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ANALYTIC FUNCTIONS IN A LACUNARY END OF A RIEMANN SURFACE

by Zenjiro KURAMOCHI

*Dédié à Monsieur M. Brelot à l'occasion
 de son 70^e anniversaire.*

Let R be a Riemann surface and $\{R_n\}(n = 0, 1, 2, \dots)$ be its exhaustion. We suppose Kerékjártó-Stoilow's topology S is defined on $R + B$, where B is the set of all ideal boundary components. Also we suppose Martin's topology M is defined over $R - R_0 + \Delta$ as follows:

$$\text{dist}(p_1, p_2) = \sup_{z \in R_1 - R_0} \left| \frac{K(z, p_1)}{1 + K(z, p_1)} - \frac{K(z, p_2)}{1 + K(z, p_2)} \right|;$$

$p_1, p_2 \in R - R_0 + \Delta$, where $K(p_0, p) = 1$, $p_0 \in R_1 - R_0$ and Δ is the set of the ideal boundary points. We denote by Δ_1 the set of all minimal boundary points in Δ . Let $\mathfrak{p}$ be a boundary component. If there exists a sequence $\{z_i\}$ in $R - R_0$ such that $z_i \xrightarrow{S} \mathfrak{p}$ (convergence relative to S) and $z_i \xrightarrow{M} p$ (relative to M), we say p lies over $\mathfrak{p}$. We denote by $\nabla(\mathfrak{p})$ the set of Martin's points over $\mathfrak{p}$. Let G be an end of a Riemann surface R with null boundary. Let

$$F_i (i = 1, 2, \dots) (F_i \cap F_j = \emptyset \text{ for } i \neq j)$$

be a compact continuum in G such that $\{F_i\}$ clusters nowhere in R and $G' = G - F$ ($F = \sum_i F_i$) is connected.

We call G' a lacunary end. Let $\mathfrak{p}$ be a boundary component

of G . If there exists a determining sequence $v_n(p)$ of p such that $\partial v_n(p)$ is a dividing cut and

$$\overline{\lim}_n \min_{z \in \partial v_n(p)} G'(z, q_0) > 0 : q_0 \in G',$$

we say F is *completely thin* at p , where $G'(z, q_0)$ is a Green's function of G' . We proved.

THEOREM 1 [1]. — *Let G be an end of a Riemann surface R with null boundary. Let F be a completely thin set at a boundary component p . If there exists an analytic function*

$$\omega = f(z) : z \in G' = G - F$$

such that its value falls on the ω -sphere and

$$\sup_w n(\omega) = n_0 < \infty,$$

then $\Delta_1 \cap \nabla(p)$ consists of at most n_0 number of points, where $n(\omega)$ is the number of times ω is covered by G' . The purpose of the present paper is to extend Theorem 1.

Let $U(\omega)$ be a lower semicontinuous function and

$$U(\omega) \leq \frac{1}{2\pi} \int_{\partial C} U(\zeta) \frac{\partial}{\partial n} G(\zeta, \omega) ds$$

for any circle C in D , we call $U(\omega)$ a *quasisubharmonic* in D , where $G(\zeta, \omega)$ is a Green's function of C .

LEMMA 1. — *Let Ω be a domain in R with a relative boundary $\partial\Omega$ consisting of at most countably infinite number of analytic curves. Let E be a compact set on $\overline{\Omega}$ of positive capacity. Let $\omega = f(z)$ be an analytic function in $\Omega + E$ and $\{\Omega_n\} (n = 0, 1, 2, \dots)$ be an exhaustion i.e. $\overline{\Omega}_n$ is compact in Ω and $\partial\Omega_n$ consists of a finite number of analytic curves. Let $n_m^0(\omega)$ be the number of points in $f^{-1}(\omega) \cap \dot{E}_m$: $\dot{E}_m = \{z : \text{dist}(E, z) \leq \frac{1}{m}\}$. Suppose there exists a number m_0 such that $\sup_w n_{m_0}^0(\omega) = N < \infty$ and $\text{dist}(f(\Omega_n - \dot{E}_{m_0}), f(E)) > 0$ for any n . Let $u(z)$ be the harmonic measure of E relative to Ω and put $U(\omega) = \sum_i u(z_i) : f(z_i) = \omega$. Then $U(\omega)$ is quasisubharmonic in $C(f(E))$ and $U(\omega) \leq N$.*

