(B) = 0$, then $c(B) = 0$, so that $\tilde{\mathfrak{L}}_\sigma^0 \subset \mathfrak{A}_c$.*

Proof. — For each $\varepsilon > 0$ let $\delta > 0$ be determined in accordance with (6. 2) (e). From 2) it follows that if $\tilde{\delta}(B) \leqq \delta$, then $c(B) \leqq \varepsilon$.

4) *To each admissible capacity c on $\mathfrak{L}_\sigma$ corresponds a sequence of numbers δ_n such that if $\{f_n\}$ is any sequence of functions*

in $\mathcal{F}$ *satisfying* $\|f_n - f_{n-1}\| \leqq \delta_n$, *then* $f_n(x)$ *converges pointwise exc.* $\mathfrak{A}_c$, *and for each* $\varepsilon > 0$, *the convergence is uniform outside some set of capacity less than* ε.

PROOF. — For a given sequence of functions f_n, let

$$A_n = \underset{x}{\mathrm{E}}\,[|f_n(x) - f_{n-1}(x)| \geqq 1/2^n].$$

If a point x belongs to no A_n with $n \geqq n_0$, then for every n with $n \geqq n_0$ and every p,

$$|f_{n+p}(x) - f_n(x)| \leqq \sum_{k=n+1}^{n+p} |f_k(x) - f_{k-1}(x)| \leqq \sum_{k=n+1}^{n+p} 1/2^k \leqq 1/2^n \text{ exc. } \mathfrak{A}.$$

Therefore $f_n(x)$ converges uniformly on the complement of $\bigcup_{k=n_0}^{\infty} A_k$, exc. $\mathfrak{A}$, for every choice of n_0. By 1) it is possible to choose δ_n so that if $\|f\| \leqq \delta_n$, then $c\left(\underset{x}{\mathrm{E}}\,[|f(x)| \geqq 1/2^n]\right) \leqq 1/2^n$; hence so that $c\left(\bigcup_{k=n_0}^{\infty} A_k\right) \leqq \sum_{k=n_0}^{\infty} c(A_k) \leqq \sum_{k=n_0}^{\infty} 1/2^k \leqq 1/2^{n_0-1}$.

REMARK 2. — The last statement is analogous and its proof is identical to the classical theorem on pointwise convergence of Cauchy sequences in a space L^p relative to a measure μ (more generally to convergence in measure). As a matter of fact, in the functional space L^p, the measure μ is equal to its capacity c_φ for $\varphi(\rho) = \rho^p$.

We shall consider now the conditions 1_b, 2_b, and 3_b of the theorem in the last section with respect to the class $\mathfrak{A}_c$ of null sets of an admissible capacity c; 2_b, 3_b, and half of 1_b are automatically satisfied.

1_b If $\|f_n\| \to 0$, then $f_n(x) \to 0$ in capacity (by 1) (b)), so that if f_n converges pointwise exc. $\mathfrak{A}_c$, it must converge pointwise to 0 exc. $\mathfrak{A}_c$.

2_b Given a Cauchy sequence $\{g_n\}$, pick a subsequence $\{f_n\}$ so that $\|f_n - f_{n-1}\| \leqq \delta_n$, where $\{\delta_n\}$ is the sequence of numbers provided by 4). By 4) the subsequence $\{f_n\}$ converges exc. $\mathfrak{A}_c$.

3_b First use 3) to find a sequence of numbers δ_n such that if $\tilde{o}(B) \leqq \delta_n$, then $c(B) \leqq 1/2^n$. If $\{B_n\}$ is a sequence of sets such that $\tilde{o}(B_n) \to 0$, then $\{B_n\}$ contains a subsequence $\{B'_n\}$

such that $\tilde{\hat{\delta}}(B'_n) \leq \hat{\delta}_n$. Let $B = \limsup B'_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B'_n$. Then for every k, $c(B) \leq c\left(\bigcup_{n=k}^{\infty} B'_n\right) \leq \sum_{n=k}^{\infty} c(B'_n) \leq \sum_{n=k}^{\infty} 1/2^n \leq \frac{1}{2^{k-1}}$. Therefore $c(B) = 0$.

The following theorems are now immediate consequences of the theorem of section 5.

THEOREM I. — *Let c be an admissible capacity on $\mathfrak{L}_\sigma$.*

(*a*) *$\mathfrak{F}$ is a functional space rel. $\mathfrak{A}_c$ if and only if $\|f\| = 0$ whenever $f(x) = 0$ exc. $\mathfrak{A}_c$.*

(*b*) *$\mathfrak{F}$ has a functional completion rel. $\mathfrak{A}_c$ if and only if $\|f_n\| \to 0$ whenever $\{f_n\}$ is a Cauchy sequence which converges pointwise to 0 exc. $\mathfrak{A}_c$.*

THEOREM II. — *Let $\mathfrak{A}'$ be an exceptional class containing $\mathfrak{A}$.*

(*a*) *If $\mathfrak{F}$ is a functional space rel. $\mathfrak{A}'$, then for each admissible capacity c, $\mathfrak{F}$ is a functional space rel. $\mathfrak{A}' \cap \mathfrak{A}_c$.*

(*b*) *If $\mathfrak{F}$ has a functional completion rel. $\mathfrak{A}'$, then for each admissible capacity c, $\mathfrak{F}$ has a functional completion rel. $\mathfrak{A}' \cap \mathfrak{A}_c$.*

COROLLARY 1. — *If $\mathfrak{F}$ has a perfect functional completion, then its exceptional class $\mathfrak{A}'$ satisfies $\tilde{\mathfrak{L}}^0_\sigma \subset \mathfrak{A}' \subset \mathfrak{A}_\Omega$.*

REMARK 3. — It is possible to form φ-capacities with the aid of the function $\tilde{\hat{\delta}}$ as easily as with the aid of the function $\hat{\delta}$. However, proposition 2) implies that for any φ the φ-capacity formed with $\tilde{\hat{\delta}}$ is identical with the φ-capacity formed with $\hat{\delta}$. Similarly, if $\mathfrak{F}$ has a functional completion, it is possible to form capacities with the aid of the δ-function for the complete class (a function which we have called $\bar{\delta}$). Suppose that there is a completion rel. $\mathfrak{A}' \supset \mathfrak{A}$, and let φ be given. In view of Theorem II and in view of the fact that our interest centers on small exceptional classes rather than on large ones, we can suppose that $\mathfrak{A}' \subset \mathfrak{A}_{c_\varphi}$. Under these conditions it follows from the lemma and the remark 1 of the last section that the φ-capacity formed with δ is identical with the φ-capacity formed with $\tilde{\hat{\delta}}$ and therefore also with the φ-capacity formed with $\hat{\delta}$.

These observations have a bearing on the existence of functional completions, or more accurately, they make clear what part of the existence problem remains open. According to Theorem II the existence of a completion rel. $\mathfrak{A}'$ for any $\mathfrak{A}'$ implies the existence of a completion rel. $\mathfrak{A}_{c_\varphi} \cap \mathfrak{A}'$ for every φ-capacity, but it is not clear whether the existence of a completion rel. some $\mathfrak{A}_{c_\varphi}$ itself is implied. Therefore, the problem is this : is the existence of a completion equivalent to the existence of a completion rel. some $\mathfrak{A}_{c_\varphi}$? By the observations of the present remark the problem is reduced to the following: does there exist a complete functional space for which the whole basic set belongs to $\mathfrak{A}_\Omega$? (A negative response to the second question is equivalent to an affirmative response to the first.)

Remark 4. — Sometimes, when the basic set $\mathcal{E}$ is topological, it is important to know that there is a functional completion whose exceptional class has some topological property, that of being generated by its Borel sets, for example. Let $\mathfrak{R}$ denote the class of sets B of the following type : for some f in $\mathfrak{F}$ and some real $\alpha > 0$ and $\beta > 0$, $\mathrm{B} = \underset{x}{\mathrm{E}}\, [\alpha < \mathrm{Re}\, f(x) < \beta]$ exc. $\mathfrak{A}$. By the classical methods of the theory of Baire functions, one proves easily that the set where a sequence $\{f_n\}$ does not converge pointwise belongs to the class $\mathfrak{R}_{\sigma\delta\sigma}$. It follows that if $\mathfrak{F}$ has a functional completion rel. $\mathfrak{A}'$, it has also a functional completion rel. $(\mathfrak{A}' \cap \mathfrak{R}_{\sigma\delta\sigma})_h$. A bound slightly better than the one in the corollary is therefore $(\mathfrak{A}_\Omega \cap \mathfrak{R}_{\sigma\delta\sigma})_h$.

Theorems I and II together with the corollary of section 5 lead immediately to the following:

Corollary 2. — *If for some admissible capacity c, $\tilde{\mathfrak{L}}^0_\sigma = \mathfrak{A}_c$, then a functional completion of $\mathfrak{F}$ exists if and only if the condition of Theorem I — (b) relative to $\mathfrak{A}_c$ is satisfied. If the last condition is satisfied, then the completion relative to $\mathfrak{A}_c = \tilde{\mathfrak{L}}^0_\sigma$ is a perfect completion.*

The interest of this corollary lies in the fact that we can prove the equality $\tilde{\mathfrak{L}}^0_\sigma = \mathfrak{A}_c$ for a large category of functional classes, described by property (6. 3) below and for a wide class of φ-capacities c_φ. In a later paper it will be shown that all usual functional classes arising in application to differential problems satisfy property (6. 3).

(6. 3) POSITIVE MAJORATION PROPERTY. — *The basic set can be written as* $\mathcal{E} = \bigcup \mathcal{E}_n$ *and constants* M_n *can be chosen so that for every* $f \in \mathfrak{F}$ *and every* n *there exists a function* $f'_n \in \mathfrak{F}$ *such that* $\|f'_n\| \leqq M_n \|f\|$ *and* $\operatorname{Re} f'_n(x) \geqq |f(x)|$ *for* $x \in \mathcal{E}_n$ *exc.* $\mathfrak{A}$.

THEOREM III. — *If* $\mathfrak{F}$ *satisfies* (6. 3) *and the capacity* $c = c_\varphi$ *is formed with a function* φ *satisfying* $\limsup_{\rho \to 0} \rho/\varphi(\rho) < \infty$ *then* $\tilde{\mathfrak{L}}^0_\sigma = \mathfrak{A}_c$.

PROOF. — We have to prove that if $B \in \mathfrak{A}_c$ then $B \in \tilde{\mathfrak{L}}^0_\sigma$. Put $B^{(n)} = B \cap \mathcal{E}_n$, hence $B = \bigcup_{n=1}^{\infty} B^{(n)}$. Take positive constants α and C such that $\rho/\varphi(\rho) < C$ for $\varphi(\rho) < \alpha$. For every positive $\varepsilon < \alpha$ we can find a covering of $B^{(n)}$, $B^{(n)} \subset \bigcup_{k=1}^{\infty} B_k^{(n)}$, such that $\sum_{k=1}^{\infty} \varphi(\delta(B_k^{(n)})) < \varepsilon$; hence $\sum_{k=1}^{\infty} \delta(B_k^{(n)}) < C\varepsilon$. Take then functions $f_{n,k} \in \mathfrak{F}$ such that $\|f_{n,k}\| < \delta(B_k^{(n)}) + \frac{\varepsilon}{2^k}$ and $|f_{n,k}(x)| \geqq 1$ for $x \in B_k^{(n)}$ exc. $\mathfrak{A}$. By property (6. 3) we have a function $f'_{n,k}$ such that $\|f'_{n,k}\| \leqq M_n \|f_{n,k}\|$ and $\operatorname{Re} f'_{n,k}(x) \geqq |f_{n,k}(x)|$ for $x \in \mathcal{E}_n$ exc. $\mathfrak{A}$. It follows that the partial sums $\sum_{k=1}^{m} f'_{n,k}$ form a Cauchy sequence $\{g_m\}$ with the properties

$$\liminf \operatorname{Re} g_m(x) \geqq 1 \qquad \text{for} \qquad x \in B^{(n)} \subset \bigcup_{k=1}^{\infty} B_k^{(n)} \text{ exc. } \mathfrak{A},$$

$$\|g_m\| \leqq \sum_{k=1}^{m} \|f'_{n,k}\| \leqq M_n \sum_{k=1}^{\infty} \|f_{n,k}\| \leqq M_n \sum_{k=1}^{\infty} \left[\delta(B_k^{(n)}) + \frac{\varepsilon}{2^k} \right] \leqq M_n (C+1)\varepsilon.$$

Hence $\tilde{\delta}(B^{(n)}) \leqq M_n (C+1)\varepsilon$ for all $\varepsilon < \alpha$ and thus $B^{(n)} \in \tilde{\mathfrak{L}}^0$ and $B = \bigcup_{n=1}^{\infty} B^{(n)} \in \tilde{\mathfrak{L}}^0_\sigma$.

