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On a generalization of de Rham lemma


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ON A GENERALIZATION OF DE-RHAM LEMMA

by Kyoji SAITO

In this short note, we give a proof of a theorem (cf. § 1) which is a generalization of a lemma due to de-Rham [1] and which was announced and used in [2].

As no proof of this theorem was available in the literature, Lê Dũng Tráng pushed me to publish it: I am grateful to him.

1. Notations and formulations of the theorem.

Let R be a noetherian commutative ring with unit. The profondeur of an ideal \( \mathfrak{a} \) of R is the maximal length \( q \) of sequences \( a_1, \ldots, a_q \in \mathfrak{a} \) with:

i) \( a_1 \) is a non-zero-divisor of R.

ii) \( a_i \) is a non-zero-divisor of \( R/a_1 R + \cdots + a_{i-1} R, i=2, \ldots, q \).

Let \( M \) be a free R-module of finite rank \( n \). We denote by \( \bigwedge^p M \) the \( p \)-th exterior product of \( M \) (with \( \bigwedge^0 M = R \) and \( \bigwedge^1 M = 0 \)).

Let \( \omega_1, \ldots, \omega_k \) be given elements of \( M \), and \( (e_1, \ldots, e_n) \) be a free basis of \( M \),

\[
\omega_1 \wedge \cdots \wedge \omega_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.
\]

We call \( \mathfrak{c} \): the ideal of \( R \) generated by the coefficients \( a_{i_1 \cdots i_k}, 1 \leq i_1 < \cdots < i_k \leq n \). (We put \( \mathfrak{c} = R \), when \( k = 0 \).)
Then we define:

\[ Z^p := \{ \omega \in \bigwedge^M \omega : \omega \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0 \} \quad p = 0, 1, 2, \ldots \]

\[ H^p := Z^p / \sum_{i=1}^{k} \omega_i \wedge \bigwedge^M \quad p = 0, 1, 2, \ldots \]

In the case when \( k = 0 \), we understand \( Z^p = 0, H^p = 0 \) for \( p = 0, 1, 2, \ldots \).

**Theorem.** — i) There exists an integer \( m \geq 0 \) such that:

\[ \alpha^m H^p = 0 \text{ for } p = 0, 1, 2, \ldots, n. \]

ii) \( H^p = 0 \) for \( 0 \leq p < \text{prof}(\alpha) \).

**2. Proof of the theorem.**

**Proof of i).** — Since \( R \) is noetherian, we have only to show for any \( \omega \in Z^p \) and any coefficients \( a_{i_1}, \ldots, a_{i_k} \),

\[ 1 \leq i_1 < \cdots < i_k \leq n, \]

there exists an integer \( m \geq 0 \) such that

\[ (a_{i_1}, \ldots, a_{i_k})^m \omega \in \sum_{i=1}^{k} \omega_i \wedge \bigwedge^M. \]

If \( a_{i_1}, \ldots, a_{i_k} \) is nilpotent, then nothing is to show. Suppose \( a_{i_1}, \ldots, a_{i_k} = a \) is not nilpotent and let \( R(\omega) \) be the localization of \( R \) by the powers of \( a = a_{i_1}, \ldots, a_{i_k} \). There is a canonical morphism \( R \rightarrow R(\omega) \) and we denote by \( [\omega] \) the image of \( \omega \in \bigwedge^M \) in \( \left( \bigwedge^M \otimes R(\omega) \right) = \bigwedge^M (M \otimes R(\omega)) \) because \( M \) is free over \( R \).

