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HARMONIC SPACES ASSOCIATED WITH ADJOINTS OF LINEAR ELLIPTIC OPERATORS

by Peter SJÖGREN

Dédié à Monsieur M. Brelot à l'occasion de son 70^e anniversaire.

1. Introduction.

The object of this paper is to improve the author's results in [7] about the distribution adjoint of an elliptic linear operator L and the associated potential theory. Under weaker hypotheses on the coefficients than in [7] we obtain again a connection with M^{me} Hervé's construction of the adjoint of a given harmonic sheaf in axiomatic potential theory. The first step in the proof of this is a regularity theorem for distribution solutions of the equation $L^*u = 0$, which we prove in a more general setting. Results of this type can be found in Agmon-Douglis-Nirenberg [1, p. 722] and Browder [4, p. 188]. Our regularity hypotheses are slightly weaker than theirs, and we assume *a priori* only that u is locally integrable, which seems to be the most natural hypothesis.

Our further results are entirely analogous to those of [7], concerning the adjoint harmonic measure and the Dirichlet problems for L and L*.

2. The regularity theorem.

We assume that L is an elliptic differential operator of order r with complex coefficients, defined by

$$\mathrm{L} = \sum_{|\alpha|\leqslant r} a_\alpha \mathrm{D}^\alpha$$

in a domain Ω_0 of $\mathbf{R}^n$, $n \geqslant 2$. (Here α is a multi-index.) The top-order coefficients of L are supposed to be locally Dini continuous in Ω_0. This means that for each compact set $\mathrm{K} \subset \Omega_0$, there exists an increasing function $w : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$|a_\alpha(x) - a_\alpha(y)| \leqslant w(|x-y|)$$

for $x, y \in \mathrm{K}$ and $|\alpha| = r$, and satisfying

$$\int_0 \frac{w(t)}{t}\,dt < \infty .$$

For $|\alpha| < r$ we assume $a_\alpha \in \mathrm{L}^\infty_{\mathrm{loc}}(\Omega_0)$. The adjoint of L is defined by

$$\mathrm{L}^* u = \sum_{|\alpha|\leqslant r} (-1)^{|\alpha|} \mathrm{D}^\alpha(a_\alpha u)$$

for any locally integrable u, where the derivatives are taken in the sense of distributions.

THEOREM 1. — *Let ω be a subdomain of Ω_0, and assume that u is defined and locally integrable in ω and satisfies*

$$\mathrm{L}^* u = \sum_{|\alpha|\leqslant r} \mathrm{D}^\alpha f_\alpha$$

in the sense of $\mathscr{D}'(\omega)$, where f_α is locally Dini continuous in ω if $|\alpha| = r$ and $f_\alpha \in \mathrm{L}^p_{\mathrm{loc}}(\omega)$ for some $p \geqslant 1$ satisfying $p > n/(r - |\alpha|)$ if $|\alpha| < r$. Then u coincides a.e. in ω with a continuous function.

To prove this theorem we need the following lemma, whose proof is based on an idea of Prof. L. Carleson.

LEMMA. — *Let* $1 < p \leqslant \infty$. *If* $u \in L^1$, $K \in L^1$, *and* $f \in L^p + L^\infty$ *are three non-negative functions in* R^n and

$$u \leqslant K * u + f \quad \text{a.e.,} \tag{2.1}$$

then $u \in L^p + L^\infty$.

Proof. — Take $M < \infty$ so large that

$$\int K_1(x)\, dx < \frac{1}{2}, \tag{2.2}$$

where $K_1 = K\chi$ and χ is the characteristic function of the set $\{x : K(x) > M\}$. Then we see from (2.1) that

$$u \leqslant K_1 * u + g \quad \text{a.e.,}$$

for some $g \in L^p + L^\infty$. By induction it follows that

$$u \leqslant K_1^{(m)} * u + \sum_1^{m-1} K_1^{(i)} * g + g \tag{2.3}$$

for $m = 1, 2, \ldots$, where $K_1^{(m)}$ is the $m - th$ power of K_1 with respect to convolution. Letting $m \to \infty$, we conclude from (2.2) that $K_1^{(m)} * u \to 0$ in L^1. Hence, we have almost everywhere convergence for some subsequence. Similarly, since

$$L^1 * (L^p + L^\infty) \subset L^p + L^\infty$$

with a corresponding inequality for the norms, the series in (2.3) converges in $L^p + L^\infty$. Choosing another subsequence, we see that u is dominated a.e. by a function in $L^p + L^\infty$, which proves the lemma.