REMARK 5. — For particular classes $\mathfrak{F}$ with property (6. 3) Theorem III may be true for larger classes of φ-capacities. In all investigated cases where the norm in $\mathfrak{F}$ was quadratic (i.e. $\mathfrak{F}$ an incomplete Hilbert space) it turned out that $\tilde{\mathfrak{L}}^0_\sigma = \mathfrak{A}_c$ with $c = c_\varphi$, $\varphi(\rho) = \rho^2$. It would be interesting to know if this is true for all functional classes with quadratic norm.

REMARK 6. — Often a strengthened version of (6. 3) holds, namely:

(6. 4) GLOBAL MAJORATION PROPERTY. — *There is a constant* M *so that for every function* f *in* $\mathfrak{F}$ *there exists a function* f' *in* $\mathfrak{F}$ *such that* $\operatorname{Re} f'(x) \geqq |f(x)|$ exc. $\mathfrak{A}$ *and* $\|f'\| \leqq M \|f\|$.

It is easy to see that if (6. 4) does hold, then $B \in \tilde{\mathfrak{L}}$ whenever $c_1(B) < \infty$, and for such B, $c_1(B) \geqq \frac{1}{M} \tilde{\delta}(B)$. The spaces L^p and the spaces of M. Riesz potentials form important examples in which $M = 1$ is a satisfactory constant. The case $M = 1$ is worth special notice. The property (6. 4) becomes then.

(6. 5) STRONG MAJORATION PROPERTY. — *For every function* f *in* $\mathfrak{F}$ *there exists a function* f' *in* $\mathfrak{F}$ *such that* $\operatorname{Re} f'(x) \geqq |f(x)|$ *exc.* $\mathfrak{A}$ *and* $\|f'\| \leqq \|f\|$.

If this property holds, $c_1(B) = \tilde{\delta}(B)$ (provided $c_1(B) < \infty$), as it is always true that $c_1(B) \leqq \tilde{\delta}(B)$.

§ 7. **Proper functional completion.** — The complete functional spaces occuring in analysis arise most often as functional completions of more elementary functional classes which consist of functions defined everywhere and which are in fact proper normed functional classes. This is not to say, however, that the complete space is proper, for in the process of completion it usually happens that some sets become exceptional. We show in this section that non-empty sets cannot become exceptional if the initial proper functional class is a proper functional space. In discussing proper functional completion we will use the notations: for each x in $\mathcal{E}$, M_x is the bound of the continuous linear functional $f(x)$, and c_x is the set function which takes the value 1 on any set containing x and 0 on any other ets.

1) *If* $\mathfrak{F}$ *is a proper functional space, then for each* x *in* $\mathcal{E}$ *and each set* B *in* $\mathfrak{L}_\sigma$, $c_x(B) \leqq M_x c_1(B)$ (*where, as always,* c_1 *is the* φ*-capacity defined by* $\varphi(t) = t$). *In particualr,* $\mathfrak{A}_{c_1} = (0)$.

Proof. — Since the left side of the inequality is 0 whenever B does not contain x, it is possible to assume that $\{x\}$ belongs to $\mathfrak{L}_\sigma$ and therefore to $\mathfrak{L}$, and that $x \in \mathrm{B}$. Given $\varepsilon > 0$, let $f \in \mathcal{F}$ be such that $|f(x)| \geq 1$ and $\|f\| \leq \delta(\{x\}) + \varepsilon$. Then

$$c_x(\mathrm{B}) = 1 \leq |f(x)| \leq \mathrm{M}_x\|f\| \leq \mathrm{M}_x[\delta(\{x\}) + \varepsilon]$$
$$= \mathrm{M}_x[c_1(\{x\}) + \varepsilon] \leq \mathrm{M}_x[c_1(\mathrm{B}) + \varepsilon].$$

Theorem I. — *If a proper functional space $\mathcal{F}$ has any functional completion, then it has a proper functional completion. A necessary and sufficient condition that a proper functional space $\mathcal{F}$ have a functional completion is that $\|f_n\| \to 0$ whenever $\{f_n\}$ is a Cauchy sequence in $\mathcal{F}$ converging to 0 at every point.*

Proof. — Theorems I and II, Section 6.

Remark. — There are simple examples of proper functional spaces which do not have a functional completion, but they are somewhat artificial. Indeed, non-existence of a functional completion of a proper functional space can be ascribed to an awkward choice either of the basic set or of the norm in the functional class. It is always possible to redetermine either of the two in such a way as to obtain a proper functional space with a completion.

In order to modify the basic set $\mathcal{E}$ we consider the abstract completion $\tilde{\mathrm{V}}$ of the normed vector space $\mathcal{F}$. Each point $x \in \mathcal{E}$ corresponds to a unique continuous linear functional X on $\tilde{\mathrm{V}}$; X is defined by the equation $\mathrm{X}(f) = f(x)$ for all $f \in \mathcal{F}$. We will think of $\mathcal{E}$ as a subset of the set $\tilde{\mathrm{V}}^*$ of all continuous linear functionals on $\tilde{\mathrm{V}}$, and in order to use notation in harmony with the notation for $\mathcal{F}$ we will write $\varphi(\mathrm{X})$ for $\mathrm{X}(\varphi)$ whenever $\varphi \in \tilde{\mathrm{V}}$ and $\mathrm{X} \in \tilde{\mathrm{V}}^*$. Then each φ in $\tilde{\mathrm{V}}$ is a function defined not only on $\mathcal{E}$ but on all of $\tilde{\mathrm{V}}^*$. In this notation the condition stated in the theorem is that $\varphi = 0$ whenever $\varphi(\mathrm{X}) = 0$ for all $x \in \mathcal{E}$. When the original basic set $\mathcal{E}$ does not have this property, additional X from $\tilde{\mathrm{V}}^*$ can be added to it so as to obtain a new basic set $\mathcal{E}'$, which does. The functions $\varphi(\mathrm{X})$ for $\varphi \in \tilde{\mathrm{V}}$ restricted to the new basic set $\mathcal{E}'$ form a complete proper functional space which can be called a « quasi-completion » of $\mathcal{F}$.

A similar process can be carried through when $\mathcal{F}$ is a functional space rel. $\mathfrak{A}_c$ for some φ-capacity $c = c_\varphi$. In this case $\mathfrak{F}$ is not a normed vector space in the proper sense of the term, but $\tilde{V}$ can be taken as the abstract completion of the normed vector space V which corresponds to $\mathfrak{F}$ through the equivalence relation $f \equiv g$ if $f = g$ exc. $\mathfrak{A}_c$. There is no way to think of $\mathcal{E}$ as a subset of $\tilde{V}^*$; nevertheless a suitable new basic set $\mathcal{E}'$ can be obtained in the form $\mathcal{E}' = \mathcal{E} \cup \mathcal{E}^*$, where $\mathcal{E}^*$ is any total subset of $\tilde{V}^*$. The manner of defining the $f \in \mathfrak{F}$ as functions on $\mathcal{E}'$ is obvious. Let c' be the φ-capacity in $\mathcal{E}'$ corresponding to the same function φ. It is not difficult to provide an argument similar to the argument following proposition 1 to show that $\mathfrak{A}_c = \mathfrak{A}_{c'}$. Nor is it difficult to use Theorem I, section 6, to show that $\mathfrak{F}$ as a functional space on the basic set $\mathcal{E}'$ has a functional completion rel. $\mathfrak{A}_c$. Such a completion can be called a « quasi-completion » rel. $\mathfrak{A}_c$. It would seem that there is considerable arbitrariness involved in the selection of $\mathcal{E}^*$. Oftentimes natural choices present themselves, however, and the idea is important in connection with measurable spaces and pseudo-reproducing kernels, subjects which will be discussed in a separate paper.

The procedure for modifying the norm in the functional class, on the other hand, is unique. We will suppose that $\mathfrak{F}$ is a proper functional space, and we will use the same notations as before for the abstract completion, the linear functionals, etc. Let $\tilde{V}_0$ be the set of all $\nu \in \tilde{V}$ such that $\nu(x) = 0$ for every $x \in \mathcal{E}$. Being the intersection of closed subspaces of $\tilde{V}$, $\tilde{V}_0$ is itself a closed subspace. Therefore the quotient space $\tilde{V}/\tilde{V}_0$ is a complete normed vector space when the norm of a quotient class C is defined by the usual formula $\|C\| = \inf \|\nu\|$ taken over all $\nu \in C$. For every $x \in \mathcal{E}$, $\nu(x)$ is constant over each quotient class C in $\tilde{V}/\tilde{V}_0$, so that every such x determines a unique linear functional, which we continue to call x on $\tilde{V}/\tilde{V}_0$. Each $x \in \mathcal{E}$ is continuous on $\tilde{V}/\tilde{V}_0$, for if M is the bound of x as a linear functional on $\tilde{V}$, and if C and $\varepsilon > 0$ are given, then there is a $\nu \in C$ such that

$$|x(C)| = |\nu(x)| \leq M\|\nu\| \leq M[\|C\| + \varepsilon];$$

and also $\|C\| = 0$ if $x(C) = 0$ for every $x \in \mathcal{E}$. If we write, as before, $C(x)$ instead of $x(C)$, then $\tilde{V}/\tilde{V}_0$ appears plainly as a complete proper functional space over the basic set $\mathcal{E}$; and it contains $\mathfrak{F}$. Therefore, if $\mathfrak{F}$ is re-normed with the norm of $\tilde{V}/\tilde{V}_0$, then, as a subspace of a complete proper functional space, it has a proper functional completion. It is clear from the definition of the norm in $\tilde{V}/\tilde{V}_0$ that the new norm of a function $f \in \mathfrak{F}$ is less than or equal to its original norm.

By a somewhat more complicated argument it is possible to prove a similar result for functional spaces rel. $\mathfrak{A}_c$. We state the result but omit the proof.

2) *Let $\mathfrak{F}$ be a normed functional class rel. $\mathfrak{A}$, and let c be an admissible capacity on $\mathfrak{L}_\sigma$. If $\mathfrak{F}$ is a functional space rel. $\mathfrak{A}_c$, then it is possible to define another norm, $\|f\|'$, on $\mathfrak{F}$ so that:* (i) $\|f\|' \leq \|f\|$; (ii) *$\mathfrak{F}$ with the norm $\|f\|'$ is a functional space rel. $\mathfrak{A}_c$ and has a functional completion rel. $\mathfrak{A}_c$.*

CHAPTER II

EXAMPLES

§ 8. Example 1. **Analytic functions.** — We take as the basic set $\mathcal{E}$ the closed unit circle in the complex plane, and we consider the class $\mathcal{F}$ of complex-valued functions continuous in the whole of $\mathcal{E}$ and analytic in its interior. We define the norm in $\mathcal{F}$ by the formula $\|f\| = \left\{\int_{\mathcal{E}} |f(x)|^2 dx\right\}^{1/2}$. $\mathcal{F}$ is a proper normed functional class. [13].

Let $\partial\mathcal{E}$ denote the boundary of $\mathcal{E}$. Each of the functions $f_n(x) = x^n$ is 1 in absolute value everywhere on $\partial\mathcal{E}$. Therefore as $\|f_n\| = \sqrt{\pi/(n+1)} \to 0$, $\delta(\partial\mathcal{E}) = 0$, and $\partial\mathcal{E} \in \mathfrak{L}^0$. By the corollary at the end of section 5, any exceptional class relative to which $\mathcal{F}$ is a functional space must contain $\partial\mathcal{E}$. In particular, $\mathcal{F}$ is not a proper functional space.

With respect to the points in the interior of $\mathcal{E}$, though, $\mathcal{F}$ acts as a proper functional space. Each of these points determines a continuous linear functional. Consider the φ-capacity c_1 (determined by $\varphi(t) = t$). Proposition 1, section 7, shows that if x is not a boundary point, then $c_1(\{x\}) \geq \dfrac{1}{M_x}$, where M_x is the bound of the linear functional determined by x [14]. From this and the last paragraph we deduce that

[13] This space is the simplest case of spaces considered extensively by S. Bergman (see [9]). Its completion, restricted to the open circle has a reproducing kernel, Bergman's kernel function,

$$K(x, y) = \frac{1}{\pi(1 - x\bar{y})^2} \qquad \text{(see [1], [4]).}$$

[14] The exact value of M_x is given by the reproducing kernel:

$$M_x = \sqrt{K(x, x)} = \frac{1}{\sqrt{\pi}(1 - x\bar{x})}.$$

$\mathfrak{A}_{c_1} = \mathfrak{L}^0 = \tilde{\mathfrak{L}}^0_\sigma =$ the class of all subsets of $\delta\mathcal{E}$. There is no difficulty in seeing from Theorem I, section 6, that there is a completion, necessarily perfect, relative to this class.

§ 9. Example 2. L^p spaces. — The Lebesgue spaces L^p are so thoroughly familiar now that the theory of functional completion cannot be expected to provide essentially new information about them. By reason of their familiarity, however, they provide an example which illustrates well the concepts which have been introduced here, especially the capacities. Reciprocally, by focusing attention at an unusual point, the theory of functional completion underscores an interesting peculiarity of L^p.