Proof. — Let $E_m : m = 1, 2, \dots$, be a closed set such that $E_m \subset \dot{E}_m$, $\text{dist}(\partial E_m, E) > 0$, $E_m \downarrow E$ and ∂E_m consists of a finite number of analytic curves. Let $\{\Omega'_n\} : n = 1, 2, \dots$ be an exhaustion of $\Omega - E$ in the direction of $\partial\Omega - E$ satisfying conditions.

a) $\partial\Omega'_n - E_m$ is compact in Ω and $(\Omega'_n \cap E_m)$ consists of a finite number of components for any n and m and $\Omega'_n \nearrow \Omega - E$ as $n \rightarrow \infty$.

$$b) \quad \partial\Omega'_n \cap \partial\Omega = \partial\Omega \cap E$$

and $\partial\Omega'_n \cap \Omega \cap E = \Omega \cap E$ for any n .

$$c) \quad \partial\Omega'_n - \dot{E}_{m_0}^0 = \partial\Omega_n - \dot{E}_{m_0}^0$$

for any n . Since $f(z)$ is analytic on $\overline{\Omega'_n - E_m}$ by a)

$$\infty > n^* = \sup n(\omega)$$

in $f(\Omega'_n) - f(E_m)$, where $n(\omega)$ is the number of points in $\Omega'_n - E_m$ lying over ω . Let $u_n(z)$ be harmonic measure of E relative to Ω'_n . Then $u'_n(z) \nearrow u(z)$. Let Ω' be a domain such that $\Omega' \subset \Omega'_n$, $\Omega' \cap E_m = \Omega'_n \cap E_m$. Let $\Omega' \nearrow \Omega'_n$. Then $\sup |u_n(z) - u'(z)|$ on $\Omega' - E_m \leq \sup u_n(z)$ on $\partial\Omega' - E_m \rightarrow 0$, where $u'(z)$ is a harmonic measure of E relative to Ω' . Hence for any $\varepsilon > 0$ we can find a domain $\Omega' \subset \Omega'_n$ such that a') $\Omega' \cap E_m = \Omega'_n \cap E_m$. b') $f(\partial\Omega' - E_m)$ intersects itself at a finite number of points $\omega_1, \omega_2, \dots, \omega_k$. c'). Any subarc of $f(\partial\Omega' - E_m)$ is covered only once by $\partial\Omega' - E_m$ except $\sum_1^k \omega_i$ and

$$|u_n(z) - u'(z)| < \frac{\varepsilon}{n^*} \quad \text{in} \quad \Omega' - E_m.$$

Hence by a), b'), c') $f(\partial\Omega' - E_m)$ divides $C(f(E_m))$ into a finite number of domains $\omega_1, \omega_2, \dots, \omega_l$. Let $n(\omega)$ be the number of points in $\Omega' - E_m$ lying over ω . Then $n(\omega) = n_i$ in ω_i and $n(\omega)$ jumps 1 in crossing $f(\partial\Omega')$. Let D be a circle in $C(f(E_m))$.

Case 1. D is contained in ω_i , then

$$U'(\omega) = \sum u'(z_i) \quad (\omega) = f(z_i)$$

is harmonic in D .