Proof of Theorem 1. — Assume n is odd, and for $y \in \omega$ and $x \in \mathbf{R}^n$ let $F(y, x - y)$ be a fundamental solution in $\mathbf{R}^n$ of the operator

$$\sum_{|\alpha|=r} a_\alpha(y) D^\alpha,$$

whose coefficients are constant, as constructed e.g. in Browder [4, Lemma 8].

For any $\varphi \in \mathcal{D}(\omega)$ we have

$$\int u L\varphi\, dx = \sum_{|\alpha| \leqslant r} (-1)^{|\alpha|} \int f_\alpha D^\alpha \varphi\, dx. \tag{2.4}$$

Take a non-negative $\zeta \in \mathcal{D}(\omega)$ and let $y \in \operatorname{supp} \zeta$ be a Lebesgue point of u. For $\rho > 0$ we put

$$\zeta_\rho(x) = \zeta(x)(1 - w((x-y)/\rho)),$$

where $w \in \mathcal{D}(\mathbf{R}^n)$ has its support in $\{|x| \leqslant 1\}$ and equals 1 in $\{|x| \leqslant 1/2\}$. Now apply (2.4) to

$$\varphi(x) = \zeta_\rho(x)\mathrm{F}(y, x-y).$$

The left-hand side of (2.4) will then be

$$\int u(x) \sum_{|\alpha|<r} a_\alpha(x) \mathrm{D}_x^\alpha(\zeta_\rho(x)\mathrm{F}(y, x-y))\, dx$$
$$+ \int u(x) \sum_{|\alpha|=r} (a_\alpha(x) - a_\alpha(y))\mathrm{D}_x^\alpha(\zeta_\rho \mathrm{F})\, dx$$
$$+ \int u(x) \sum_{|\alpha|=r} a_\alpha(y)\mathrm{D}_x^\alpha(\zeta_\rho \mathrm{F})\, dx = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.$$

Since we have precise estimates of the derivatives of $\zeta_\rho - \zeta$ and F, we easily find that

$$\mathrm{I}_1 + \mathrm{I}_2 = \int u(x) \sum_{|\alpha|<r} a_\alpha(x)\mathrm{D}_x^\alpha(\zeta \mathrm{F})\, dx$$
$$+ \int u(x) \sum_{|\alpha|=r} (a_\alpha(x) - a_\alpha(y))\mathrm{D}_x^\alpha(\zeta \mathrm{F})\, dx + o(1)$$

as $\rho \to 0$, where the integrals are absolutely convergent because of the Lebesgue property of u at y and the Dini continuity of the a_α for $|\alpha| = r$.

Setting $\mathrm{B}_\rho = \{x : |x-y| < \rho\}$, we have

$$\mathrm{I}_3 = \int u(x) \sum_{|\alpha|=r} a_\alpha(y)\mathrm{D}_x^\alpha(\zeta \mathrm{F})\, dx$$
$$+ \int_{\mathrm{B}_\rho} u(x) \sum_{|\alpha|=r} a_\alpha(y)\mathrm{D}_x^\alpha((\zeta_\rho - \zeta)\mathrm{F})\, dx = \mathrm{I}_4 + \mathrm{I}_5.$$

