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Exotic characteristic classes and subfoliations


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EXOTIC CHARACTERISTIC CLASSES
AND SUBFOLIATIONS

by Luis A. CORDERO and P.M. GADEA (*)

1. Introduction.

The development in the last years of the study of topological invariants associated to a foliated structure on a differentiable manifold(***) (usually called exotic characteristic classes of the foliation) has been well known.

Within the general context of this study, the following problem appears in a canonical way: let $M$ be a differentiable manifold on which two foliations $F_1$ and $F_2$ are defined, and such that $F_1 \subset F_2$, that is, every leaf of $F_2$ is, itself, foliated by leaves of $F_1$; briefly, $F_1$ is said to be subfoliation of $F_2$; in fact, this geometrical structure on $M$ can be described as a special type of multifoliate structure (in the sense of Kodaira-Spencer ({8})); now, we present two questions: 1) does a relation exist between exotic classes of $F_1$ and $F_2$?, and 2) is it possible to give a topological obstruction to the existence of such a structure on $M$?

In this paper we give the answer to these questions, by studying the problem through a more general situation and using Lehmann's techniques ({9}, {10}). For this purpose, we consider the following situation: let $Q_i$, $i = 1, 2$, be two $G_i$-principal fibre bundles over $M$, and let $\Pi : Q_1 \rightarrow Q_2$ be a morphism of principal fibre bundles (over the identity of $M$); by an appropriate choice of connections on these fibre bundles we point out a relation between the images of Lehmann's exoticism associated to those connections (Theorem 4.5); in the special case of $F_1$ and $F_2$, two foliations as above, that relation gives the answer to our questions: every exotic characteristic class of $F_2$ is also an

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(**) Always manifolds will mean paracompact differentiable manifold of class $C^\infty$. 
exotic characteristic class of $F_1$; in fact, this result can be expressed as a topological obstruction $F_1$ to be a subfoliation of $F_2$.

2. Notations and basic concepts.

Let $M$ be a differentiable manifold. We shall denote $\mathfrak{X}(M)$ the Lie algebra of vector fields and $A^*(M)$ the exterior algebra of differential forms on $M$.

Given a $G$-principal fibre bundle $E \to M$, $G$ being the structural Lie group, $\omega$ indistinctly denotes an (infinitesimal) connection on the bundle or the 1-form of that connection; $I(G)$ is the algebra of invariant polynomials over the Lie algebra $\mathfrak{g}$ of $G$; $I(G)$ is a graded algebra, $I(G) = \oplus_{k \geq 0} I^k(G)$ and $I^+(G)$ denotes its maximal ideal

$$I^+(G) = \oplus_{k \geq 1} I^k(G).$$

Denote by $\lambda_\omega : I(G) \to A^*(M)$ the Chern-Weil homomorphism, defined by $\lambda_\omega(f) = f(\Omega)$, for $f \in I(G)$ and $\Omega$ being the curvature form of $\omega$. If $I = [0, 1]$ is the unit interval, $\int_0^1 : A^k(M \times I) \to A^{k-1}(M)$ denotes the integration along the fibre of $M \times I \to M$. If $\omega'$ is another connection on $E$, we write $[\omega, \omega']$ the connection on $E \times I \to M \times I$ defined by

$$[\omega, \omega']\left(\frac{\partial}{\partial t}\right) = 0, [\omega, \omega']|_{E \times \{t\}} = t \omega' + (1 - t) \omega$$

and by $\Delta_{\omega, \omega'} : I^k(G) \to A^{2k-1}(M)$ the composition $\int_0^1 \lambda_{[\omega, \omega']}$. As it is well known, $\lambda_\omega$ induces an homomorphism $\lambda : I(G) \to H^{even}(M, \mathbb{R})$ which is independent of $\omega$.