Case 2. $D = (\omega_i + \omega_{i+1} + f(\partial\Omega')) \cap D$. Suppose

$$n_{i+1} \geq n_i.$$

Then $n_{i+1} = n_i + 1$. There exists a domain G' (or sum of domains denoted by G' also) of n_i leaves of disks and another domain G'' of one leaf such that $f(G'') = \omega_{i+1} \cap D$. Put $U''(\omega) = \Sigma u'(z_i) : f(z_i) = \omega : z_i \in G'$. Then $U''(\omega)$ is harmonic. Put $U'''(\omega) = u'(z) : \omega = f(z) : z \in G''$ and $U'''(\omega) = 0$ on $D - \omega_{i+1}$. Then since $u'(z) = 0$ on $\partial\Omega'$ and $U'''(\omega) = 0$ on $\partial\omega_i \cap D$ by $(\partial\omega_{i+1} \cap D) \subset f(\partial\Omega' - E_m)$ and

$$\partial\omega_i \cap f(E_m) = 0.$$

$U'''(\omega)$ is continuous and subharmonic in D and

$$U'(\omega) = U''(\omega) + U'''(\omega)$$

is continuous and subharmonic in D .

Case 3. $D - f(\Omega')$ consists of a finite number of domains, in this case similarly as before $U'(\omega)$ is continuous and subharmonic. Put $U'(\omega) = 0$ in $C(f(E_m))$. Then $U'(\omega)$ is continuous and subharmonic in $C(f(E_m))$. Now

$$0 \leq U_n(\omega) - U'(\omega) \leq \Sigma u_n(z_i) - u'(z_i) \leq n^* \cdot \frac{\varepsilon}{n^*} = \varepsilon$$

in $f(\Omega') - f(E_m)$. Let $\varepsilon \rightarrow 0$. Then $U'(\omega)$ uniformly $\rightarrow U_n(\omega)$ in $f(\Omega'_n) - f(E_m)$ and $U_n(\omega)$ is continuous and subharmonic in $C(f(E_m))$ by putting $U_n(\omega) = 0$ in $C(f(E_m))$. For any given number n by (c) there exists a number $m(n)$ such that

$$\text{dist}(f(\partial E_m) - f(E)) < \text{dist}((f(\Omega_n - \dot{E}_{m_0}), f(E)) : m > m(n).$$

Then $f^{-1}(\omega) \cap \Omega'_n$ (for $\omega \in \overline{f(E_m)}$ and $m > m(n)$) consists of only points in $\dot{E}_{m_0}$, whence $f^{-1}(\omega) : \omega \in \overline{f(E_m)} : m > m(n)$ consists of at most N number of points. By $u(z) \leq 1$, $U(\omega) \leq N$ in $\overline{f(E_m)}$. Since $U_n(\omega)$ is continuous and subharmonic in $C(f(E_m))$, by the maximum principle $U_n(\omega) \leq N$ in $C(f(E_m))$. Let $m \rightarrow \infty$. Then $U_n(\omega)$ is continuous and subharmonic and $\leq N$ in $C(f(E))$. Let $n \rightarrow \infty$. Then

$U_n(\omega) \nearrow U(\omega)$ by a) and $U(\omega)$ is lower semicontinuous and $\leq N$ in $C(f(E))$. Evidently

$$U(\omega) \leq \frac{1}{2\pi} \int_{\partial C} U(\zeta) \frac{\partial}{\partial n} G(\zeta, \omega) ds$$

for any circle C in $C(f(E))$. Hence $U(\omega)$ is quasisubharmonic and $\leq N$ in $C(f(E))$.