These integrals are absolutely convergent, because the terms involving $\mathrm{D}_x^\alpha \mathrm{F}$ cancel. Further,

$$\text{(2.5)} \quad \mathrm{I}_5 = u(y) \int_{\mathrm{B}_\rho} \sum_{|\alpha|=r} a_\alpha(y) \sum_{\beta<\alpha} \frac{\alpha\,!}{\beta\,!(\alpha-\beta)\,!}$$
$$\mathrm{D}_x^{\alpha-\beta}(\zeta_\rho - \zeta)\mathrm{D}_x^\beta \mathrm{F}\, dx + o(1)$$

as $\rho \to 0$, where $\alpha\,! = \Pi \alpha_i\,!$, etc. It is easily seen that this integral, restricted to $\mathrm{B}_{\rho/2}$, tends to 0 with ρ. Due to the definition of F, we get

$$\int_{B_\rho} \sum_{|\alpha|=r} a_\alpha(y) D^\alpha(\zeta_\rho - \zeta) F \, dx = -\zeta(y).$$

Hence, (2.5) implies

$$I_5 = (-1)^r u(y)\zeta(y) + o(1),$$

if we integrate by parts in $B_\rho \setminus B_{\rho/2}$ and use the equation

$$\sum_{\beta<\alpha} (-1)^{|\beta|} \frac{\alpha!}{\beta!(\alpha-\beta)!} = (-1)^{r+1}$$

for $|\alpha| = r$, which follows from the binomial theorem. (For this argument, cf. [7, proof of Lemma 1].)

Using a similar technique, we see that the right-hand side of (2.4) with our φ equals

$$\text{(2.6)} \quad \overset{*}{\int} \sum_{|\alpha|\leqslant r} (-1)^{|\alpha|} f_\alpha D_x^\alpha(\zeta F) \, dx + \sum_{|\alpha|=r} c_\alpha(y)\zeta(y) f_\alpha(y) + o(1)$$

as $\rho \to 0$, where the c_α are continuous functions and

$$\overset{*}{\int} dx = \lim_{\eta\to 0^+} \int_{|x-y|>\eta} dx.$$

The terms in (2.6) for which $|\alpha| < r$ are absolutely convergent integrals and depend continuously on y. For $|\alpha| = r$, the Dini continuity of the f_α implies that the corresponding integrals in (2.6) are continuous in y, since $D_x^\alpha F(y, x-y)$ is a variable Calderón-Zygmund kernel.

Collecting terms and letting $\rho \to 0$, we get (cf. [1, p. 722])

$$\begin{aligned}
\text{(2.7)} \quad u(y)\zeta(y) = {} & (-1)^{r+1} \int u(x) \sum_{|\alpha|<r} a_\alpha(x) D_x^\alpha(\zeta F) \, dx \\
& + (-1)^{r+1} \int u(x) \sum_{|\alpha|=r} (a_\alpha(x) - a_\alpha(y)) D_x^\alpha(\zeta F) \, dx \\
& + (-1)^{r+1} \int u(x) \sum_{|\alpha|=r} a_\alpha(y) D_x^\alpha(\zeta F) \, dx \\
& + \overset{*}{\int} \sum_{|\alpha|\leqslant r} (-1)^{r-|\alpha|} f_\alpha(x) D_x^\alpha(\zeta F) \, dx \\
& + (-1)^r \sum_{|\alpha|=r} c_\alpha(y)\zeta(y) f_\alpha(y)
\end{aligned}$$

for a.a. y in supp ζ. Due to the properties of the convolution product and the behaviour of F, the first and third integrals in (2.7) are in $L^p_{loc}(\omega)$ for $p < n/(n-1)$. The same is true of the second integral, except possibly for the term

$$\int u(x)\zeta(x) \sum_{|\alpha|=r} (a_\alpha(x) - a_\alpha(y))D^\alpha_x F(y, x-y)\, dx.$$

But this is a function of y which is dominated by a convolution $|u\zeta| * K$ where $K \in L^1$ and $K \geqslant 0$ in $\mathbf{R}^n$. Hence, our Lemma implies that $u \in L^p_{loc}(\omega)$ for $p < n/(n-1)$, since ζ is arbitrary. Then we can improve the estimates of the terms in (2.7), and after repeated applications of our Lemma we get that $u \in L^\infty_{loc}(\omega)$. Using (2.7) again, we then see that u coincides a.e. in ω with a continuous function.