Let $J \subseteq I(G)$ be a homogeneous ideal; $\omega$ is said a $J$-connection if $\lambda_\omega(f) = 0$ for every $f \in J$. If $P$ denotes a property of the degree of elements of $I(G)$, $J(P)$ denotes the homogeneous ideal generated by the elements satisfying the property $P$. For example, if $\dim M = n$, every connection on $E$ is a $J\left(\geq \left\lceil \frac{n}{2} \right\rceil\right)$-connection.
If $G = \text{Gl}(q, \mathbb{R})$, it is $I(G) = \mathbb{R}[c_1, \ldots, c_q]$, where $c_1, \ldots, c_q$ are the usual generators given by

$$\det(I + tA) = 1 + \sum_{i=1}^{q} c_i(A)t^i, \text{ for every } A \in \mathfrak{gl}(q, \mathbb{R})$$

If $Q \to M$ is a vector bundle, $\nabla$ denotes the derivation law of a linear connection on $Q$. Thus, every metric connection on $Q$ is a $J(\text{odd})$-connection.

If $Q \to M$ is the normal bundle of a foliation on $M$, of codimension $q$, and $\nabla$ is a basic connection on $Q$ (in the sense of Bott ({1})), then $\nabla$ is a $J(>q)$-connection.

3. The Lehmann's exoticism (((9)), ((10))).

Let $E$ be a $G$-principal fibre bundle on $M$. Consider $J, J'$ homogeneous ideals of $I(G)$; if $f \in I^k(G)$, we write

$$\bar{f} = f \pmod{J}, \bar{f}' = f \pmod{J'}$$

and introduce a graduation on the quotient algebras $I(G)/J, I(G)/J'$ by $\deg \bar{f} = \deg \bar{f'} = 2k$, for every $f \in I^k(G)$; also, we shall denote $\Lambda(I^+(G))$ the exterior algebra over $\mathbb{R}$ generated by the elements of $I^+(G)$ and define a graduation on $\Lambda(I^+(G))$ by $\deg f = 2k - 1$, for every $f \in I^k(G), k > 0$. Then, consider the graded algebra

$$\hat{W}(J, J') = I(G)/J \otimes_\mathbb{R} I(G)/J' \otimes_\mathbb{R} \Lambda(I^+(G))$$

and $I(G)/J, I(G)/J', \Lambda(I^+(G))$ are canonically identified to subalgebras of $\hat{W}(J, J'); I^+(G)$ can be identified to one part of $\Lambda(I^+(G)) \subset \hat{W}(J, J')$ by the isomorphism

$$h : I^+(G) \to \Lambda^1(I^+(G))$$

and, if $G = \text{Gl}(q, \mathbb{R})$, we write $h_i = h(c_i)$.

$\hat{W}(J, J')$ is endowed with a structure of graded differential algebra by defining a differential (of degree 1)

$$d(\bar{f}) = d(\bar{f'}) = 0, \quad \text{for } f \in I(G)$$

$$d(f) = \bar{f} - \bar{f'}, \quad \text{for } f \in I^+(G)$$
and, clearly, $d^2 = 0$.

If $\omega$ is a $J$-connection and $\omega'$ is a $J'$-connection on $E$, a homomorphism of graded algebras $\rho_{\omega, \omega'} : W(J, J') \to A^*(M)$ is defined by

$$
\rho_{\omega, \omega'}(f) = \lambda_{\omega}(f)
$$

and, in cohomology, $\rho_{\omega, \omega'}$ induces a homomorphism of graded algebras

$$
\rho_{\omega, \omega'}^* : H^*(\tilde{W}(J, J')) \to H^*(M, \mathbb{R})
$$

The elements of $\text{Im} \rho_{\omega, \omega'}^*$ are said to be the exotic characteristic classes associated to $J$, $J'$, $\omega$ and $\omega'$.

Let $J \subset I(G)$ be a homogeneous ideal and $\omega_0$, $\omega_1$ two $J$-connections on $E$; $\omega_0$ and $\omega_1$ are said to be differentiably $J$-homotopic if there does exist a $J$-connection $\tilde{\omega}$ on $E \times I \to M \times I$ such that

$$
\tilde{\omega}|_{E \times \{0\}} = \omega_0, \quad \tilde{\omega}|_{E \times \{1\}} = \omega_1
$$

and, in a more general form, $\omega_0$ and $\omega_1$ are said to be $J$-homotopic if there does exist a finite sequence $\omega_0 = \omega_0, \omega_1, \ldots, \omega_k = \omega_1$ of $J$-connections such that, for every $i = 0, 1, \ldots, k - 1$, $\omega_i$ and $\omega_{i + 1}$ are differentiably $J$-homotopic. A set $C$ of connections on $E$ is said to be $J$-connected if it is not-empty and any two connections in $C$ are $J$-homotopic.