We are concerned here with obtaining L^p as a functional completion of a subspace composed of elementary functions. In the case of a measure on an abstract set there are no new problems coming specifically from the functional completion point of view. The natural choice for the space of elementary functions is the space of linear combinations of characteristic functions of measurable sets of finite measure. The passage from this space to its completion with respect to the L^p norm is completely standard. The proof that the perfect completion is the usual L^p requires nothing (beyond the definition of « perfect ») from the theory of functional completion.

The situation is different in the case of a measure on a topological space. Here the natural choice for the space of elementary functions is usually a space of continuous functions. Since the continuous functions are defined everywhere (not almost everywhere), it is not at all evident that the sets of measure 0 form the exceptional class for the perfect completion. Indeed this is not true in general, as we shall show in the succeeding paragraphs.

We take as the basic set $\mathcal{E}$ an arbitrary locally compact Hausdorff space. The two σ-rings used in topological-measure theoretic investigations in locally compact spaces are the Borel σ-ring, which is generated by the compact sets, and the Baire σ-ring, which is generated by thec ompact G_δ's [15]. In all ordinary topological spaces, for example separable spaces

[15] We use the terminology of Halmos [18].

or metric spaces, these two σ-rings are identical, but in general they are distinct. The measures usually considered are regular Borel measures, those defined on the Borel sets and having the additional properties: (i) the measure of each compact set is finite; (ii) (regularity) the measure of each Borel set is the infimum of the measures of the open Borel sets containing it [16]. We suppose given such a measure on $\mathcal{E}$, and we call it μ. We denote by C the class of continuous real valued functions on $\mathcal{E}$ which vanish outside a compact set. For each real number $p \geq 1$ we define $||f||_p = \left\{\int |f(x)|^p d\mu\right\}^{1/p}$, and we denote by C_p the class C with this expression as pseudonorm. It is well known that relative to the exceptional class $\mathfrak{A}_\mu$ of subsets of Borel sets of μ-measure 0, C_p is a functional space, and that it posesses a functional completion, $L^p(\mu)$, relative to $\mathfrak{A}_\mu$. We shall illustrate some of the general theorems of this paper by re-proving the existence of L^p, by finding the perfect completion, and by computing some of the capacities.

Because the capacities themselves are outer measures, and because the classes with which they are associated, the $\mathfrak{A}_c$, $\mathfrak{K}^0_\sigma$, etc., are all hereditary classes, there is some advantage in extending the measure μ so that it is an outer measure too, defined on the hereditary σ-ring $\mathfrak{K}_\sigma$ on which all capacities are defined.

C_p is a proper normed functional class if and only if there is no open Baire set in $\mathcal{E}$ of measure 0. In general the smallest exceptional class relative to which C_p is a normed functional class is the class of subsets of open Baire sets of mesure 0. We call this exceptional class $\mathfrak{A}$, and we consider C_p as a normed functional class rel. $\mathfrak{A}$. In this case $\mathfrak{K}$ is the class of sets which are contained exc. $\mathfrak{A}$ in a compact set, (or, equivalently, in a compact Baire set) and $\mathfrak{K}_\sigma$ is the hereditary σ-ring generated by the compact sets (or, compact Baire sets.) We effect the extension of μ to $\mathfrak{K}_\sigma$ by the standard device of setting $\mu(B')$ equal to the infimum of the numbers $\mu(B)$ taken over all Borel sets B containing B'.

If $||f_n||_p \to 0$, then $f_n \to 0$ in measure, and each sequence

([16]) For example, BOURBAKI considers only measures of this type in its presentation of integration theory [10].

which converges to 0 in measure contains a subsequence which converges to 0 almost everywhere. Therefore C_p is a functional space rel. $\mathfrak{A}_\mu$, and we can apply the corollary at the end of section 5 to conclude that $\mathfrak{L}^0_\sigma \subset \mathfrak{A}_\mu$.

Let K be any compact set in $\mathcal{E}$. Then $K \in \mathfrak{L}$, as we have mentioned above. We prove now that $\delta(K)^p = \mu(K)$. First let f be any function in C_p which is ≥ 1 on K exc. $\mathfrak{A}$. Then $\int |f(x)|^p d\mu \geq \mu(K)$. As $\delta(K)^p$ is the infimum of the numbers on the left side, $\delta(K)^p \geq \mu(K)$. On the other hand, it is possible to find a function $f \in C_p$ which is ≥ 1 on K and which is such that $\mu(K) \geq \int |f(x)|^p d\mu - \varepsilon$ for arbitrarily small $\varepsilon > 0$. Therefore $\delta(K)^p \leq \mu(K)$, and so $\delta(K)^p = \mu(K)$.

Suppose that B is any set in $\mathfrak{L}$. It is easy to see that there is a decreasing sequence of non-negative functions $f_n \in C_p$ such that $f_n \geq 1$ on B exc. $\mathfrak{A}$ and such that $\|f_n\|_p \to \delta(B)$. Let $K_n = \mathop{\mathsf{E}} [f_n(x) \geq 1]$ and let $K = \bigcap_{n=1}^{\infty} K_n$. Then $K \supset B$ exc. $\mathfrak{A}$, and we can write

$$\delta(B)^p = \lim \|f_n\|_p^p \geq \lim \mu(K_n) = \mu(K) = \delta(K)^p \geq \delta(B)^p,$$

so equality holds throughout. We have proved the following statement.

1) *For each set* $B \in \mathfrak{L}$ *there is a compact* G_δ, K, *such that* $K \supset B$ *exc.* $\mathfrak{A}$ *and such that* $\delta(B)^p = \delta(K)^p = \mu(K) \geq \mu(B)$.

From this it follows that μ is an admissible capacity on $\mathfrak{L}_\sigma$. We can use Theorem I section 6, to prove that there is a functional completion rel. $\mathfrak{A}_\mu$. Suppose that $\{f_n\}$ is a Cauchy sequence which converges pointwise to 0 exc. $\mathfrak{A}_\mu$. By Fatou's lemma, $\int |f_n(x)|^p d\mu \leq \liminf_{m \to \infty} \int |f_n(x) - f_m(x)|^p d\mu$, and the right side can be made arbitrarily small by a suitable choice of n. Therefore $\|f_n\|_p \to 0$ and this is the condition of the theorem.

Now we can employ Theorem III of section 6 to obtain the existence of a perfect completion, for C_p has the strong majoration property: a majorant for an arbitrary function $f(x)$ is the function $|f(x)|$. We obtain also from Theorem III the exceptional class for the perfect completion, the class $\tilde{\mathfrak{L}}^0_\sigma$. As we have mentioned, the completion rel. $\mathfrak{A}_\mu$ is not necessarily perfect; that is, it is not necessarily true that $\mathfrak{A}_\mu = \tilde{\mathfrak{L}}^0_\sigma$. According to Remark 4, section 6, the existence of a completion rel. $\mathfrak{A}_\mu$ implies the existence of a completion rel. $(\mathfrak{A}_\mu \cap \mathfrak{R}_{\sigma\delta\sigma})_h$. The latter class does give the perfect completion and may be smaller than $\mathfrak{A}_\mu$ itself, as we shall see. In this example the class $\mathfrak{R}$, which is composed in general of all sets of the type $\underset{x}{\mathrm{E}}[\alpha < \mathrm{Re} f(x) < \beta \text{ exc. } \mathfrak{A}]$ for $\alpha > 0$, $\beta > 0$, is the class of sets equal exc. $\mathfrak{A}$ to a bounded open Baire set ([17]).

It is possible to identify the perfect completion itself, as well as its exceptional sets. The standard device which we used to extend the original Borel measure μ to an outer measure serves to extend any measure defined on a σ-ring to an outer measure defined on the class of all subsets of sets in the σ-ring. Let μ_0 denote first the restriction of μ to the σ-ring of Baire sets, then its own extension by this scheme to an outer measure on $\mathfrak{L}_\sigma$. In general the outer measures μ and μ_0 are different([18]).

By the general theory of Baire measures it can be proved without any difficulty that $\mathfrak{A}_{\mu_0} = (\mathfrak{A}_\mu \cap \mathfrak{R}_{\sigma\delta\sigma})_h$; and most of the rest of this section will be given to proving that $\tilde{\mathfrak{L}}^0_\sigma = \mathfrak{A}_{\mu_0}$, that the perfect completion of C_p is $\mathrm{L}^p(\mu_0)$, and that for any set $\mathrm{B} \in \mathfrak{L}_\sigma$, $\mu_0(\mathrm{B}) = c_p(\mathrm{B})$, where c_p is the φ-capacity defined by the function $\varphi(t) = t^p$.

Let B be an arbitrary set in $\mathfrak{L}_\sigma$, and let $\{\mathrm{B}_n\}$ be a sequence of sets in $\mathfrak{L}$ such that $\mathrm{B} \subset \bigcup_{n=1}^{\infty} \mathrm{B}_n$ exc. $\mathfrak{A}$ and such that

$$c_p(\mathrm{B}) \geqq \sum_{n=1}^{\infty} \delta(\mathrm{B}_n)^p - \varepsilon.$$

By 1) there is a sequence $\{\mathrm{K}_n\}$ of compact G_δ's such that for each n, $\mathrm{K}_n \supset \mathrm{B}_n$ exc. $\mathfrak{A}$ and $\delta(\mathrm{B}_n)^p = \delta(\mathrm{K}_n)^p = \mu(\mathrm{K}_n) = \mu_0(\mathrm{K}_n)$. It follows easily that $c_p(\mathrm{B}) \geqq \mu_0(\mathrm{B})$.

([17]) A set is bounded if it is contained in some compact set. Hence every bounded set is in $\mathfrak{L}$.

([18]) It is obvious from the construction that $\mu(\mathrm{B}) = \mu_0(\mathrm{B})$ for all Baire sets and for all compacts sets, that $\mu(\mathrm{B}) \leqq \mu_0(\mathrm{B})$ for all sets, and that the class $\mathfrak{A}_{\mu_0}$ is exactly the class of subsets of Baire sets of μ-measure 0. All compact sets are measurable with respect to the outer measure μ, but this is not necessarily true of μ_0 (see Halmos [18]).

Next we establish the opposite inequality, and in addition we prove that $\mu_0(B) \geqq \tilde{\delta}(B)^p$ for bounded sets B. From the latter will follow the relation $\mathfrak{A}_{\mu_0} \subset \tilde{\mathfrak{L}}^0_\sigma$.

Let G be an arbitrary bounded open Baire set and let $G = \bigcup_{n=1}^{\infty} K_n$ be a representation of G as a union of increasing compact sets ([19]). For each n choose a function f_n in C_p with the properties:

(i) $f_n(x) = 1$ if $x \in K_n$;
(ii) $f_n(x) = 0$ if $x \notin G$;
(iii) $0 \leqq f_n(x) \leqq 1$.

The sequence $\{f_n\}$ is a Cauchy sequence since it converges pointwise and is dominated by the characteristic function of G which is integrable. As $\lim f_n(x) = 1$ for every $x \in G$, $\tilde{o}(G) \leqq \lim \|f_n\|_p \leqq \mu(G)^{1/p}$. Thus $\mu_0(G) \geqq \mu(G) \geqq \tilde{\delta}(G)^p \geqq c_p(G)$ ([20]). By a passage to the limit in which the regularity of μ_0 is used we get $\mu_0(B) \geqq \tilde{\delta}(B)^p \geqq c_p(B)$ for every bounded set B. In the light of this fact the relation $\mathfrak{A}_{\mu_0} \subset \tilde{\mathfrak{L}}^0_\sigma$ is obvious. Furthermore, an arbitrary Baire set B can be written as a disjoint union of bounded Baire sets B_n. Therefore

$$c_p(B) \leqq \sum_{n=1}^{\infty} c_p(B_n) \leqq \sum_{n=1}^{\infty} \mu_0(B_n) = \mu_0(B).$$

Combining this with the inequality $c_p(B) \geqq \mu_0(B)$ already proved, and with the regularity of μ_0 we obtain finally $c_p(B) = \mu_0(B)$ for any set $B \in \mathfrak{L}_\sigma$. The one remaining assertion, that $L^p(\mu_0)$ is the perfect completion of C_p, is now clear.

It can be proved that the capacity c_p is identical with $(c_1)^p$. In fact, we know from the strong majoration property that $B \in \tilde{\mathfrak{L}}$ if and only if $c_1(B) < \infty$; and if $B \in \tilde{\mathfrak{L}}$, then $c_1(B) = \tilde{\delta}(B)$. Using the lemma in section 5 it is easy to show that because the strong majoration property is present $\tilde{\delta}$ is identical with $\bar{o}$,

([19]) It is known that every open Baire set is a countable union of compact sets, and that conversely every open set which is a countable union of compact sets is a Baire set. See Halmos [18].