Since the ideal in \( R(\omega) \) generated by the coefficients of

\[ [\omega_1] \wedge \cdots \wedge [\omega_k] \]

contains the image of \( a = a_{i_1}, \ldots, a_{i_k} \) in \( R(\omega) \), it coincides with \( R(\omega) \) and we may consider

\[ [\omega_1], \ldots, [\omega_k] \]

as a part of free basis of \( M \otimes R(\omega) \). We add some other elements \( [e_1], \ldots, [e_{n-k}] \) such that

\[ [\omega_1], \ldots, [\omega_k], [e_1], \ldots, [e_{n-k}] \]
give a basis of $M \otimes R(\omega)$. Then any element

$$[\omega] \in \bigwedge^p (M \otimes R(\omega))$$

can be developed in the form:

$$[\omega] = \sum_{l+m=p} \sum_{1 \leq i_1 < \cdots < i_l \leq k} a_{i_1 \cdots i_k} [\omega_{i_1}] \wedge \cdots \wedge [\omega_{i_l}] \wedge [e_{j_1}] \wedge \cdots \wedge [e_{j_m}].$$

Then the fact $[\omega] \wedge [\omega_1] \wedge \cdots \wedge [\omega_k] = 0$ is equivalent to the existence of some $\eta_i' \in \bigwedge (M \otimes R(\omega))$ with

$$[\omega] = \sum_{i=1}^{k} \eta_i' \wedge [\omega_i].$$

Let us take $\eta_i \in \bigwedge M$ and $m_1 \geq 0$ with $\eta_i' = a^{-m_i} [\eta_i]$.

Then we have:

$$a^{-m} \omega - \sum_{i=1}^{k} \eta_i' \wedge \omega_i = a^{-m} [\omega] - \sum_{i=1}^{k} [\eta_i'] \wedge [\omega_i] = 0.$$

By the definition of $R(\omega)$, there exists some $m_2 \geq 0$ such that

$$a^{m_2} \left(a^{-m} \omega - \sum_{i=1}^{k} \eta_i' \wedge \omega_i\right) = 0$$

in $\bigwedge^p M$.

This completes the proof of i).

**Proof of ii.** We prove it by double induction on $(p, k)$ for $p, k \geq 0$.

a) In the case $k = 0$, the assertion is trivially true by the definition of $H^p$.

b) Case $p = 0$.

Let $\omega \in \bigwedge^0 M = R$ with $\omega \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0$. The fact $p = 0 < \text{prof} (\mathcal{A})$ implies the existence of $a \in \mathcal{A}$, which is non-zero-divisor of $R$. Since $a\omega = 0$, we get $\omega = 0$.

c) Case $0 < p < \text{prof} (\mathcal{A})$ and $0 < k$.

The induction hypothesis is then, that for $(p-1, k)$ and $(p, k-1)$ the assertion ii) of the theorem is true.

Let $a \in \mathcal{A}$ be a non-zero-divisor of $R$. According to i), there exists an integer $m > 0$ with $a^m H^p = 0$. Since $a^m \in \mathcal{A}$ is again a non-zero-divisor of $R$, we may assume that $m=1$. 
We denote by $\bar{\omega}$ the image of $\omega \in \bigwedge^p M$ in
\[
\left( \bigwedge^p M \right) \otimes_R R/aR \simeq \bigwedge^p (M \otimes_R R/aR).
\]

For $\omega \in \mathbb{Z}^p$, we have a presentation:

\[\sum_{i=1}^{k} \eta_i \wedge \omega_i, \quad \text{with} \quad \eta_i \in \bigwedge^{p-1} M.\]

We have then: $0 = \sum_{i=1}^{k} \bar{\eta}_i \wedge \bar{\omega}_i$.

For any $1 \leq j \leq k$, we get:

\[
\bar{\eta}_j \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k = \left( \sum_{i=1}^{k} \bar{\eta}_i \wedge \bar{\omega}_i \right) \\
\wedge \left( (-1)^{j-1} \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_j \wedge \cdots \wedge \bar{\omega}_k \right) = 0.
\]

Here the symbol $\wedge$ means, we omit the corresponding term.