**Proposition 3.1.** — $\text{Im} \rho_{\omega, \omega'}^*$ depends only on the $J$-connected component of $\omega$ and the $J'$-connected component of $\omega'$.

In particular, if $C$ is the set of basic connections on the transversal bundle $Q$ of a $q$-codimensional foliation on $M$ and $C'$ is the set of metric connections on $Q$, Lehmann shows that $C$ is $J(>q)$-connected and $C'$ is $J(\text{odd})$-connected; moreover, in this case $\tilde{W}(J(>q))$, $J(\text{odd}))$ has the same cohomology that its subalgebra

$$
W_{O_q} = \mathbb{R} \left[ c_1, \ldots, c_q \right] / (J(>q) \otimes \mathbb{R} \Lambda (h_1, h_3, \ldots, h_{(q)})
$$

where $(q)$ denotes the largest odd integer $\leq q$ and $h_i = h(c_i)$. Therefore
Proposition 3.2. — The homomorphism $\rho_{\nabla \nabla}^*: H^*(W_{\nabla}) \rightarrow H^*(M, R)$ does not depend on the choice of $\nabla \in C$ and $\nabla' \in C'$.


In this paragraph we shall consider the following situation: let $Q_i \rightarrow M$ be a $G_i$-principal fibre bundle ($i = 1,2$) and let

\[ \xymatrix{ Q_1 \ar[rr] \ar[rd] & & Q_2 \ar[ld] \ar[rr] & & \Pi \ar[ld] \ar[rd] & & M \ar[ll] } \]

a homomorphism of principal fibre bundles; also, denote

\[ \Pi : G_1 \rightarrow G_2 \]

the corresponding homomorphism of Lie groups and assume that $\Pi$ is surjective but not a submersion in general, e.g.

\[ d\Pi : G_1 \rightarrow G_2 \]

is not of maximal rank in general; the linear mapping $d\Pi$ permits to define:

Definition 4.1. — If $f \in \mathfrak{I}^k(G_2)$, $i(f)$ is defined by

\[ i(f)(X_1 \otimes \ldots \otimes X_k) = f(d\Pi(X_1) \otimes \ldots \otimes d\Pi(X_k)), \]

for every $X_j \in G_1$, $j = 1,2, \ldots, k$.

A direct application of this definition shows

Proposition 4.2. — For every $f \in \mathfrak{I}(G_2)$, $i(f) \in \mathfrak{I}(G_1)$ and

\[ i : \mathfrak{I}(G_2) \rightarrow \mathfrak{I}(G_1) \]

is a homomorphism of graded algebras. Moreover, if $d\Pi$ is of maximal rank, then $i$ is injective.
Let $J_2 \subseteq I(G_2)$ be an homogeneous ideal and $J_1$ an arbitrary homogeneous ideal of $I(G_1)$, such that $J_1 \supseteq i(J_2)$ (in particular, $J_1$ could be thought as the homogeneous ideal generated by the elements of $i(J_2)$).

**Theorem 4.3.** — Let $\omega_1$ be a connection in $Q_1$, and $\Omega_1$ its curvature form. Then:

a) there is a unique connection $\omega_2$ in $Q_2$ such that the horizontal subspaces of $\omega_1$ are mapped into horizontal subspaces of $\omega_2$ by $\Pi$.

b) if $\Omega_2$ is the curvature form of $\omega_2$, then

$$\Pi^{*}\omega_2 = d \Pi \cdot \omega_1$$

$$\Pi^{*}\Omega_2 = d \Pi \cdot \Omega_1$$

c) if $\omega_1$ is a $J_1$-connection, then $\omega_2$ is a $J_2$-connection.