([20]) For any set $B \in \tilde{\mathfrak{L}}$, $\tilde{\delta}(B)^p \geqq c_p(B)$. This general property of capacities is a simple consequence of Remark 3, section 6.

the δ-function for the complete space; and $\tilde{\mathfrak{L}}$ is identical with $\overline{\mathfrak{L}}$. In this example the composition of $\overline{\mathfrak{L}}$ is evident. It is the class of all sets of finite μ_0-measure. The function $\bar{\delta}$ is easy to calculate too: if $B \in \overline{\mathfrak{L}}$, then $\bar{\delta}(B)^p = \mu_0(B)$. Thus, if $c_1(B)$ is finite, then $B \in \tilde{\mathfrak{L}} = \overline{\mathfrak{L}}$ and $c_1(B)^p = \tilde{\delta}(B)^p = \bar{\delta}(B)^p = \mu_0(B)$; while if $c_1(B) = +\infty$ then $B \notin \tilde{\mathfrak{L}} = \overline{\mathfrak{L}}$ and $\mu_0(B) = +\infty$.

The following theorem gives a summary of the main points of interest in this example.

Theorem I. — *The space $L^p(\mu_0)$ is the perfect functional completion of the space C_p. The exceptional sets are the sets of μ_0-measure 0; equivalently they are the subsets of Baire sets of μ-measure 0. The class $\mathfrak{L}_\sigma$ is the hereditary σ-ring generated by the compact sets. $(c_1)^p$, c_p, and μ_0 are identical outer measures on $\mathfrak{L}_\sigma$. $\tilde{\mathfrak{L}} = \overline{\mathfrak{L}}$ is the class of sets of finite μ_0-measure. On this class $\tilde{\delta}^p = \bar{\delta}^p = \mu_0$.*

Remark 1. — We have stated in an earlier part of the paper that the classes $\tilde{\mathfrak{L}}_\sigma^0$ and $\mathfrak{L}_\sigma^0$ are different in general. For an example take the space C_p with $\mathcal{E}$ the interval $0 \leqq x \leqq 1$ and μ Lebesgue measure. $\tilde{\mathfrak{L}}_\sigma^0$ is the class of all sets of Lebesgue measure 0; $\mathfrak{L}_\sigma^0$ is the class of all subsets of sets F_σ of Lebesgue measure 0. As each subset of an F_σ of measure 0 is of first category, and as there are sets of measure 0 which are not of first category, the example is established.

§ 10. Example 3. Some spaces of harmonic functions and Fatou's theorem. — In the first part of this section the basic set $\mathcal{E}$ is the closed sphere with center 0 and radius R in n-dimensional space E_n; $\partial\mathcal{E}$ is its boundary; θ, φ etc., refer to points on the boundary; $d\theta$, $d\varphi$, etc., to the n-1-dimensional measure on the boundary. ω_n is the area of the surface of the unit sphere in E_n. $h(\theta, x)$ is the Poisson kernel for $\mathcal{E}$:

$$h(\theta, x) = \frac{R^2 - |x|^2}{\omega_n R\,|\theta - x|^n}.$$

The functional class which is to be considered is the class $\mathcal{F}$

of all complex-valued functions continuous in $\mathcal{E}$ and harmonic in the interior of $\mathcal{E}$. The norm in $\mathcal{F}$ is defined by

$$\|f\|_p = \left\{ \int_{\partial\mathcal{E}} |f(\theta)|^p \, d\theta \right\}^{1/p},$$

where p is fixed and satisfies $1 \leq p < \infty$ [21].

The object of the section is to show how the well known theorem of Fatou on the boundary values of harmonic functions can be proved by means of capacities. In the course of the development of capacities it was shown that each convergent sequence in a functional space contains a subsequence which converges pointwise uniformly outside a set of arbitrarily small capacity. Thus each function in the completion of a space composed of continuous functions is continuous outside a set of arbitrarily small capacity. Once the sets of small capacity are identified in the example at hand, it becomes clear that each function in the completion has non-tangential boundary values almost everywhere.

If $f(\theta)$ is a continuous function defined on $\partial\mathcal{E}$, then the Poisson formula $f(x) = \int_{\partial\mathcal{E}} h(\theta, x) f(\theta) \, d\theta$ defines a function $f(x)$ in the interior of $\mathcal{E}$. $f(x)$ is harmonic there, and if it is extended to the boundary by assigning $f(\theta)$ as boundary values, the resulting function f is continuous throughout $\mathcal{E}$. Thus the class $\mathcal{F}$ is exactly the class of functions $f(x)$ obtained as follows: f is determined by a unique continuous function $f(\theta)$ by the equations $f(x) = \int_{\partial\mathcal{E}} h(\theta, x) f(\theta) \, d\theta$ if $x \in$ interior of $\mathcal{E}$; $f(x) = f(\theta)$ if $x = \theta$.

Consider the class $\overline{\mathcal{F}}$ of functions determined in the same way by functions $f(\theta)$ defined almost everywhere and in L^p on $\partial\mathcal{E}$. With the norm $\|f\|_p = \left\{ \int_{\partial\mathcal{E}} |f(\theta)|^p \, d\theta \right\}^{1/p}$ $\overline{\mathcal{F}}$ is a complete functional space relative to the exceptional class $\mathfrak{A}$ of subsets of $\partial\mathcal{E}$ of n-1-dimensional measure 0. It is clear that $\mathcal{F}$ is contained in $\overline{\mathcal{F}}$ and that $\mathcal{F}$ is dense in $\overline{\mathcal{F}}$. Thus $\overline{\mathcal{F}}$ is a functional completion of $\mathcal{F}$.

We can use Theorem III of section 6 on positive majorants to prove the existence of a perfect completion. For each

[21] The case $p = \infty$ has no interest here, for $\|f\|_\infty = \sup_{\theta \in \partial\mathcal{E}} |f(\theta)| = \sup_{x \in \mathcal{E}} |f(x)|$, and $\mathcal{F}$ is already a complete proper functional space.

function $f \in \mathfrak{F}$ the function $f^+(x) = \int_{\partial\mathcal{E}} h(\theta, x)|f(\theta)|\,d\theta$ belongs to $\mathfrak{F}$ and is a positive majorant for f. In addition, $||f||_p = ||f^+||_p$. Therefore, by the theorem quoted, $\mathfrak{A}_{c_1} = \tilde{\mathfrak{L}}^0_\sigma$, and since it is established that there is some functional completion, there is a completion, necessarily perfect, relative to $\mathfrak{A}_{c_1} = \tilde{\mathfrak{L}}^0_\sigma$. It is easy to show that $\tilde{\mathfrak{L}}^0_\sigma = \mathfrak{A}$: if $A \subset \partial\mathcal{E}$ is a set of n-1-dimensional measure 0, then there is a sequence $\{f_n(\theta)\}$ of continuous functions on $\partial\mathcal{E}$ such that $\int_{\partial\mathcal{E}} |f_n(\theta)|^p\,d\theta \to 0$ and $f_n(\theta) \to \infty$ for each $\theta \in A$; the sequence $f_n(x) = \int_{\partial\mathcal{E}} h(\theta, x)f_n(\theta)\,d\theta$ is such that $||f_n||_p \to 0$ while $f_n(\theta) \to \infty$ for each $\theta \in A$, so $A \in \tilde{\mathfrak{L}}^0_\sigma$, and $\tilde{\mathfrak{L}}^0_\sigma \supset \mathfrak{A}$. The opposite inclusion is trivial. We can conclude that $\bar{\mathfrak{F}}$ is the perfect completion of $\mathfrak{F}$.

The connections between the values on $\partial\mathcal{E}$ of a function $f \in \bar{\mathfrak{F}}$ and the values in the interior of $\mathcal{E}$ form the subject of Fatou's theorem. Actually the assertion of Fatou is that $f(x) \to f(\theta)$ pointwise a.e. under suitable conditions. For the sake of completeness, we proceed to show first the well known fact that a certain convergence in mean takes place. Define $f_r(\varphi) = T_r f(\varphi) = \int_{\partial\mathcal{E}} h(\theta, r\varphi)f(\theta)\,d\theta = f(r\varphi)$ for each function $f(\theta)$ belonging to L^p on $\partial\mathcal{E}$, and each $r < 1$. The mean convergence which takes place is that $\lim_{r\to 1} ||f - f_r||_p = 0$.

The proof is a classical one which we will reproduce only in brief. Since 1 is harmonic, the Poisson formula gives $\int_{\partial\mathcal{E}} h(\theta, x)\,d\theta = 1$. Since $h(\theta, x)$ is a harmonic function of x, the mean value theorem gives $\int_{\partial\mathcal{E}} h(\theta, r\varphi)\,d\varphi = 1$. These two facts in conjunction with Holder's inequality give

$$\begin{aligned}\int_{\partial\mathcal{E}} |f_r(\varphi)|^p\,d\varphi &= \int_{\partial\mathcal{E}} \left| \int_{\partial\mathcal{E}} h(\theta, r\varphi)f(\theta)\,d\theta \right|^p d\varphi \\ &\leq \int_{\partial\mathcal{E}} \int_{\partial\mathcal{E}} h(\theta, r\varphi)|f(\theta)|^p\,d\theta\,d\varphi = \int_{\partial\mathcal{E}} |f(\theta)|^p\,d\theta,\end{aligned}$$

from which it follows that the transformations T_r are a uniformly bounded family of linear transformations from L^p on $\partial\mathcal{E}$ to L^p on $\partial\mathcal{E}$. In order to show that $T_r f \to f$ in L^p for each $f \in L^p$ it is enough to show that this happens on a dense set of f. For the dense set take the continuous functions.

The functions in the complete class $\overline{\mathfrak{F}}$ can be characterized in another way. Suppose that f is a function defined only in the interior of $\mathcal{E}$. According to the preceding paragraph there is at most one function in $\overline{\mathfrak{F}}$ which coincides with f in the interior of $\mathcal{E}$ (at most one function up to equivalence in $\overline{\mathfrak{F}}$, that is). It is therefore clear how the phrase « f belongs to $\overline{\mathfrak{F}}$ » should be interpreted when f is defined only in the interior of $\mathcal{E}$.

1) *If $p > 1$, then $\overline{\mathfrak{F}}$ consists of all harmonic functions f defined in the interior of $\mathcal{E}$ and having the property that*

$$\sup_{0<r<1} \int_{\partial\mathcal{E}} |f(r\theta)|^p \, d\theta < \infty.$$

2) *If $p \geq 1$, then $\overline{\mathfrak{F}}$ consists of all harmonic functions f defined in the interior of $\mathcal{E}$ and having the property that there exists a sequence $r_n \nearrow 1$ such that the functions $f_{r_n}(\theta) = f(r_n\theta)$ converge weakly in L^p on $\partial\mathcal{E}$.*

Proof. — If an f satisfies the condition in 1), then it satisfies the condition in 2), for bounded sets are weakly compact in L^p, $p > 1$.

Suppose that the condition in 2) is satisfied for a certain function f and some sequence $r_n \nearrow 1$, and let g be the weak limit of f_{r_n}. Since $g \in \mathrm{L}^p$ on $\partial\mathcal{E}$, when we have shown that $f(x) = \int_{\partial\mathcal{E}} h(\theta, x)\, g(\theta)\, d\theta$ it will follow that $f \in \overline{\mathfrak{F}}$. Let x be a fixed point in the interior of $\mathcal{E}$. Then $h(\theta, x)$ is a continuous function of θ. Thus $\int_{\partial\mathcal{E}} h(\theta, x)\, g(\theta)\, d\theta = \lim_{r_n \to 1} \int_{\partial\mathcal{E}} h(\theta, x) f(r_n\theta)\, d\theta$. On the other hand, $f(r_n x)$, r_n fixed, is a function harmonic in the interior of $\mathcal{E}$ and continuous in $\mathcal{E}$; that is, $f(r_n x)$ is a function in $\mathfrak{F}$. Therefore $f(r_n x) = \int_{\partial\mathcal{E}} h(\theta, x) f(r_n\theta)\, d\theta$. Finally, as f is continuous at x, $f(x) = \lim_{r_n \to 1} f(r_n x)$.

In the statement and proof of the fundamental proposition which comes next, and in the rest of the section, we shall use the following terminology. The set C of points θ in $\partial\mathcal{E}$ satisfying $|\theta - \varphi| \leq \rho$ is the circle with center φ and radius ρ; $|\mathrm{C}|$ and $\rho(\mathrm{C})$ denote the n-1-dimensional measure of C and the radius of C. For each $x \neq 0$, θ_x is the point $\frac{R}{|x|}x$. If C is a circle on $\partial\mathcal{E}$ and x is a point interior to $\mathcal{E}$ and on the normal to $\partial\mathcal{E}$ through the center of C, then the cone with vertex x and base C is the set generated by joining x to each point of C. The axis of the cone is the normal to $\partial\mathcal{E}$ through the center of C. The angle of the cone is the maximum angle between the axis and any line joining x to a point of C.