Lemma 1 is very simple but interesting. For example we apply it to the theory of value distribution. Then we have

PROPOSITION. — *Let $\omega = f(z)$ be an analytic function in $(0 <) r < |z| \leq 1$ such that $|f(z)| \leq 1$ and $f(e^{i\theta})$ covers $|\omega| = 1$ n_0 times as θ varies from 0 to 2π . Then $f^{-1}(\omega)$: $|\omega| < 1$ consists at most $m \left(\leq \frac{n_0}{\alpha} \right)$ number of points in $r^{1-\alpha} < |z| < 1$.*

In fact, let $\Omega = \{1 > |z| > r\}$ and $E = \{|z| = 1\}$. Then $u(z) = 1 - \frac{\log |z|}{\log r}$. Let $z_i (i = 1, 2, \dots)$ be a point in

$$r^{1-\alpha} < |z| < 1$$

such that $\omega = f(z_i)$. Then $u(z_i) > \alpha$ and

$$U(\omega) = \Sigma u(z_i) \leq n_0.$$

Hence we have the proposition.

LEMMA 2. — *Let Ω be a domain in $R - R_0$ and let*

$$F_i (i = 1, 2, \dots)$$

be a compact set clustering nowhere in R . Put

$$\Omega' = \Omega - F : F = \Sigma F_i.$$

Let $\varphi(\zeta)$ be a non negative continuous function on $\partial\Omega - F$. Let $U(z)$ be the least positive harmonic function in Ω' such that $U(\zeta) = \varphi(\zeta)$ on $\partial\Omega - F$. Then

$$U(z) = \frac{1}{2\pi} \int_{\partial\Omega - F} \varphi(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds,$$

where $G(\zeta, z)$ is a Green's function of Ω' .

Put $\Omega'_n = (\Omega' \cap R_n) - F$. Let $U_n(z)$ be a harmonic function in Ω'_n such that $U_n(z) = \varphi(\zeta)$ on $(\partial\Omega - F) \cap R_n = 0$ on $(F \cap \partial\Omega) + (\partial R_n \cap \Omega) + (F \cap \Omega)$. Then $U_n(z) \nearrow U(z)$. Let $G_n(\zeta, z)$ be a Green's function of Ω'_n . Then

$$U_n(z) = \frac{1}{2\pi} \int_{(\partial\Omega - F) \cap R_n} \varphi(\zeta) \frac{\partial}{\partial n} G_n(\zeta, z) ds.$$

Since $\frac{\partial}{\partial n} G_n(\zeta, z) \nearrow \frac{\partial}{\partial n} G(\zeta, z)$ on $\partial\Omega$, we have Lemma 2.

Let $D_1 \supset D_2$ be two domains. Let U be a positive harmonic function in D_1 . We denote by rU the greatest subharmonic function in D_2 vanishing on ∂D_2 not larger than U . Let V be a positive harmonic function in D_2 vanishing on ∂D_2 except at most a set of capacity zero. We denote by sV the least positive superharmonic function in D_1 larger than V . Then the following are well known.

rU and sV (for $sV < \infty$) are harmonic and $rsrU = rU$ and $srsV = sV$.

Let U be minimal in D_1 . Then if $rU > 0$, $srU = U$ and rU is minimal in D_2 .

Let V be minimal in D_2 . If $sV < \infty$, $rsV = V$ and sV is minimal in D_1 .

If $U_n \nearrow U$, $rU = \lim rU_n$.

In the sequel we suppose R is a Riemann surface with null boundary and Martin's topology M is defined over $R - R_0 + \Delta$ by $K(z, p) = \frac{G(z, p)}{G(p, p_0)}$, where p_0 is a fixed point in $R_1 - R_0$ and $G(z, p)$ is a Green's function of $R - R_0$. We remark there exist consts. M and N such that $M > G(z, p_0) > N > 0$ in $R - R_2$. Let G be an end in $R - R_0$ and let G' be a lacunary end such that

$$G' = G - F : F = \Sigma F_i.$$

Degree of irregularity $\delta(p)$ ($p \in G + \Delta_1$) at p . Let $G'(z, q_0)$ (q_0 is a fixed point in $G' \cap R_{n_0}$) be a Green's function of G' . We define $\delta(p)$ as follows:

$$\delta(p) = \overline{\lim}_{\substack{M \\ z \rightarrow p}} G'(z, q_0) : q_0 \in G' \cap R_{n_0}.$$