Since the ideal of $R/aR$ generated by the coefficients of $\bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k$ is equal to $\mathcal{A}/aR$ and

\[
\text{prof } \mathcal{A}/aR = \text{prof } \mathcal{A} - 1 \geq p - 1 \geq 0,
\]

we can apply to $\bar{\eta}_j$ the induction hypothesis for $(p - 1, k)$; there exist $\xi_{ji} \in \bigwedge^p M$, $j, i = 1, \ldots, k$, such that

\[
\bar{\eta}_j = \sum_{i=1}^{k} \bar{\xi}_{ji} \wedge \bar{\omega}_i, \quad j = 1, \ldots, k.
\]

Lifting back this relation to $\bigwedge^p M$, we find some $\xi_j \in \bigwedge^p M$, $j = 1, \ldots, k$, such that

\[
\eta_j = \sum_{i=1}^{k} \xi_{ji} \wedge \omega_i + a\xi_j \quad j = 1, \ldots, k.
\]

Replacing $\eta_j$ in the presentation (*) by this, we obtain:

\[
a \left( \omega - \sum_{j=1}^{k} \xi_j \wedge \omega_j \right) = \sum_{i,j=1}^{k} \xi_{ji} \wedge \omega_i \wedge \omega_j.
\]

Multiplying by $\omega_2 \wedge \cdots \wedge \omega_k$, we have:

\[
a \left( \omega - \sum_{i=1}^{k} \xi_i \wedge \omega_i \right) \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0.
\]
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Since $a$ is a non-zero-divisor of $R$, we have:

$$\left( \omega - \sum_{i=1}^{k} \zeta_i \wedge \omega_i \right) \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0.$$ 

Now since the ideal $\mathfrak{a}'$ generated by the coefficients of $\omega_2 \wedge \cdots \wedge \omega_k$ contains the ideal $\mathfrak{a}$, we have $\text{prof } \mathfrak{a}' \geq \text{prof } \mathfrak{a} \geq p$. Again by the induction hypothesis for $(p, k-1)$, we find some $\theta_j \in \wedge M$, $j = 2, \ldots, k$ with

$$\omega - \sum_{i=1}^{k} \zeta_i \wedge \omega_i = \sum_{j=2}^{k} \theta_j \wedge \omega_i.$$ 

This ends the proof of ii).

3. Remark.

We can formulate the theorem in § 2, for a more general class of modules $M$ than the one of free modules, as follows.

Let $M$ be a $R$-finite module with homological dimension $\text{hd}_R(M) \leq 1$, and $\omega_1, \ldots, \omega_k$ be elements of $M$. Since $\text{hd}_R(M) \leq 1$, we have a free resolution:

$$0 \to L_1 \to L_2 \to M \to 0.$$ 

Let $\tilde{\omega}_1, \ldots, \tilde{\omega}_k$ be some liftings of $\omega_1, \ldots, \omega_k$ in $L_2$ and $\tilde{e}_1, \ldots, \tilde{e}_m$ be images in $L_2$ of a free basis $e_1, \ldots, e_m$ of $L_1$. Let $\mathfrak{a}$ be the ideal of $R$ generated by coefficients of $\tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_k \wedge \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_m$.

Since $\mathfrak{a}$ can be considered as a Fitting ideal of the following resolution:

$$L_1 \oplus R^k \to L_2 \to M \to \sum_{i=1}^{k} R \omega_i \to 0.$$ 

we obtain the following lemma.

**Lemma.** — $\mathfrak{a}$ does only depend on $M$ and $\omega_1, \ldots, \omega_k$ and does depend neither on the choice of $\tilde{\omega}_1, \ldots, \tilde{\omega}_k$ and $e_1, \ldots, e_m$ nor on the resolution of $M$, we have used.

Let us define again:

$$H^p = \left\{ \omega \in \wedge^p M : \omega \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0 \right\} \wedge_{i=1}^{k} \wedge^p M.$$
Then we obtain again: i) $\alpha^m H^p = 0$, $p = 0, 1, 2, \ldots$ for some $m > 0$ and ii) $H^p = 0$ for $0 \leq p < \text{prof } \alpha$.

For the proof we have only to apply the theorem to $L_2$ and $\omega_1, \ldots, \omega_k, \tilde{e}_1, \ldots, \tilde{e}_m$.

**BIBLIOGRAPHY**


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