If n is even, there may occur a logarithmic term in F, which we avoid in the following way. Take $x_0 \in \omega$, write $D_i = \partial/\partial x_i$, and let the polynomial

$$P(D) = P(D_1, \ldots, D_n)$$

be the principal part of L at x_0. Then

$$\overline{P}(D) = P(-D_1, D_2, \ldots, D_n)$$

is an operator with constant coefficients in $\mathbf{R}^n$, and the composition $L\overline{P}(D)$ is elliptic and of order 2r. Further, the principal part of $L\overline{P}(D)$ at x_0 is a polynomial

$$Q(D_1^2, D_2, \ldots, D_n).$$

It is easily seen that

$$(L\overline{P}(D))^* u = (-1)^r \sum_{|\alpha|\leqslant r} D^\alpha \overline{P}(D) f_\alpha \tag{2.8}$$

in the distribution sense. We now identify $\mathbf{R}^n$ with the hyperplane $x_{n+1} = 0$ in $\mathbf{R}^{n+1}$ and denote points in $\mathbf{R}^{n+1}$ by (x, x_{n+1}).

Then put

$$\tilde{L} = L\overline{P}(D) + Q(D_1^2 + D_{n+1}^2, D_2, \ldots, D_n) - Q(D_1^2, D_2, \ldots, D_n),$$

which is an operator in $\mathbf{R}^{n+1}$ of order 2r whose restriction to $\mathbf{R}^n$ is $L\overline{P}(D)$. It is elliptic at $(x_0, 0)$ and thus in a neighbourhood U of $(x_0, 0)$ in $\mathbf{R}^{n+1}$. Further, we can write

$$\tilde{L} = L\overline{P}(D) + D_{n+1}R(D, D_{n+1}),$$

where R is a polynomial in $n + 1$ variables. Putting

$$\varphi(x, x_{n+1}) = u(x) \qquad \text{and} \qquad g_\alpha(x, x_{n+1}) = f_\alpha(x),$$

we conclude from (2.8) as in [7, proof of Lemma 1] that

$$(\tilde{L})^*\varphi = (-1)^r \sum_{|\alpha|\leqslant r} D^\alpha \overline{P}(D) g_\alpha$$

in the sense of distributions in U. For $|\alpha| < r$ the g_α do not quite satisfy the regularity assumptions made in the proof of the odd case, but since φ and the g_α are independent of x_{n+1}, we still get the necessary estimates of the terms in (2.7), and the proof carries over. It follows that φ and thus also u coincide a.e. with continuous functions, and Theorem 1 is proved.

3. Second-order operators.

From now on we assume that

$$L = \Sigma a^{ij}\partial^2/\partial x_i \partial x_j + \Sigma a^i \partial/\partial x_i + a$$

is an elliptic second-order operator in Ω_0 with real coefficients satisfying the regularity assumptions of Section 2. As to notions from potential theory, we follow the terminology used in [7].

For any subdomain $\omega \subset \Omega_0$, the Sobolev space $W^{2,p}(\omega)$ consists of those functions which belong to $L^p(\omega)$ together with their derivatives of order 1 and 2. For $p > n$, any function in $W^{2,p}_{\text{loc}}(\omega)$ coincides a.e. in ω with a function of class $C^{(1)}$. A continuous function u is called L-harmonic in ω if $u \in W^{2,p}_{\text{loc}}(\omega)$ with $n < p < \infty$ and $Lu = 0$ a.e. in ω. As Bony [3] has shown, we then get a harmonic sheaf satisfying the axioms of Brelot.

Let $\Omega \subset \Omega_0$ be a subdomain in which there exists a positive L-potential. In [2, Chap. 4], Boboc and Mustaţă construct for any given $y \in \Omega$ an L-potential P_y in Ω of support $\{y\}$ in the case when the coefficients of L are Hölder continuous. Their proof will carry over to our case, since the minimum principle and the existence and uniqueness theorems for the Dirichlet problem still hold there, as shown by Bony [3]. Boboc and Mustaţă assume that the coefficient a is non-positive, but this is no restriction because of a well-known argument found e.g. in [7, proof of Lemma 2]. Theorem 4.4 in [2] will also generalize, showing that, locally, L-potentials with supports consisting of one given point are proportional. Because of Theorem 16.4 in Hervé [5], this is then also true globally in Ω. If Ω is of class $C^{(2)}$, it follows, e.g. from an argument involving approximation of the coefficients of L, that $G^{\Omega}(x, y) = P_y(x)$ is the Green's function of L in Ω (cf. [7, proof of Lemma 2]). For any Ω, the function P_y defines a fundamental solution.