We see at once $\delta(p) > 0$ for $p \in G'$ and $\delta(p) > 0$ ($p \in \Delta_1$)

if and only if there exists a sequence $\{z_i\}$ such that $G'(z, z_i) \rightarrow$ a positive harmonic function. Let $\delta'(p)$ the one defined with respect to $G'(z, q_0)$. Then since there exists an exhaustion $\{R_n\}$ such that $\partial R_n \cap F = 0$, there exists a const. K such that

$$\frac{1}{K} \delta(p) \leq \delta'(p) \leq K \delta(p) \quad \text{for } p \in \Delta_1.$$

Let p^1 and p^2 in Δ_1 . If there exists a sequence of curves $\{\gamma_i\}$ ($i = 1, 2, \dots$) with two end points p_i^1 and p_i^2 such that $p_i^1 \xrightarrow{M} p^1, p_i^2 \xrightarrow{M} p^2$ and γ_i tends to the ideal boundary of R and

$$\overline{\lim}_{i=\infty} \min_{z \in \gamma_i} G'(z, q_0) > 0,$$

we say p^1 and p^2 are *chained*. Let p^0 and p^{n_0} . If there exists $p^1, p^2, \dots, p^{n_0-1}$ such that p^j and p^{j+1} are chained: $j = 0, 1, 2, \dots, n_0 - 1$, we say p^0 and p^{n_0} are *kindred*. It is clear if p^i and p^j are kindred, p^i and p^j lie over the same boundary component. Kindredness does not depend on the choice of q_0 .

Definition of $G(z, p)$ and $G'(z, p): p \in G + \Delta_1$.

LEMMA 3. — *Suppose Martin's topology is defined on*

$$R - R_0 + \Delta,$$

G is an end in $R - R_0$ and $G' = G - F$ be a lacunary end. Let $G(z, z_i)$ and $G'(z, z_i)$ be Green's functions of $R - R_0$ and G' respectively. Then:

1) Let $\{z_i\}$ be any sequence such that $z_i \xrightarrow{M} p \in G + \Delta_1$. Then $G(z, z_i)$ converges to a uniquely determined positive minimal harmonic function in $R - R_0$ denoted by $G(z, p)$ and $G(z, p) = \alpha K(z, p)$, where $\alpha = 2\pi \int_{\partial R_0} \frac{\partial}{\partial n} K(z, p) ds$ and $N' = \min_{z \in \partial R_1} G(z, p_0) < \alpha < M' = \max_{z \in \partial R_1} G(z, p_0)$.

2) Let $\{z_i\}$ be a sequence such that $z_i \xrightarrow{M} p \in G + \Delta_1$ and

$$G'(z, z_i) \rightarrow G'(z, \{z_i\}).$$

Then $G'(z, \{z_i\}) = \beta rG(z, p)$, with $0 \leq \beta < 1$ and where the operation r concerns domains $R - R_0$ and G' . Let $\{z_i\}$

be a sequence such that $z_i \xrightarrow{M} p \in G + \Delta_1$ and

$$G'(z_i, q_0) \rightarrow \delta(p) > 0.$$

Then $G'(z, z_i) \rightarrow$ a uniquely determined positive minimal harmonic function denoted by $G'(z, p)$ and

$$G'(z, p) \geq \frac{\delta(p)}{M} rG(z, p),$$

where $M = \max_{z \in \partial R_{n_0} + 1} G(z, q_0)$

Proof of 1. — Let $\{z'_i\}$ be a subsequence of $\{z_i\}$ such that $G(z, z'_i) \rightarrow$ a harmonic function $G(z)$. Then

$$G(z)/M' \leq K(z, p).$$

By the minimality of $K(z, p)$, $G(z) = \alpha K(z, p)$. On the other hand, by the compactness of ∂R_0 $\int_{\partial R_0} \frac{\partial}{\partial n} G(z) ds = 2\pi$. Hence $\alpha = 2\pi / \int_{\partial R_0} \frac{\partial}{\partial n} K(z, p) ds$. Now $\{z'_i\}$ is an arbitrary sequence for which $G(z, z_i)$ converges. Hence $G(z, z_i) \rightarrow$ a uniquely determined harmonic function denoted by $G(z, p)$.