3) *To each angle* α, $0 < \alpha < \frac{\pi}{2}$, *corresponds a constant* $k > 0$ *such that for every* $f \geqq 0$ *in* $\overline{\mathfrak{F}}$ *and every* x *in the interior of* $\mathcal{E}$ *there is a cone with vertex* x *and angle* $\geqq \alpha$ *with the property that the average of* $f(\theta)$ *over the base of the cone is* $\geqq kf(x)$ [22].

Proof. — For each $\rho > 0$, write C_ρ for the circle with center θ_x and radius ρ, and put $I(\rho) = \int_{C_\rho} f(\theta)\, d\theta$. Let ρ_0 be such that C_{ρ_0} is the base of the cone with vertex x and angle α, and set $m(x) = \sup\limits_{\rho \geqq \rho_0} I(\rho)/\rho^{n-1}$. Since the ratio $|C|/\rho(C)^{n-1}$ is bounded above and bounded away from 0 by constants depending only on the dimension, the inequality to be proved takes the form $m(x) \geqq kf(x)$.

If ρ_1 is an arbitrary number $\geqq \rho_0$, then

$$f(x) = \frac{R^2 - |x|^2}{\omega_n R} \int_{C_{\rho_1}} \frac{f(\theta)}{|\theta - x|^n}\, d\theta + \frac{R^2 - |x|^2}{\omega_n R} \int_{\partial\mathcal{E} - C_{\rho_1}} \frac{f(\theta)}{|\theta - x|^n}\, d\theta = I_1 + I_2.$$

Using the majoration $|\theta - x| \geqq R - |x|$, we obtain

$$I_1 \leqq \frac{R + |x|}{\omega_n R} \frac{1}{(R - |x|)^{n-1}} I(\rho_1) \leqq \frac{2}{\omega_n} \left(\frac{\rho_1}{R - |x|}\right)^{n-1} \frac{I(\rho_1)}{\rho_1^{n-1}}.$$

Using the majoration $|\theta - x| \geqq 1/2\, |\theta - \theta_x|$, we obtain

$$I_2 \leqq \frac{2^{n+1}(R - |x|)}{\omega_n} \int_{\rho_1}^{2R} \frac{dI(\rho)}{\rho^n}$$

[22] The calculation of the best possible constant will be given elsewhere. The constant which appears below in formula (10.2) has the correct order of magnitude for $\alpha \to \pi/2$.

Now,

$$\int_{\rho_1}^{2R} \frac{dI(\rho)}{\rho^n} = \frac{I(\rho)}{\rho^n}\bigg|_{\rho_1}^{2R} + n\int_{\rho_1}^{2R} \frac{I(\rho)}{\rho^{n+1}}d\rho \leqq \frac{I(\rho)}{\rho^n}\bigg|_{\rho_1}^{2R} + nm(x)\int_{\rho_1}^{2R}\frac{d\rho}{\rho^2}.$$

Thus

$$I_2 \leqq \frac{2^{n+1}(R-|x|)}{\omega_n}\left[\frac{I(2R)}{(2R)^n} - \frac{I(\rho_1)}{\rho_1^n} + \frac{nm(x)}{\rho_1} - \frac{nm(x)}{2R}\right]$$

so

$$(10.\ 1) \qquad I_1 + I_2 \leqq -\frac{2^n(n-1)(R-|x|)m(x)}{\omega_n R}$$
$$+\frac{2}{\omega_n}\left[\left(\frac{\rho_1}{R-|x|}\right)^{n-1} - 2^n\left(\frac{R-|x|}{\rho_1}\right)\right]\frac{I(\rho_1)}{\rho_1^{n-1}} + \frac{2^{n+1}n}{\omega_n}\left(\frac{R-|x|}{\rho_1}\right)m(x).$$

To complete the evaluation we use the simple geometric inequality $\sin\alpha \leqq \frac{\rho_0}{R-|x|} \leqq \tan\alpha$. Putting $\rho_1 = \rho_0$ in (10. 1), and dropping the obviously negative terms, we obtain

$$(10.\ 2) \qquad f(x) \leqq I_1 + I_2 \leqq \frac{2}{\omega_n}\left[\tan^{n-1}\alpha + \frac{2^n n}{\sin\alpha}\right]m(x).$$

A different evaluation of (10. 1) is better when α is not too large. If $\alpha \leqq \arctan 2$, then $\frac{\rho_0}{R-|x|} \leqq 2$, and it is possible to choose $\rho_1 \geqq \rho_0$ so that $\frac{\rho_1}{R-|x|} = 2$. Using this ρ_1, we obtain from (10. 1)

$$(10.\ 3) \qquad f(x) \leqq I_1 + I_2 \leqq \frac{2^n n}{\omega_n}m(x) \quad \textit{whenever} \quad \alpha \leqq \arctan 2.$$

In the course of the next proposition we shall use a general covering theorem not unlike a part of the Vitali theorem. It has some intrinsic interest, so we shall present it as a lemma separate from the present line of discussion.

LEMMA 1. — *Let* Π *be a family of spheres in a metric space. For each sphere* $S \in \Pi$ *let* S' *denote the sphere whose center is the center of* S *and whose radius is four times the radius of* S. *If* Π *has the two properties listed below, then there is a disjoint sequence, perhaps finite,* $\{S_n\} \subset \Pi$ *such that* $\bigcup_{S\in\Pi} S \subset \bigcup_{n=1}^{\infty} S'_n$. *The properties are the following:*

(i) *The radii of the spheres in* Π *are bounded, and all are* $\neq 0$.

(ii) *If a sequence of spheres in* Π *is disjoint, then the sequence of radii converges to* 0.

PROOF. — The sequence $\{S_n\}$ is defined inductively. Let M_0 be the least upper bound of the radii of the spheres in Π, and let S_1 be a sphere in Π whose radius is larger than $(2/3)M_0$. If $S_1, \ldots, S_k$ have already been defined, let M_k be the least upper bound of the radii of the spheres in Π which do not meet any of $S_1, \ldots, S_k$ and let S_{k+1} be a sphere in Π which does not meet any of $S_1, \ldots, S_k$ and whose radius is larger than $(2/3)M_k$.

Suppose that the point x lies in a sphere $S \in \Pi$ which meets the sphere S_k but no sphere S_i with $i < k$. If r is the radius of S, and if r_k is the radius of S_k, then $r \leqq M_{k-1} < \frac{3}{2} r_k$. Thus, if y is a point common to S and S_k, and if x_k is the center of S_k, then $d(x, x_k) \leqq d(x, y) + d(y, x_k) \leqq 2r + r_k < 4r_k$ and so $x \in S'_k$. On the other hand, every point $x \in \bigcup_{S \in \Pi} S$ lies in a sphere which meets some S_k. In fact, two cases arise. If the inductive procedure for defining S_k cannot be continued beyond some finite k_0, then all spheres in Π must meet one of $S_1, \ldots, S_{k_0}$. If the inductive procedure can be continued, then there are infinitely many S_k and their radii $r_k \to 0$. But we have seen above that if S does not meet one of $S_1, \ldots, S_{k-1}$, then $r < \frac{3}{2} r_k$.

Turning once again to the harmonic functions we provide a notation for use in the next proposition. Given a set $B \subset \mathcal{E}$ and an angle α we write B_α for the set of points $\theta \in \partial\mathcal{E}$ which lie either in B itself or in the base of some cone of angle α with vertex in B.

4) *To each angle* α, $0 < \alpha < \frac{\pi}{2}$, *corresponds a constant* k *such that for every set* $B \subset \mathcal{E}$, $|B_\alpha| \leqq k c_1(B)^p$, *where* $|B_\alpha|$ *denotes the* $n-1$*-dimensional measure of* B_α.

PROOF. — For sets in $\partial\mathcal{E}$ the present capacities are the same as the capacities determined by the functional space L^p on $\partial\mathcal{E}$. By virtue of the discussion we have made of the latter spaces we can write for any set $B \subset \partial\mathcal{E}$, $|B| = c_1(B)^p$. It follows easily that for the remainder of the proof we can assume that B lies entirely in the interior of $\mathcal{E}$.

From the fact that the strong majoration property holds it follows that the set functions c_1 and $\tilde{\delta}$ are equal [23]. In addition, we have seen earlier that $\tilde{\delta}$ is essentially the δ-function for $\overline{\mathcal{F}}$ [24]. Hence, if m is an arbitrary number larger than $c_1(\mathrm{B})$, then there is a function $f \geqq 0$ in $\mathcal{F}$ such that $m > \|f\|$ and such that $f(x) \geqq 1$ for every $x \in \mathrm{B}$. Let k_1 be the constant of proposition 3), and for each $x \in \mathrm{B}$ let $\mathrm{B}(x)$ be the base of a cone with vertex x, with angle $\geqq \alpha$, and with the mean value property of proposition 3). Let k' be a constant such that $|\mathrm{S}'| \leqq k'|\mathrm{S}|$ whenever S and S′ are circles in $\partial\mathcal{E}$ with the same center and with $\rho(\mathrm{S}') = 4\rho(\mathrm{S})$. Let $\mathrm{B}'(x)$ be the circle with the same center as $\mathrm{B}(x)$ and with $\rho[\mathrm{B}'(x)] = 4\rho[\mathrm{B}(x)]$. By virtue of the covering theorem there is a disjoint sequence $\{\mathrm{B}(x_n)\}$ such that $\mathrm{B}_\alpha \subset \bigcup_{n=1}^{\infty} \mathrm{B}'(x_n)$. Therefore

$$|\mathrm{B}_\alpha| \leqq \sum_{n=1}^{\infty} |\mathrm{B}'(x_n)| \leqq k' \sum_{n=1}^{\infty} |\mathrm{B}(x_n)| = k'|\mathrm{A}|,$$

where $\mathrm{A} = \bigcup_{n=1}^{\infty} \mathrm{B}(x_n)$.

Furthermore, the mean value of f over A is $\geqq k_1$, since the mean value is $\geqq k_1$ over each $\mathrm{B}(x_n)$, and these are disjoint. From this it follows by Hölder's inequality that

$$|\mathrm{A}| \leqq (1/k_1)^p \|f\|^p.$$

and hence that $|\mathrm{B}_\alpha| \leqq k'(1/k_1)^p m^p$. As m is any number $> c_1(\mathrm{B})$, 4) results.

It is simple now to derive Fatou's theorem. Let α' be a given angle, $0 < \alpha' < \frac{\pi}{2}\cdot$ For each point $\theta \in \partial\mathcal{E}$ let $\mathrm{K}_{\theta,\alpha'}$ be a closed cone extending into $\mathcal{E}$ from the vertex θ and touching $\partial\mathcal{E}$ only at θ; let $\mathrm{K}_{\theta,\alpha'}$ have angle α' and axis the normal to $\partial\mathcal{E}$ through θ. Fatou's theorem asserts that if f is a function in $\overline{\mathcal{F}}$, then for almost every θ, f is continuous in $\mathrm{K}_{\theta,\alpha'}$. It is proved as follows.

Let A be the set of points θ for which f is not continuous in

[23] Remark 6 at the end of section 6.

[24] The lemma and remark 1 in section 5.

$K_{\theta,\alpha'}$. For each $\varepsilon > 0$ let B^ε be a set such that $c_1(B^\varepsilon) < \varepsilon$ and such that f is continuous outside B^ε. Then for each $\varepsilon > 0$, and each $\theta \in A$, B^ε contains points of $K_{\theta,\alpha'}$, arbitrarily close to θ. This implies that $A \subset (B^\varepsilon)_\alpha$ for every α satisfying $\alpha' < \alpha < \frac{\pi}{2}$. Therefore $|A| \leq |(B^\varepsilon)_\alpha| \leq kc_1(B^\varepsilon)^p \leq k\varepsilon^p$, so finally $|A| = 0$.

The results which we have described are not restricted to the sphere. They are valid, and large parts of their proofs as well, for all closed domains with sufficiently smooth boundaries. A brief discussion of the situation follows.

We shall suppose that the basic set $\mathcal{E}$ is a closed bounded domain in Euclidean space E_n, and we shall suppose that the boundary $\partial\mathcal{E}$ is a C^1 hypersurface ([25]). This ensures that at each point of $\partial\mathcal{E}$ there is a tangent plane to $\partial\mathcal{E}$, and that the tangent plane turns continuously. Also there is a normal to $\partial\mathcal{E}$ at each point φ of $\partial\mathcal{E}$; n_φ denotes the unit vector in the direction of the exterior normal at φ. If x is any point of E_n, there is at least one point θ on $\partial\mathcal{E}$ which minimizes the distance $|\varphi - x|$, $\varphi \in \partial\mathcal{E}$. The line determined by x and any minimizing point ($\neq x$) is normal to $\partial\mathcal{E}$. In case there is only one minimizing point we shall call it θ_x. θ_x is a continuous function of x on the set where it is defined. In general θ, φ etc., refer to points on $\partial\mathcal{E}$; $d\theta$, $d\varphi$, etc. refer to the n-1-dimensional measure which is definable on $\partial\mathcal{E}$ in the classical manner; $|E|$ where E is a set $\subset \partial\mathcal{E}$ refers also to this measure. If C is any circle on $\partial\mathcal{E}$, and circles are defined as they were before, $\rho(C)$ denotes its radius.