**Proof.** — a) and b) are well-known results (see Kobayashi-Nomizu, vol I ((7)), p. 79).

In order to prove c), we have to show that, if $f \in J_2$ with $\deg f = k$, then $\lambda f_2 (f) = 0$, e.g.

$$f(\Omega_2) (Y_1 \otimes \ldots \otimes Y_{2k}) = 0, \text{ for } Y_1, \ldots, Y_{2k} \in \mathfrak{X}(Q_2)$$

But it suffices to show that when $Y_i, i = 1, \ldots, 2k$, is horizontal with respect to $\omega_2$ and, in this case, there exist $X_1, \ldots, X_{2k} \in \mathfrak{X}(Q_1)$ such that $d\Pi(X_i) = Y_i$ for every $i = 1, 2, \ldots, 2k$. But $i(f) \in J_1$, then

$$0 = i(f) \left( \Omega_1 \right) (X_1 \otimes \ldots \otimes X_{2k}) =$$

$$= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} i(f) \left( \Omega_1 \right)(X_{\sigma(1)}, X_{\sigma(2)}) \otimes \ldots \otimes \Omega_1(X_{\sigma(2k-1)}, X_{\sigma(2k)}) =$$

$$= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f(d\Pi(X_{\sigma(1)}, X_{\sigma(2)})) \otimes \ldots \otimes d\Pi(X_{\sigma(2k-1)}, X_{\sigma(2k)})) =$$

$$= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f((\Pi^{*}\Omega_2)(X_{\sigma(1)}, X_{\sigma(2)}) \otimes \ldots \otimes (\Pi^{*}\Omega_2)(X_{\sigma(2k-1)}, X_{\sigma(2k)})) =$$

$$= \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f(\Omega_2(d\Pi(X_{\sigma(1)})), d\Pi(X_{\sigma(2)})) \otimes \ldots \otimes (\Omega_2(d\Pi(X_{\sigma(2k-1)})), d\Pi(X_{\sigma(2k)}))) =$$
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\[ = \frac{1}{(2k)!} \sum_{\sigma} \epsilon_{\sigma} f (\Omega_{2}(Y_{\sigma(1)}, Y_{\sigma(2)}) \otimes \cdots \otimes \Omega_{2}(Y_{\sigma(2k-1)}, Y_{\sigma(2k)})) = \]

\[ = f(\Omega_{2}) (Y_{1} \otimes \cdots \otimes Y_{2k}) \]

**Remark.** — Note that, if \( \overline{J}_{2} \) is another homogeneous ideal of \( I(G_{2}) \) with \( \overline{J}_{2} \supset J_{2} \), it might happen that \( \omega_{2} \) be, in fact, a \( \overline{J}_{2} \)-connection.

Now, let \( J_{2}, J'_{2} \) (respect. \( J_{1}, J'_{1} \)) homogeneous ideales of \( I(G_{2}) \) (respect. \( I(G_{1}) \)) such that

\[ J_{1} \supset i(J_{2}) \quad \text{and} \quad J'_{1} \supset i(J'_{2}) \]

By virtue of Theorem 4.3, given \( \omega_{1} \) a \( J_{1} \)-connection and \( \omega'_{1} \) a \( J'_{1} \)-connection, there exist \( \omega_{2} \) a \( J_{2} \)-connection and \( \omega'_{2} \) a \( J'_{2} \)-connection satisfying the condition b) in the Theorem. Then, consider the graded differential algebras

\[ \hat{W}_{1}(J_{1}, J'_{1}) = I(G_{1})/J_{1} \otimes_{R} I(G_{1})/J'_{1} \otimes_{R} \Lambda(I^{+}(G_{1})) \]

\[ \hat{W}_{2}(J_{2}, J'_{2}) = I(G_{2})/J_{2} \otimes_{R} I(G_{2})/J'_{2} \otimes_{R} \Lambda(I^{+}(G_{2})) \]

The homomorphism \( i : I(G_{2}) \to I(G_{1}) \) induces canonically a new homomorphism of graded algebras

\[ \overline{i} : \hat{W}_{2}(J_{2}, J'_{2}) \to \hat{W}_{1}(J_{1}, J'_{1}) \]