Proof of 2. — Let $\{z_i\}$ be a sequence such that $G'(z, z_i) \rightarrow$ a harmonic function $G'(z, \{z_i\})$. Then by 1)

$$G'(z, \{z_i\}) \leq G(z, p)$$

and we have $G'(z, \{z_i\}) \leq rG(z, p)$. By the minimality of $rG(z, p)$ $G'(z, \{z_i\}) = \beta rG(z, p) : 0 \leq \beta < 1$. Let $\{z'_i\}$ be a subsequence of $\{z_i\}$ such that $G'(z, z'_i)$ converges and $\lim_i G'(z'_i, q_0) = \delta(p)$. In this case β attains the greatest value β^* given by

$$\delta(p)/rG(q_0, p)$$

Now $\{z'_i\}$ is an arbitrary subsequence with

$$\lim G'(z'_i, q_0) = \delta(p).$$

Hence $G'(z, z_i) \rightarrow$ a uniquely determined positive minimal harmonic function in $G' - p$ denoted by $G'(z, p)$. Now

$\lim_i G'(z_i, q_0) \geq \frac{\delta(p)}{M} G(q_0, p)$. Hence

$$G'(q_0, p) \geq \frac{\delta(p)}{M} G(q_0, p) \geq \frac{\delta(p)}{M} rG(q_0, p) \quad \text{and} \quad \beta^* \geq \frac{\delta(p)}{M}.$$

Thus we have 2).

We shall discuss the behaviour of Green's functions of a planar domain. Let Ω be a domain in the z -sphere such that Ω has a Green's function $G(z, p)$. Let t_0 be a fixed point in Ω and $\nu(t_0)$ be a neighbourhood of t_0 in Ω and put $\delta(p) = \overline{\lim}_{z \rightarrow p} G(z, t_0) : p \in \bar{\Omega}$. Then $\delta(p)$ is upper semicontinuous in $\bar{\Omega}$ and $\delta(p) \leq \max_{z \in \partial\nu(t_0)} G(z, t_0)$. We see $\delta(p) > 0 : p \in \partial\Omega$ if and only if p is irregular. We introduce Martin's topology over $\Omega + \Delta$ by $K(z, p^M) : p^M \in \Omega + \Delta$ with $K(t_0, p^M) = 1$. By Brelot's theorem [2] there exists only one point p^M on p for $\delta(p) > 0$ and p^M is minimal. We denote by $p^M = \varphi(p)$. Then also this implies $\varphi(p)$ is continuous at p with $\delta(p) > 0$. Clearly $K(z, p^M)$ is continuous with respect to p^M . Hence $K(z, \varphi(p))$ is continuous at p with $\delta(p) > 0$ and we denote p^M by p simply in the following. Let $\{z_i\}$ be a sequence such that $z_i \rightarrow p$, $G(z_i, t_0) \geq \varepsilon_0 > 0$. Then there exists a subsequence $\{z'_i\}$ with $G(z, z'_i) \rightarrow$ a harmonic function $G(z)$. Then $G(z) \leq \frac{M}{\varepsilon_0} K(z, p) : M = \max_{z \in \partial\nu(t_0)} G(z, t_0)$. By the minimality of $K(z, p)$ $G(z) = \alpha K(z, p) : 0 < \alpha < \infty$. Let $\{z_i\}$ be a sequence such that $z_i \rightarrow p$, $G(z_i, t_0) \rightarrow \delta(p)$. Then we see easily $G(z, z_i) \rightarrow$ a harmonic function $\tilde{G}(z) = \bar{\alpha} K(z, p)$ and $\bar{\alpha}$ is the maximal value and $\tilde{G}(z)$ is the limit of $\{G(z, z_i)\}$ such that $z_i \rightarrow p$ and