Furthermore, we shall restrict $\partial\mathcal{E}$ to be a $C^{1,1}$, hypersurface, i. e. one whose normals are Lipschitzian:

$$(10.\ 4) \qquad 1/r_0 = \sup_{\theta \neq \varphi} \frac{\left|\sin\left(1/2\, \widehat{n_\theta n_\varphi}\right)\right|}{1/2\,|\theta - \varphi|} < \infty,$$

where $\widehat{n_\theta n_\varphi}$ denotes the angle between n_θ and n_φ ([26]). The

([25]) To say that $\partial\mathcal{E}$ is a C^1 hypersurface is to say that each point of $\partial\mathcal{E}$ has an n-dimensional neighborhood V which can be mapped in 1-1 fashion on an n-dimensional cube by a transformation T such that: *a*) T and T^{-1} are both C^1 transformations with non-vanishing Jacobians; *b*) $T(\partial\mathcal{E} \cap V)$ is the intersection of the cube with one of the coordinate hyperplanes.

([26]) The usual definition is that $\sup \frac{\widehat{n_\theta n_\varphi}}{|\theta - \varphi|} = M < \infty$; this is obviously equivalent to (10. 4).

constant r_0 will be called the *radius* of $\partial\mathcal{E}$. We list here the essential properties of such boundaries. Proofs will be given in a separate note.

a) The number r_0 and the two numbers r_0' and r_0'' defined below are all equal.

(10. 5) $r_0' = \sup \rho$ *taken over* $\rho \geqslant 0$ *such that for any two points* $\varphi \neq \theta$ *on* $\partial\mathcal{E}$ *the normal segments* $[\varphi - \rho n_\varphi;\ \varphi + \rho n_\varphi]$ *and* $[\theta - \rho n_\theta;\ \theta + \rho n_\theta]$ *are disjoint.*

(10. 6) $r_0'' = \sup \rho$ *taken over* $\rho \geqslant 0$ *such that for each* $\theta \in \partial\mathcal{E}$ *the tangent spheres of radius* ρ *with centers at* $\theta + \rho n_\theta$ *and* $\theta - \rho n_\theta$ *have no points in the interior and exterior of* $\mathcal{E}$ *respectively.*

It is the exterior tangent sphere of radius r_0 which intervenes in most subsequent calculations. When we speak simply of the tangent sphere at θ we mean this one. We write y_θ for its center $\theta + r_0 n_\theta$.

b) If $\alpha(\varphi, \theta)$ *denotes the angle between* n_φ *and the direction* $\overrightarrow{\varphi\theta}$ *then for any two points* φ *and* θ *on* $\partial\mathcal{E}$, $|\cos\alpha(\varphi,\theta)| \leqq \dfrac{|\varphi - \theta|}{2r_0}$.

c) For every point $x \in \mathcal{E}$ *within distance* r_0 *of* $\partial\mathcal{E}$ *and every point* θ *on* $\partial\mathcal{E}$ *the following inequality is satisfied:*

$$(10.\ 7) \qquad 0 \leqq |x - y_\theta| - r_0 - |x - \theta_x| \leqq \frac{|\theta - \theta_x|^2}{r_0}.$$

The significance of the inequality will appear upon examination of *g)* below and the proof which is given after this list of properties.

For each number r, $0 \leqq r < r_0$, and each point θ on $\partial\mathcal{E}$ we define $z(r, \theta)$ to be the point $\theta - r n_\theta$. For each number r, $0 \leqq r < r_0$, we define the parallel hypersurface to $\partial\mathcal{E}$ at distance r, for which we write $\partial\mathcal{E}_r$, to be the set of all points $z(r, \theta)$. $\partial\mathcal{E}_r$ is the boundary of a closed domain $\mathcal{E}_r$ contained in the interior of $\mathcal{E}$.

d) Each hypersurface $\partial\mathcal{E}_r$ *is of class* $C^{1,1}$ *with radius* $\geqq r_0 - r$. *For fixed* r *the transformation* $\theta \to z(r, \theta)$ *is a* 1-1 *continuous transformation of* $\partial\mathcal{E}$ *onto* $\partial\mathcal{E}_r$. *It possesses a Jacobian which is* *bounded and bounded away from* 0, *and these bounds are uniform with respect to* r *for* $r \leqq r' < r_0$; *the Jacobians converge uniformly to* 1 *as* $r \to 0$.

e) There are constants K_1 *and* K_2 *such that for every circle* C *on* $\partial\mathcal{E}$ $K_1\rho(C)^{n-1} \leqq |C| \leqq K_2\rho(C)^{n-1}$. *If* $r' < r_0$ *is fixed, the constants can be chosen so that the same inequality is valid on each* $\partial\mathcal{E}_r$, $r \leqq r'$.

f) The Green's function $G(y, x)$ *for the domain* $\mathcal{E}$ *exists. For each fixed* x *in the interior of* $\mathcal{E}$ *the function* $G(y, x)$ *as a function of* y *has a normal derivative* $-h(\theta, x)$ *at every point* θ *of* $\partial\mathcal{E}$. *If* θ *is fixed,* h *is harmonic in* x, *if* x *is fixed,* h *is continuous in* θ. *The Poisson formula holds with respect to the kernel* h: $f(x) = \int_{\partial\mathcal{E}} h(\theta, x) f(\theta)\, d\theta$ *for every function* f *continuous in* $\mathcal{E}$ *and harmonic in the interior.*

f') If $h_r(\psi, x)$ *denotes the kernel for the domain* $\mathcal{E}_r$, *then for fixed* x *the functions* $h_r[z(\theta, r), x]$ *converge uniformly as* $r \to 0$ to $h(\theta, x)$ (27).

g) $h(\theta, x)$ *satisfies the following inequality, obtained by the method of comparison domains*

$$0 \leqq h(\theta, x) \leqq \frac{|y_\theta - x|^2 - r_0^2}{r_0 \omega_n |\theta - x|^n}.$$

The function on the right is the Poisson kernel for the exterior of the tangent sphere at θ. *(Note the inequality in c)).*

We consider the functional space $\mathcal{F}$ of all complex valued functions continuous in $\mathcal{E}$ and harmonic in the interior of $\mathcal{E}$; we define the norm by $\|f\|_p = \left\{ \int_{\partial\mathcal{E}} |f(\theta)|^p d\theta \right\}^{1/p}$, where p is fixed and satisfies $1 \leqq p < \infty$.

It is obvious that there is no difficulty in showing that the perfect completion of $\mathcal{F}$ is the space $\mathcal{F}$ of Poisson integrals of functions L^p on $\partial\mathcal{E}$. This goes as it did before.

The various assertions about the manner in which the functions in $\overline{\mathcal{F}}$ assume boundary values require some comment. The first step is to define the analogues of the transforma-

(27) Both properties *f)* and *f')* are obtained by constructing h and h_r by the classical method of integral equations. See for example KELLOGG [19].

tions T_r. For each r, $0 \leqq r < r_0$, and each function $f(\theta)$ in L^p on $\partial\mathcal{E}$ we put $T_r f(\varphi) = f_r(\varphi) = \int_{\partial\mathcal{E}} h[\theta, z(r, \varphi)] f(\theta)\, d\theta$. It must be established that there is a constant K' such that for every $f \in L^p$, $\int_{\partial\mathcal{E}} |f_r(\varphi)|^p\, d\varphi \leqq K'^p \int_{\partial\mathcal{E}} |f(\theta)|^p\, d\theta$. Once this is done it follows by the argument we have used before that $\lim\limits_{r \to 0} \|f - f_r\|_p = 0$. In other words, the values of f on the parallel surface to $\partial\mathcal{E}$ at distance r converge in mean of order p to the values of f on $\partial\mathcal{E}$.

It is true, and for the same reason as before, that

$$\int_{\partial\mathcal{E}} h(\theta, x)\, d\theta = 1.$$

It is no longer true that $\int_{\partial\mathcal{E}} h[\theta, z(r, \varphi)]\, d\varphi = 1$, but the integral is $\leqq K''$ for some K'' independent of r and θ, and this is just as good; however, proof is required.

We will continue to use the notations we have used through the section. For example, if we are considering a given point θ on the boundary, then for any number r we write C_r for the circle with center θ and radius r; etc.

Let r and θ be fixed. Then by property g),

$$\int_{\partial\mathcal{E}} h[\theta, z(r, \varphi)]\, d\varphi \leqq \int_{C_r} \frac{|y_\theta - z(r, \varphi)|^2 - r_0^2}{\omega_n r_0 |\theta - z(r, \varphi)|^n}\, d\varphi + \int_{\partial\mathcal{E} - C_r} \frac{|y_\theta - z(r, \varphi)|^2 - r_0^2}{\omega_n r_0 |\theta - z(r, \varphi)|^n}\, d\varphi = I_1 + I_2.$$

Now, if $|\theta - \varphi| \leqq r$, then $|y_\theta - z(r, \varphi)| - r_0 \leqq 2r$, and in any case $|y_\theta - z(r, \varphi)| + r_0 \leqq D + 2r_0$ where D is the diameter of $\mathcal{E}$. Therefore,

$$I_1 \leqq \frac{2(D + 2r_0)}{\omega_n r_0} \frac{1}{r^{n-1}} |C_r| \leqq \frac{2(D + 2r_0)}{\omega_n r_0} K_2.$$

To evaluate I_2 we write

$$I_2 \leqq \frac{D + 2r_0}{\omega_n r_0} \int_{\partial\mathcal{E} - C_r} \frac{|y_\theta - z(r, \varphi)| - r_0}{|\theta - z(r, \varphi)|^n}\, d\varphi$$
$$\leqq \frac{D + 2r_0}{\omega_n r_0} \int_{\partial\mathcal{E} - C_r} \frac{|z(r, \varphi) - \varphi|}{|\theta - z(r, \varphi)|^n}\, d\varphi + \frac{1}{r_0} \frac{D + 2r_0}{\omega_n r_0} \int_{\partial\mathcal{E} - C_r} \frac{|\theta - \varphi|^2}{|\theta - z(r, \varphi)|^n}\, d\varphi$$

if we make use of c).

Since $|\theta - \varphi| \geqq r$ implies $|\theta - z(r, \varphi)| \geqq |\theta - \varphi|/2$, the two last integrals are majorated by

$$r\int_r^{D} 2_u \frac{d|C_\rho|}{\rho^n} \quad \text{and} \quad \int_r^{D} 2^n \frac{d|C_\rho|}{\rho^{n-2}} \quad \text{respectively.}$$

Integration by parts shows immédiately that these expressions are indeed bounded by bounds independent of r and θ.

The proof of proposition 2) was entirely special to the sphere. A proof which will yield the statement of 2) in this more general case can be based upon f') in the following way. Let f be a harmonic function for which the functions $f_{r_n}(\varphi) = f[z(r_n, \varphi)]$ converge weakly for some sequence $r_n \to 0$. If $g(\varphi)$ is the weak limit, and if x is fixed in the interior of $\mathcal{E}$, then

$$\int_{\partial\mathcal{E}} h(\theta, x) g(\theta)\, d\theta = \lim \int_{\partial\mathcal{E}} h(\theta, x) f_{r_n}(\theta)\, d\theta.$$

On the other hand, if ψ_{r_n} denotes the variable on $\partial\mathcal{E}_{r_n}$, then

$$f(x) = \int_{\partial\mathcal{E}_{r_n}} h_{r_n}(\psi_{r_n}, x) f(\psi_{r_n})\, d\psi_{r_n} = \int_{\partial\mathcal{E}} h_{r_n}[z(r_n, \theta), x] f_{r_n}(\theta) J_{r_n} d\theta,$$

where J_{r_n} is the Jacobian of the transformation $\theta \to z(r_n, \theta)$. Now because of the weak convergence of the f_{r_n} and the uniform convergence of the Jacobians to 1, and of the h_{r_n} to h, we deduce that $f(x) = \int_{\partial\mathcal{E}} h(\theta, x)\, g(\theta)\, d\theta$, and hence $f \in \overline{\mathcal{F}}$.

There is nothing at all to impede the extension of the key proposition 3). We shall not repeat the proof, for with the original proof and the calculations used to show $\|T_r f\|$ bounded as a model, the reader will not find it difficult. One remark will suffice: 3) should be proved only for x within distance r_0 of $\partial\mathcal{E}$, but as each set of small capacity is included within this strip, the restriction is harmless. Proposition 4) is valid as it stands, as is Fatou's theorem.