**Proposition 4.4.** — The following diagram is commutative

\[ \begin{array}{ccc}
\hat{W}_{1}(J_{1}, J'_{1}) & \xrightarrow{\overline{i}} & \hat{W}_{2}(J_{2}, J'_{2}) \\
\rho \omega_{1} \omega'_{1} & & \rho \omega_{2} \omega'_{2} \\
\downarrow \Lambda^{*(M)} & & \downarrow \Lambda^{*(M)} \\
\end{array} \]

(4.1)

**Proof.** — It suffices to prove the commutativity for \( \overline{f} = f(\text{mod } J_{2}) \), a \( \overline{J}_{2} \)-homogeneous element, with \( f \in I(G_{2}) \), and \( \Delta \omega_{2}, \omega'_{2} = \Delta \omega_{1}, \omega'_{1} \cdot \overline{i} \).

If \( \overline{i} : I(G_{2})/J_{2} \to I(G_{1})/J_{1} \) denotes, once more, the mapping given by \( \overline{i}(f) = \overline{f} \), we have

\[ \rho \omega_{2} \omega'_{2} (\overline{f}) = \lambda \omega_{2} (f) = f(\Omega_{2}) \]
and

\[ \rho_{\omega_1} (\mu (\bar{f})) = \rho_{\omega_1} (\bar{i} (f)) = i (f) (\Omega_1) \]

and it is clear that \( f(\Omega_2) \) and \( i(f) (\Omega_1) \) define the same element of \( A^*(M) \). In a similar way, the commutativity is proved for \( \bar{f} \).

Now, we consider

\[ Q_1 \times I \xrightarrow{\pi \times id} Q_2 \times I \]

where \([\omega_2, \omega_1']\) is the unique connection in \( Q_2 \times I \) which might be obtained form \([\omega_1, \omega_1']\) in \( Q_1 \times I \) through Theorem 4.3; hence, the following diagram is commutative.

\[ I(G_2) \xrightarrow{i} I(G_1) \]

\[ \lambda_{[\omega_2, \omega_1]}, \lambda_{[\omega_1, \omega_1']} \]

\[ \Delta_{\omega_2 \omega_2'}, \Delta_{\omega_1 \omega_1'} \]

\[ A^*(M \times I) \]

\[ A^*(M) \]

\[ \int_0^1 \]

Remark. — If \( J_2 \) and \( J_2' \) are homogeneous ideales of \( I(G_2) \) such that \( J_2 \supset J_2', J_2' \supset J_2' \), and the connections \( \omega_2, \omega_2' \) are not only \( J_2 \)-and \( J_2' \)-connections but \( \bar{J}_2 \)- and \( \bar{J}_2' \)-connections, respectively, and if

\[ \eta : \hat{W}_2 (J_2, J_2') \rightarrow \hat{W}_2 (\bar{J}_2, \bar{J}_2') \]

is the canonical projection, (4.1) can be enlarged to a new commutative diagram.
**Theorem 4.5.** — Diagram (4.1) induces, in cohomology, a new commutative diagram

\[
\begin{array}{c}
\hat{W}_1(J_1, J'_1) \\ \rho_{\omega_1 \omega_1'} \\
\end{array} \quad \begin{array}{c}
\tilde{\rho} \\
\end{array} \quad \begin{array}{c}
\hat{W}_2(J_2, J'_2) \\ \eta \\
\end{array}
\]

\[
\begin{array}{c}
A^*(M) \\ \rho_{\omega_2 \omega_2'} \\
\end{array} \quad \begin{array}{c}
\rho_{\omega_2 \omega_2'} \\
\end{array} \quad \begin{array}{c}
\hat{W}_2(J_2, J'_2) \\
\end{array}
\]

**Theorem 4.5.** — Diagram (4.1) induces, in cohomology, a new commutative diagram

\[
\begin{array}{c}
H^*(\hat{W}_1(J_1, J'_1)) \\ \rho_{\omega_1 \omega_1'} \\
\end{array} \quad \begin{array}{c}
\tilde{T}^* \\
\end{array} \quad \begin{array}{c}
H^*(\hat{W}_2(J_2, J'_2)) \\ \rho_{\omega_2 \omega_2'} \\
\end{array}
\]

\[
\begin{array}{c}
H^*(M, \mathbb{R}) \\
\rho_{\omega_1 \omega_1'} \\
\end{array}
\]