$$G(z_i, z) \rightarrow \overline{\lim}_{w \rightarrow p} G(w, z)$$

for any z . We make $\tilde{G}(z)$ correspond to p and denote it by $G(z, p) : p \in \partial\Omega : \delta(p) > 0$. Thus the domain of definition of p of $G(z, p)$ is extended to $\Omega + \{p \in \partial\Omega : \delta(p) > 0\}$: This fact means $G(z, p)$ is upper semicontinuous with respect to p . Let $p \in \bar{\Omega}$ with $\delta(p) > 0$. Then by $K(t_0, p) = 1$ we have $G(z, p) = \delta(p)K(z, p)$. Let μ be a positive mass distribution

over $\Omega + \{p \in \partial\Omega : \delta(p) > 0\}$: Then a potential

$$\int G(z, q) d\mu(q) \quad \text{and} \quad \delta(q)\mu(q)$$

are defined well. Then we have

LEMMA 4. — 1) Let $\{z_i\}$ be a sequence such that $z_i \rightarrow p$ and $G(z, z_i) \rightarrow$ a harmonic function $G(z, \{z_i\})$. Then

$$G(z, \{z_i\}) \leq G(z, p).$$

2) Let $\nu(p)$ be a neighbourhood of p , then there exists a const. L such that $G(z, p) < L$ on $C \nu(p)$.

3) Let $U(z)$ be a potential $U(z) = \int_{\bar{\Omega}} G(z, q) d\mu(q)$ and $\int d\mu(q) < \infty$.

If $G(z, p) \leq U(z)$, $\lim_n \int_{\nu_n(p)} d\mu(q) \geq 1$:

$$\nu_n(p) = \left\{ |z - p| < \frac{1}{n} \right\}.$$

Proof. — 1) is evident. We shall prove 2). Let $\{z_i\}$ be a sequence such that $G(z_i, t_0) \rightarrow \delta(p)$ and $z_i \rightarrow p$. Then $G(z, z_i) \rightarrow G(z, p)$. Let $\nu'(t_0) = \{z : |z - t_0| < r'\}$ such that $\nu'(t_0) \in \Omega$. Let $G'(z, z_i)$ be a Green's function in $\Omega - \nu'(t_0)$. Let $H_i(z)$ be the least positive harmonic function in

$$\Omega - \nu'(t_0)$$

such that $H_i(z) = G(z, z_i)$ on $\partial\nu'(t_0)$. Then

$$G(z, z_i) - H_i(z) = G'(z, z_i).$$

Since $\partial\nu'(t_0)$ is compact, $H_i(z) \rightarrow H(z)$, where $H(z)$ is the least positive harmonic function in $\Omega - \nu'(t_0)$ such that $H(z) = G(z, p)$. Whence $G'(z, z_i) \rightarrow$ a uniquely determined function denoted by $G'(z, p)$. On the other hand, there exists no singular minimal point on planar domains (this is equivalent to there exists no bounded minimal positive harmonic function). Whence $\sup_z G(z, p) = \infty$. But

$$\sup_z H(z) \leq \max_{z \in \partial\nu(t_0)} G(z, p).$$

Hence $G'(z, p) > 0$. Let $\nu(t_0) = \{|z - t_0| < r\}$ such that $r > r'$ and $\nu(t_0) \subset \Omega$. Now

$$\min_{z \in \partial\nu(t_0)} G'(z, z_i) = N'_i \rightarrow N' = \min_{z \in \partial\nu(t_0)} G'(z, p)$$

and $\max_{z \in \partial\nu(t_0)} G(z, z_i) = M_i \rightarrow M = \max_{z \in \partial\nu(t_0)} G(z, p)$. Clearly