§ 11. Example 4. **Potentials of order α of M. Riesz.** — In this last example we shall discuss the potentials of order α of Marcel Riesz. Among the many papers on the subject especially relevant to our needs are those of O. Frostman [17], M. Riesz [23], H. Cartan [12, 13] and J. Deny [15]. The paper of Deny even gives explicitly several of the functional space properties of the spaces of potentials, but through most of the paper the prevailing interest lies in measures or in distributions, and not in their potentials.

In the course of the discussion we shall prove that our φ-capacity c_2, formed with the function $\varphi(t) = t^2$, is exactly the classical outer capacity. This can be taken as justification of our use of the term capacity.

The basic set $\mathcal{E}$ is Euclidean n-dimensional space, $n \geq 2$. We designate by K_α the kernel of order α of M. Riesz:

$$(11.\ 1) \quad K_\alpha(x) = K_{n,\alpha}(x) = \frac{1}{H_n(\alpha)}|x|^{\alpha-n} \quad \text{for} \quad 0 < \alpha < n,$$

$$(11.\ 1') \quad H_n(\alpha) = \frac{\pi^{n/2}\, 2^\alpha \Gamma\left(\dfrac{\alpha}{2}\right)}{\Gamma\left(\dfrac{n-\alpha}{2}\right)}.$$

Fundamental in the theory of potentials with respect to these kernels is the composition formula established by Riesz:

$$(11.2) \quad K_{\alpha+\beta}(x-y) = \int K_\alpha(x-z)\, K_\beta(z-y)\, dz \text{ if } \alpha+\beta < n.$$

Let us write Ω_α^+ for the class of all positive Borel measures μ on $\mathcal{E}$ with the property:

$$(11.\ 3) \quad \|\mu\|^2 = \iint K_\alpha(x-y)\, d\mu(y)\, d\mu(x) < \infty,$$

and let us write Ω_α for the class of differences of measures in Ω_α^+. It can be shown that the integral (11. 3), which is called the energy integral, is finite and strictly positive for every non-zero measure $\mu \in \Omega_\alpha$ (see O. Frostman [17] and H. Cartan [12]). The value of the integral is called the energy of μ. A measure in Ω_α is said to be of finite energy. With the usual definitions of addition of measures and of multiplication of a measure by a real number the class Ω_α is a (real) vector space. On it the function $\|\mu\|$ defined by the integral (11. 3) is a quadratic norm. The space Ω_α is not complete in this norm. However, an important theorem of H. Cartan (for $0 < \alpha \leq 2$) and of J. Deny (for the remaining α) asserts that the subset Ω_α^+, which is a convex cone in Ω_α, is complete ([28]).

([28]) Deny's proof is based upon the theory of the Fourier transform in the space of distributions of L. SCHWARTZ [24]. It is possible to obtain through direct analysis of the energy integral (11.3) a proof which does not make use of distributions.

Now we define the corresponding space of potentials, the actual functional space in which we are interested. First, the exceptional class $\mathfrak{A}_\alpha$ is to consist of all sets A for which there is a measure $\mu \in \Omega_\alpha^+$ such that the integral $\int K_\alpha(x-y)\,d\mu(y)$ is infinite for every $x \in A$. Given a measure $\mu \in \Omega_\alpha$ we define its potential of order α as follows.

$$(11.\ 4) \qquad K_\alpha\mu(x) = \int K_\alpha(x-y)\,d\mu(y).$$

for every x for which the integrand is integrable. It is obvious from the definition of $\mathfrak{A}_\alpha$ that $K_\alpha\mu(x)$ is defined exc. $\mathfrak{A}_\alpha$. We write $\mathcal{F}_\alpha$ for the functional class rel. $\mathfrak{A}_\alpha$ consisting of all functions $K_\alpha\mu(x)$ for $\mu \in \Omega_\alpha$. It is not difficult to prove that every function of class C^∞ vanishing outside of a compact is equal everywhere to the potential of order α $(0 < \alpha < n)$ of a measure μ_α belonging to all Ω_β, $0 < \beta < n$ (29).

We shall make use of an exceptional class, to be called $\mathfrak{A}_{\Omega_\alpha}$ and to consist of all subsets of the G_δ-sets which have measure 0 for every measure $\nu \in \Omega_\alpha$. Although this class seems to be different form the class $\mathfrak{A}_\alpha$, we shall finish by showing that the two are identical. The proof is difficult, however, and for the moment we are content to observe that $\mathfrak{A}_\alpha \subset \mathfrak{A}_{\Omega_\alpha}$. The argument for the latter proceeds as follows. If μ and ν belong to Ω_α^+, then the potential $K_\alpha\mu(x)$ is lower semi-continuous and the integral $(\mu, \nu) = \int K_\alpha\mu(x)\,d\nu$ is finite. Therefore the set of points at which $K_\alpha\mu(x)$ is infinite is a set G_δ of ν-measure 0.

From $\mathfrak{A}_\alpha \subset \mathfrak{A}_{\Omega_\alpha}$ follows that $K_\alpha\mu(x) = 0$ exc. $\mathfrak{A}_\alpha$ if and only if $||\mu|| = 0$. Therefore if we define $||K_\alpha\mu|| = ||\mu||$, $\mathcal{F}_\alpha$ becomes a normed functional class with quadratic norm. We shall see presently that $\mathcal{F}_\alpha$ is a functional space rel. $\mathfrak{A}_\alpha$ and that it has a functional completion rel. $\mathfrak{A}_\alpha$.

For an arbitrary closed set $A \subset \mathcal{E}$ let Γ_A denote the convex cone of measures in Ω_α^+ which are supported by A. Γ_A is closed in Ω_α^+ and hence complete. By arguments standard in the theory of Hilbert space it can be shown that corresponding to any $\mu \in \Omega_\alpha$ there is a unique $\mu' \in \Gamma_A$ which minimizes the distance from μ to elements of Γ_A; i.e. $||\mu - \mu'|| = \min ||\mu - \nu||$

(29) The measure μ_α is given by a density $f_\alpha(x)$ of class C^∞ and $O(|x|^{-n-\alpha})$ for $|x| \to \infty$.

over all $\nu \in \Gamma_A$. μ' is called the result of sweeping the measure μ onto A. It can be shown that $K_\alpha\mu'(x) = K_\alpha\mu(x)$ a.e. (μ'), and that $K_\alpha\mu'(x) \geq K_\alpha\mu(x)$ for $x \in A$ exc. $\mathfrak{A}_{\Omega_\alpha}$. Since Ω_α^+ contains each restriction of Lebesgue measure to a compact set, a particular consequence is that $K_\alpha\mu'(x) \geq K_\alpha\mu(x)$ a.e. in the Lebesgue sense. It can be shown further that $\|\mu'\| \leq \|\mu\|$; in addition, if μ'' is the result of sweeping $-\mu$ onto A, then $\|\mu' + \mu''\| \leq \|\mu\|$ [30]. Consider the special case $A = \mathcal{E}$. We have $\|K_\alpha(\mu' + \mu'')\| \leq \|K_\alpha\mu\|$, and also

$$K_\alpha(\mu' + \mu'')(x) \geq |K_\alpha\mu(x)| \text{ exc. } \mathfrak{A}_{\Omega_\alpha}.$$

In the next paragraph we shall see that the inequality holds exc. $\mathfrak{A}_\alpha$, and this will yield the strong majoration property.

From a lemma of Frostman ensues the fact, that if μ belongs to Ω_α^+, then at every point x the mean value of $K_\alpha\mu$ over the sphere with center x and radius r converges as $r \to 0$ to $K_\alpha\mu(x)$ (whether the latter is finite or infinite). From this it is clear that if $\mu \in \Omega_\alpha$ then at every point x exc. $\mathfrak{A}_\alpha$ the mean value of $K_\alpha\mu$ over the sphere with center x and radius r converges as $r \to 0$ to $K_\alpha\mu(x)$. Hence, if μ and ν belong to Ω_α, and if $K_\alpha\nu(x) \geq K_\alpha\mu(x)$ almost everywhere with respect to Lebesgue measure, then $K_\alpha\nu(x) \geq K_\alpha\mu(x)$ exc. $\mathfrak{A}_\alpha$. The strong majoration property in $\mathcal{F}_\alpha$ results.

With the aid of the strong majoration property it is easy to show that $\mathcal{F}_\alpha$ is a functional space rel. $\mathfrak{A}_\alpha$. Let $\{\mu_n\}$ be a sequence of measures converging to 0 in norm. For each n let $\bar{\mu}_n$ be such that $K_\alpha\bar{\mu}_n$ is a positive majorant for $K_\alpha\mu_n$ with the same norm, and from the sequence $\{\bar{\mu}_n\}$ pick a subse-

[30] We make use of the following result which is valid in abstract Hilbert space. *If Γ is a closed convex cone with vertex at the origin, and if μ' and μ'' are respectively the points of Γ at minimum distance from μ and $-\mu$, then $\|\mu' + \mu''\| \leq \|\mu\|$, whatever be the vector μ.*

PROOF. — Since $\mu - \mu'$ is orthogonal to μ' and since $-\mu - \mu''$ is orthogonal to μ'', the inequality to be proved takes the form

$$\|\mu\|^2 (\cos^2\theta + \cos^2\varphi + 2\cos\theta\cos\varphi\cos\psi) \leq \|\mu\|^2,$$

where θ, ψ, φ, are the angles between μ and μ', μ' and μ'', μ'' and $-\mu$ respectively. Since $0 \leq \theta$, $\varphi \leq \pi/2$ and $\pi \leq \theta + \psi + \varphi$, it is sufficient to prove that

$$\cos^2\theta + \cos^2\varphi - 2\cos\theta\cos\varphi\cos(\theta + \varphi) \leq 1.$$

The left side of this inequality is $= \sin^2(\theta + \varphi)$.

quence $\{\bar{\mu}_{n_k}\}$ such that $\sum_{k=1}^{\infty}||\bar{\mu}_{n_k}|| < \infty$. As Ω_α^+ is complete there is a $\bar{\mu} \in \Omega_\alpha^+$ such that $\bar{\mu} = \sum_{k=1}^{\infty}\bar{\mu}_{n_k}$. It can be shown that if ω_n belongs to Ω_α^+ and if the sequence $\{\omega_n\}$ converges to ω, then for every x, $K_\alpha\omega(x) \leqq \liminf K_\alpha\omega_n(x)$ (31). If the sequence $\{\omega_n\}$ is increasing, then for every x, $K_\alpha\omega(x) \geqq \sup K_\alpha\omega_n(x)$, so that $K_\alpha\omega(x) = \lim K_\alpha\omega_n(x)$. Applied to the partial sums of the series $\sum_{k=1}^{\infty}\bar{\mu}_{n_k}$, this gives $K_\alpha\bar{\mu}(x) = \sum_{k=1}^{\infty}K_\alpha\bar{\mu}_{n_k}(x)$ for every x (where the value $+\infty$ must be admitted, of course). Finally, therefore, exc. $\mathfrak{A}_\alpha$ we have $|K_\alpha\mu_{n_k}(x)| \leqq K_\alpha\bar{\mu}_{n_k}(x) \to 0$.

We are prepared to show that the functions δ and $\tilde{\delta}$ are identical. One consequence of this will be that $\mathfrak{L} = \tilde{\mathfrak{L}}$ and $\mathfrak{L}_\sigma^0 = \tilde{\mathfrak{L}}_\sigma^0$. Let $\{\mu_n\}$ be a Cauchy sequence of measures such that for a given set $B \in \tilde{\mathfrak{L}}$, $\liminf|K_\alpha\mu_n(x)| \geqq 1$ on B exc. $\mathfrak{A}_\alpha$ and $\lim||\mu_n|| \leqq \tilde{\delta}(B) + \varepsilon$. Let μ_n' and μ_n'' denote the results of sweeping μ_n and $-\mu_n$, respectively, on $\mathfrak{E}$. Then each of the sequences μ_n' and μ_n'' is Cauchy, so the sequence $\bar{\mu}_n = \mu_n' + \mu_n''$ is Cauchy, and because of the completeness of Ω_α^+ it has a limit $\bar{\mu}$. As $\mathcal{F}_\alpha$ is a functional space rel. $\mathfrak{A}_\alpha$, $\{\bar{\mu}_n\}$ contains a subsequence $\{\bar{\mu}_{n_k}\}$ such that $K_\alpha\bar{\mu}(x) = \lim K_\alpha\bar{\mu}_{n_k}(x)$ exc. $\mathfrak{A}_\alpha$. Therefore we have exc. $\mathfrak{A}_\alpha$, $K_\alpha\bar{\mu}(x) \geqq \liminf|K_\alpha\mu_n(x)| \geqq 1$ and at the same time $\delta(B) \leqq ||\bar{\mu}|| \leqq \lim||\mu_n|| \leqq \tilde{\delta}(B) + \varepsilon$.