Hence

\[
\text{Im } \rho_{\omega_2 \omega_2'} \subset \text{Im } \rho_{\omega_1 \omega_1'}
\]

Moreover, \(\text{Im } \rho_{\omega_2 \omega_2'}\) does not change when \(\omega_1\) (respect. \(\omega_1'\)) runs over its \(J_1\)-connected component (respect. \(J_1'\)-connected component).

**Proof.** — The commutativity of this diagram is evident from that of (4.1), and this fact implies trivially (4.3).

In order to prove the last assertion, it suffices to show that if \(\omega_1\) (respect. \(\omega_1'\)) runs over its \(J_1\)-connected (respect. \(J_1'\)-connected) component, then \(\omega_2\) (respect. \(\omega_2'\)) does it over its \(J_2\)-connected (respect. \(J_2'\)-connected) component.

For that, let \(\bar{\omega}_1\) be a connection in \(Q_1\) differentiably \(J_1\)-homotopic to \(\omega_1\) and let \(\bar{\omega}_2\) be the connection in \(Q_2\) corresponding to \(\bar{\omega}_1\) through Theorem 4.3; \(\bar{\omega}_2\) is a \(J_2\)-connection. Now, consider the connection \(\tilde{\omega}\) in \(Q_1 \times I \to M \times I\) which defines the \(J_1\)-homotopy between \(\omega_1\) and \(\bar{\omega}_1\); \(\tilde{\omega}\) is also a \(J_1\)-connection and its corresponding connection in \(Q_2 \times I\) through Theorem 4.3 is a \(J_2\)-connection which
defines a $J_2$-homotopy between $\omega_2$ and $\bar{\omega}_2$. All these facts can be easily checked by a direct calculation.

5. Application to subfoliations.

The geometric situation which we have described in § 1 is a particular case of multifoliate structure on the manifold $M$ and is defined as follows: consider the set $P = \{1, 2, 3\}$ with the usual order, $1 < 2 < 3$, and suppose $\dim M = n$. Now, we define a mapping

$$\alpha = \{1, 2, \ldots, n\} \rightarrow P$$

and, thus, $\{\alpha\}$ is $P$-multifoliate and we have determined the subgroup $G_p \subset \text{Gl} (n, \mathbb{R})$ of matrices

$$
\begin{pmatrix}
* & * & * \\
0 & * & * \\
0 & 0 & *
\end{pmatrix}
= P_1, P_2
$$

Let us suppose given an integrable $G_p$-structure on $M$; then, on $M$, there exist two foliations $F_1, F_2$ of dimensions $p_1, p_2$, respectively, and such that every leaf of $F_2$ is, itself, foliated by leaves of $F_1$. This fact is equivalent to the existence of two vector subbundles $E_i \subset TM, i = 1, 2$, and an injective morphism

If $N_i = TM/E_i, i = 1, 2$, is the normal bundle of $F_i$, there is canonically defined a surjective morphism

$$
\begin{array}{ccc}
E_1 & \longrightarrow & E_2 \\
\downarrow & & \downarrow \\
M & & M
\end{array}
$$

$$
\begin{array}{ccc}
N_1 & \longrightarrow & N_2 \\
\downarrow & & \downarrow \\
M & & M
\end{array}
$$
Denote $q_i = n - p_i = \text{codim } F_i$, $i = 1, 2$; it is possible to choose a covering $\{U\}$ of $M$ which trivializes simultaneously $N_1$ and $N_2$, and a local basis of sections of $N_1$

$$\omega^1, \ldots, \omega^{q_2}, \omega^{q_2+1}, \ldots, \omega^{q_1}$$

in such form that $\omega^1, \ldots, \omega^{q_2}$ is a local basis of sections of $N_2$; it is clear that this choice can be done compatibly with $p : N_1 \to N_2$. Moreover, as $E_1$ and $E_2$ are completely integrable