Using this P_y, we introduce the sheaf of L^*-harmonic functions as defined in Hervé [5]. It satisfies the axioms of Brelot. Since the behaviour of $P_y(x)$ as $x \to y$ is similar to that of the Newtonian kernel, the results of Hervé [5, p. 568] will carry over. This means that thinness of subsets of Ω and regularity of boundary points of open subsets in the L- or L^*-theory will coincide with the same notions in classical potential theory.

We then have the following results.

THEOREM 2. — *Let $\omega \subset \Omega$ be an open set. A function u is locally integrable in ω and satisfies $L^*u = 0$ or $L^*u \leqslant 0$ in the sense of $\mathcal{D}'(\omega)$ if and only if u coincides a.e. in ω with a function which is L^*-harmonic or L^*-superharmonic, resp., in ω.*

THEOREM 3. — *Let ω be a relatively compact subdomain of Ω of class $C^{(2)}$. Then the L^*-harmonic measure σ_y^{ω}, $y \in \omega$, of ω is given by*

$$d\sigma_y^{\omega}(x) = -\frac{\partial G^{\omega}(x, y)}{\partial \nu_x}\, dS_x$$

for $x \in \partial\omega$. The density $-\partial G^{\omega}(x, y)/\partial \nu_x$ is continuous and strictly positive on $\partial\omega$.

Here G^{ω} is the Green's function of L in ω, and $\partial/\partial\nu$ is the exterior conormal derivative on $\partial\omega$. Finally, dS is the area measure of $\partial\omega$.

THEOREM 4. — *Let* ω *be a relatively compact subdomain of* Ω_0 *of class* $C^{(2)}$. *Consider the problems*

$$(3.1) \quad u \in W^{2,p}(\omega), \qquad Lu = f \text{ a.e. in } \omega, \qquad u = 0 \text{ on } \partial\omega,$$

and

$$(3.2) \quad \nu \in L^1_{loc}(\omega), \qquad L^*\nu = g \quad \text{in the sense of } \mathscr{D}'(\omega), \quad \nu = \varphi$$

on $\partial\omega$, *where* $f \in L^p(\omega)$ *with* $p > n$, $g \in L^q(\omega)$ *with* $q > n/2$, *and* φ *is continuous on* $\partial\omega$. *Then either* (3.1) *and* (3.2) *are both uniquely solvable, or else the corresponding homogeneous problems have the same finite number of linearly independent solutions* u_i *and* ν_i, $i = 1, \ldots, m$. *In the second case* (3.1) *is solvable if and only if*

$$\int_\omega f\nu_i \, dx = 0, \; i = 1, \ldots, m,$$

and (3.2) *if and only if*

$$\int_\omega g u_i \, dx + \int_{\partial\omega} \varphi \frac{\partial u_i}{\partial \nu} \, dS = 0, \; i = 1, \ldots, m.$$

Under these hypotheses, any function in $W^{2,p}(\omega)$ has a well-defined trace on $\partial\omega$, so the boundary value condition in (3.1) makes sense, and so does that in (3.2), because any solution of $L^*\nu = g$ is continuous in ω if g is as in (3.2).

Indications of proof. — Using Theorem 1, we prove Theorem 2 by a method which is quite analogous to that of [7, Theorem 1]. The necessary Schauder estimates for Sobolev spaces can be found e.g. in Miranda [6, Sec. 37], and they are also used to prove the formula in Theorem 3, following the method of [7, Sec. 5]. The continuity of $d\sigma_y^\omega/dS$ is also obtained in this way, and to show that this density is strictly positive, we observe that Theorem 3, IV of [6] and its proof generalize to functions in $W^{2,p}$ for $p > n$. As to Theorem 4, finally, we can repeat the argument of [7, Sec. 6].

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