If A is any set in $\mathfrak{E}$, then the measures $\mu \in \Omega_\alpha^+$ such that $K_\alpha\mu(x) \geqq 1$ on A exc. $\mathfrak{A}_\alpha$ form a closed convex set. Call it Γ_A^*. A closed convex set in a Hilbert space necessarily contains a point at minimum distance from the origin (or from any other point). Thus the infimum, $\inf||\mu||$ taken over $\mu \in \Gamma_A^*$, is a minimum, ie. is assumed; for each $A \in \mathfrak{L}$ there is a measure $\mu \in \Omega_\alpha^+$ such that $K_\alpha\mu(x) \geqq 1$ on A exc. $\mathfrak{A}_\alpha$ and such that $||\mu|| = \delta(A)$. An immediate consequence is that $\mathfrak{A}_\alpha = \mathfrak{L}_\sigma^0 \; (= \tilde{\mathfrak{L}}_\sigma^0)$. Γ_A^* will be used again later.

The next step is to obtain the relation between our capacities and the classical capacity (of order α). One of the many common definitions of the classical capacity is as follows.

(31) See H. Cartan [13]. The simple proof is based upon the fact that K_α is lower semi-continuous.

(11. 5) *If* C *is a compact set, then* $\gamma(C)$, *the capacity of* C, *is the number* $\|\mu_C\|^2$, *where* μ_C *minimizes the expression* $\|\mu\|^2 - 2\mu(C)$ *among all measures* $\mu \in \Omega_\alpha^+$ *supported by* C. μ_C *is called the capacitary distribution of* C. *If* A *is an arbitrary set, then* $\gamma_i(A)$, *the inner capacity of* A, *is the supremum of the numbers* $\gamma(C)$ *over all compact sets* $C \subset A$. *If* A *is an arbitrary set, then* $\gamma_0(A)$, *the outer capacity of* A, *is the infimum of the numbers* $\gamma_i(G)$ *over all open sets* $G \supset A$.

It is well known that the capacitary distribution exists for any compact set C and is uniquely determined by C. μ_C is the result of sweeping onto C an arbitrary measure ν whose potential is equal to 1 everywhere on C. Consequently, the potential of μ_C is $\geqq 1$ on C exc. $\mathfrak{A}_{\Omega_\alpha}$ and is equal to 1 a.e. (μ_C). If ν is any measure in Ω_α^+ such that $K_\alpha\nu(x) \geqq 1$ on C exc. $\mathfrak{A}_{\Omega_\alpha}$, then

$$\|\mu_C\|\|\nu\| \geqq (\mu_C, \nu) = \int K_\alpha\nu(x)\, d\mu_C \geqq \int K_\alpha\mu_C(x)\, d\mu_C = \|\mu_C\|^2,$$

so $\|\nu\| \geqq \|\mu_C\|$, and we have the following formula.

(11. 6) $\gamma(C) = \inf \|\nu\|^2$ *taken over all* $\nu \in \Omega_\alpha^+$ *such that* $K_\alpha\nu(x) \geqq 1$ *on* C *exc.* $\mathfrak{A}_{\Omega_\alpha}$. *The minimizing measure is* μ_C.

It is convenient to express the capacity also as the square of the distance from a certain convex set to the origin. The measures $\mu \in \Omega_\alpha^+$ such that $K_\alpha\mu(x) \geqq 1$ on C exc. $\mathfrak{A}_{\Omega_\alpha}$ form a closed convex set similar to Γ_C^*. Call this new set Γ_C^{**}. By virtue of (11.6) it is plain that $\gamma(C)$ is the square of the distance from Γ_C^{**} to the origin, and that μ_C is the point in Γ_C^{**} closest to the origin.

Suppose that an open set G is written as the union of an increasing sequence $\{C_n\}$ of compact sets, and suppose that the sequence $\{\mu_{C_n}\}$ of capacitary distributions is bounded (as they must be if G has finite inner capacity). Then there exists a subsequence $\{\mu_{C_{n_k}}\}$ converging weakly to a measure $\mu \in \Omega_\alpha^+$. For each k, all $\mu_{C_{n_i}}$ with $i \geqq k$ belong to $\Gamma_{C_{n_k}}^{**}$, so, as $\Gamma_{C_{n_k}}^{**}$ is closed and convex, μ belongs to $\Gamma_{C_{n_k}}^{**}$. Hence

$K_\alpha\mu(x) \geqq 1$ on C_{n_k} exc. $\mathfrak{A}_{\Omega_\alpha}$, and so $K_\alpha\mu(x) \geqq 1$ on G exc. $\mathfrak{A}_{\Omega_\alpha}$. By taking mean-values it follows that $K_\alpha\mu(x) \geqq 1$ on G everywhere. Now, $\|\mu\|^2 \leqq \liminf \|\mu_{C_{n_k}}\|^2 \leqq \gamma_i(G)$; and $\|\mu\|^2 \geqq \gamma_i(G)$ is obvious from (11. 6). We have proved:

1) *If* $\gamma_i(G) < \infty$ *for an open set* G, *then there is a* $\mu \in \Omega_\alpha^+$ *such that* $K_\alpha\mu(x) \geqq 1$ *on* G *everywhere and such that*

$$\gamma_0(G) = \gamma_i(G) = \|\mu\|^2.$$

The same argument (up to the point where the mean-values are taken) applied to δ and $\mathfrak{A}_\alpha$ gives a similar result which will be important.

2) *If* A *is the union of an increasing sequence of sets* A_n *then* $\delta(A) = \lim \delta(A_n)$ *whenever* $A \in \mathfrak{L}$; $A \in \mathfrak{L}$ *whenever each* $A_n \in \mathfrak{L}$ *and* $\lim \delta(A_n) < \infty$.

It is plain from (11. 6) that $\gamma(C) \leqq \delta(C)^2$ for any compact set C. It follows immediately from 2) that $\gamma_0(G) = \gamma_i(G) \leqq \delta(G)^2$ and from 1) that $\gamma_0(G) = \gamma_i(G) \geqq \delta(G)^2$ whenever G is an open set in $\mathfrak{L}$; and it follows also that $G \in \mathfrak{L}$ whenever $\gamma_i(G) < \infty$. An immediate consequence is that if A is any set with $\gamma_0(A) < \infty$, then $A \in \mathfrak{L}$ and $\delta(A)^2 \leqq \gamma_0(A)$. To obtain the converse, let $A \in \mathfrak{L}$, and let $\mu \in \Omega_\alpha^+$ be such that $K_\alpha\mu(x) \geqq 1$ on A exc. $\mathfrak{A}_\alpha$ and such that $\|\mu\| = \delta(A)$. For each $\eta < 1$, let $G_\eta = \underset{x}{E}\,[K_\alpha\mu(x) > \eta]$. Then for each $\eta < 1$, G_η is an open set, $G_\eta \supset A$ exc. $\mathfrak{A}_\alpha$, and $\delta(G_\eta) \leqq \frac{\|\mu\|}{\eta}$. If A_0 is any set in $\mathfrak{A}_\alpha$, then there is a measure $\nu \in \Omega_\alpha^+$, $\|\nu\| = 1$, such that $K_\alpha\nu(x) = +\infty$ at every point x of A_0. Setting $G'_\varepsilon = \underset{x}{E}\left[K_\alpha\nu(x) > \frac{1}{\varepsilon}\right]$ we have $G'_\varepsilon \supset A_0$, and $\delta(G'_\varepsilon) \leqq \varepsilon$. Now, taking, $A_0 = A - G_\eta$ we obtain an open set $G_\eta \cup G'_\varepsilon$ containing A and such that $\delta(G_\eta \cup G'_\varepsilon) \leqq \delta(G_\eta) + \delta(G'_\varepsilon) \leqq \frac{\|\mu\|}{\eta} + \varepsilon$, a number as close as we please to $\delta(A)$ [32]. It follows that if $A \in \mathfrak{L}$, then $\delta(A) = \inf \delta(G)$, the infimum being taken over all open sets in $\mathfrak{L}$ containing A. And from this and the previous discussion follows immediately the next statement.

[32] The sub-additivity of δ results from the fact that $\delta(A) = \tilde{\delta}(A) = c_1(A)$ whenever $c_1(A) < \infty$. The second equality comes from the strong majoration property.

3) $A \in \mathfrak{L}$ *if and only if* $\gamma_0(A) < \infty$. *If* $A \in \mathfrak{L}$, *then* $\delta(A)^2 = \gamma_0(A)$.

4) $\gamma_0(A) = c_2(A)$ *for every set* A (*where* c_2 *is the* φ-*capacity formed with the function* $\varphi(t) = t^2$.

Proof. — Suppose that $A \subset \bigcup_{n=1}^{\infty} A_n$ with $A_n \in \mathfrak{L}$. Then $\gamma_0(A) \leq \sum_{n=1}^{\infty} \gamma_0(A_n) = \sum_{n=1}^{\infty} \delta(A_n)^2$, and because $\{A_n\}$ is any sequence covering A, $\gamma_0(A) \leq c_2(A)$. Now, if $\gamma_0(A)$ is finite, then $A \in \mathfrak{L}$ and $\gamma_0(A) = \delta(A)^2 \geq c_2(A)$.

The natural question to approach next is that of the relationship between the inner and outer capacities. A necessary preliminary result is the following, obtained directly from 2) and 3).

5) *If* A *is the union of an increasing sequence of sets* A_n, *then* $\gamma_0(A) = \lim \gamma_0(A_n)$.

With the aid of 5) and a theorem of G. Choquet we are able to state [33]:

6) *If* A *is any analytic set, then* $\gamma_i(A) = \gamma_0(A)$. *In particular* $\mathfrak{A}_{\Omega_\alpha} = \mathfrak{A}_\alpha$.

It has not been proved explicitly yet that the space $\mathcal{F}_\alpha$ has a functional completion. We bring the example to an end by doing that and by exhibiting a representation of the functions in the perfect completion.

With the aid of the Riesz composition formula, (11. 2), it is easy to see that if $\mu \in \Omega_\alpha^+$, then for every x, $K_\alpha\mu(x) = K_{\alpha/2}f(x)$, where $f = K_{\alpha/2}\mu$, and where $K_\beta g$ for any function g signifies the potential of order β of the measure whose density with respect to Lebesgue measure is g. Furthermore, $||\mu||^2 = \int |f(x)|^2 dx$. It follows that for any $\mu \in \Omega_\alpha$, and for $f = K_{\alpha/2}\mu$, we have

$$(11.\ 7) \qquad K_\alpha\mu(x) = K_{\alpha/2}f(x) \text{ exc. } \mathfrak{A}_\alpha, \ \textit{and} \ ||\mu||^2 = \int |f(x)|^2 dx.$$

[33] G. Choquet has developed an abstract and very general theory of capacity in topological spaces. The crucial properties of the present set functions γ, γ_i, and γ_0 by virtue of which Choquet's theorem is applicable are the following: (see Choquet [14 *a*].

a) γ is an increasing non-negative set function defined on all compact sets.

b) Given a compact set C and an $\varepsilon > 0$ there is an open set $G \supset C$ such that $\gamma(C') \leq \gamma(C) + \varepsilon$ whenever $C \subset C' \subset G$.

c) γ_i and γ_0 are constructed from γ as in (11. 5).

d) γ_0 satisfies 5).

Let A be the set of points where $K_{\alpha/2}f(x) = +\infty$ for some non-negative square integrable function f. Because of the lower semi-continuity of $K_{\alpha/2}f(x)$, A is a set G_δ. It is immediately seen that if $\mu \in \Omega_\alpha^+$ then the integral $\int K_{\alpha/2}f(x)\,d\mu(x)$ is finite. It follows that $\mu(A) = 0$. In other words, $A \in \mathfrak{A}_{\Omega_\alpha} = \mathfrak{A}_\alpha$. The class of subsets of sets where the potentials $K_{\alpha/2}f(x)$, $f \geq 0$ and square integrable, become infinite is exactly the class $\mathfrak{A}_\alpha$. It can be proved easily by methods we have already used that the class $\bar{\mathfrak{F}}_\alpha$ of functions $K_{\alpha/2}f(x)$, f square integrable, is a functional space rel. $\mathfrak{A}_\alpha$ when given the norm $\|K_{\alpha/2}f\| = \left\{\int |f(x)|^2\,dx\right\}^{1/2}$. It is evident that this class is complete, and by (11.7) it contains $\mathfrak{F}_\alpha$ as a subclass. Indeed, $\bar{\mathfrak{F}}_\alpha$ is the perfect completion of $\mathfrak{F}_\alpha$; the only remaining point, that of the density of $\mathfrak{F}_\alpha$ in $\bar{\mathfrak{F}}_\alpha$, is easy to settle with the aid of the fact that every infinitely differentiable function which is 0 outside a compact set is the potential of order $\alpha/2$ of a measure $\mu \in \Omega_\alpha$. This is a fact we have mentioned before (see the paragraph after (11. 4)).

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