$$d\omega^i = \theta^i_j \wedge \omega^j, \ i, j = 1, 2, \ldots, q_2$$

$$d\omega^a = \theta^a_j \wedge \omega^j + \theta^a_b \wedge \omega^b, \ a, b = q_2 + 1, \ldots, q_1$$

and the matrix of 1-forms

$$\theta = \begin{pmatrix} 0 & \theta^i_j \\ \theta^a_i & \theta^a_b \end{pmatrix}$$

is the 1-form of a connection in $N_1$, which is basic with respect to $F_1$, and

$$\theta' = (\theta^i_j)$$

is the 1-form of a connection in $N_2$, basic with respect to $F_2$. If $\nabla$ (respect. $\nabla'$) denotes to derivation law associated to $\theta$ (respect. $\theta'$), the following diagram commutes

$$\begin{array}{ccc}
\Gamma(N_1) & \xrightarrow{\nabla} & \Gamma(T^*M \otimes N_1) \\
p \downarrow & & \downarrow 1 \otimes p \\
\Gamma(N_2) & \xrightarrow{\nabla'} & \Gamma(T^*M \otimes N_2)
\end{array} \quad (5.1)$$

Similarly, if we consider a weakly-compatible Riemannian metric (see Vaisman (11)) on the multifoliate manifold $M$, it is possible to define two metric connections $\tilde{\nabla}$ and $\nabla'$ on $N_1$ and $N_2$ respectively, which permit to write a new commutative diagram like (5.1) (in particular, by using the techniques introduced in (4)), it is possible to write the global expression of these connections.

By another part, consider the Lie groups $G_1$ and $G_2$ given as follows: $G_1 \subset GL(q_1, \mathbb{R})$ is the group of all matrices
with \( A \in GL(q_2, \mathbb{R}) \), and \( G_2 = GL(q_2, \mathbb{R}) \), and the homomorphism

\[
G_1 \longrightarrow G_2 \\
\begin{array}{c}
\text{m} \\
\rightarrow
\end{array}
\longrightarrow A
\]

Next, consider \( I(G_1) \) and \( I(G_2) \) and their homogeneous ideals given by

Ideals of \( I(G_1) \): \( J_1 = J(>q_1), J'_1 = J(\text{odd}) \)

Ideals of \( I(G_2) \): \( J_2 = J(>q_1), J'_2 = J(\text{odd}), J_2 = J(>q_2) \)

Clearly, \( \nabla \) (respect. \( \nabla' \)) is a \( J_1 \)-connection (respect. \( J_2 \)-connection) and \( \widetilde{\nabla} \) (respect. \( \widetilde{\nabla}' \)) is a \( J'_1 \)-connection (respect. \( J'_2 \)-connection); in fact, \( \nabla' \) is a \( J_2 \)-connection.

Under these assumptions, we can use the results of § 4 and state

**Proposition 5.1.** - The following diagram commutes

\[
\begin{array}{ccc}
\hat{H}^{\ast}(W_1(J_1, J'_1)) & \xrightarrow{i^{\ast}} & \hat{H}^{\ast}(W_2(J_2, J'_2)) \\
\downarrow \rho_{\nabla, \nabla}' & & \downarrow \eta^{\ast} \\
H^{\ast}(M, \mathbb{R}) & \leftarrow & H^{\ast}(\hat{W}_2(J_2, J'_2))
\end{array}
\]

Hence, \( \text{Im} \rho_{\nabla, \nabla}' \subseteq \text{Im} \rho_{\nabla, \nabla}' \), e.g. the set of exotic classes of \( F_2 \) is a subset of the set of exotic classes of \( F_1 \).

This result permits us to give a topological obstruction to \( F_1 \) be a subfoliation of \( F_2 \), as follows:

**Corollary 5.2.** - A necessary condition for \( F_1 \) be a subfoliation of \( F_2 \) is that every exotic class of \( F_2 \) be also an exotic class of \( F_1 \).

At last, note that if \( F_2 \) is given by \( E_2 = TM \), e.g. if \( F_2 \) has the manifold \( M \) as unique leaf, that obstruction is trivial.
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