$$M_i \geq N'_i, G'(z, z_i)/N'_i \quad \text{and} \quad G(z, z_i)/M_i$$

have log singularities with coefficients $1/N'_i$ and $1/M_i$ respectively and $G'(z, z_i)/N'_i \geq G(z, z_i)/M_i$ on $\partial\nu(t_0)$. Hence by the maximum principle and by letting $i \rightarrow \infty$ we have $G'(z, p)/N' \geq G(z, p)/M$ in $C\nu(t_0)$. Let $\tilde{\Omega} = z\text{-sphere} - \nu'(t_0)$ and let $\tilde{G}(z, p)$ be a Green's function of $\tilde{\Omega}$. Then

$$\tilde{G}(z, p) \geq G'(z, p).$$

Clearly there exists a const. L such that

$$LN'/M \geq \tilde{G}(z, p) \geq G'(z, p)$$

on $C\nu(p)$. Hence $L \geq G(z, p)$ on $C\nu(p)$ for any neighbourhood $\nu(p)$. Hence we have 2).

Proof of 3). *Case 1.* $p \in \Omega$. Let $\nu(p) = \{z: |z - p| < 1/n_0\}$ such that $\nu(p) \subset \Omega$. Then $G(z, q) = G'(z, q) + H(z, q)$ or $G(z, q)$ according as $q \in \nu(p)$ or $q \notin \nu(p)$, where $G'(z, q)$ is a Green's function of $\nu(p)$ and $H(z, q)$ and $G(z, q): q \notin \nu(p)$ are least positive harmonic functions in $\nu(p)$ such that $H(z, q) = G(z, q)$ and $G(z, q) = G(z, q)$ on $\partial\nu(p)$. Since for any q and any neighbourhood $\nu(q)$ there exists a const. $L(q, \nu(q))$ such that $G(z, q) < L(q, \nu(q))$ on $C\nu(q)$ and since $\partial\nu(p)$ is compact, there exists a const. L such that $H(z, q) \leq L$ and $G(z, q)(q \notin \nu(p)) \leq L$ on

$$\nu_{n_1}(p) \subset \nu(p): n_1 > n_0.$$

Hence:

$$\begin{aligned} G'(z, p) \leq G(z, p) \leq U(z) &= \int_{\nu(p)} G'(z, q) d\mu(q) \\ &+ \int_{\nu(p)} H(z, q) d\mu(q) + \int_{C\nu(p)} G(z, q) d\mu(q) \\ &\leq \int_{\nu(p)} G'(z, q) d\mu(q) + L \int d\mu \quad \text{in } \nu_{n_1}(p). \end{aligned}$$

Let $H_n(z)$: $n > n_0$, be a harmonic function in $\nu(p) - \nu_n(p)$ such that $H_n(z) = 0$ on $\partial\nu(p)$ and $= L'$ on $\partial\nu_n(p)$: $L' = L \int d\mu$. Then $U(z) \leq \int_{\nu(p)} G'(z, q) d\mu(q) + H_n(z)$ on $\partial\nu_n(p)$. Since $G'(z, p) = 0$ on $\partial\nu(p)$,

$$G'(z, p) \leq \int_{\nu(p)} G'(z, q) d\mu(q) + H_n(z)$$

on $\partial\nu(p) + \partial\nu_n(p)$. $G'(z, p)$ is harmonic and

$$\int_{\nu(p)} G'(z, q) d\mu(q) + H_n(z)$$

is superharmonic in $\nu(p) - \nu_n(p)$. By the maximum principle and by letting $n \rightarrow \infty$ $G'(z, p) \leq \int_{\nu(p)} G'(z, q) d\mu(q)$. This implies $1 \leq \int_{\nu(p)} d\mu(q)$. Now $\nu(p)$ is arbitrary and

$$\lim_n \int_{\nu_n(p)} d\mu \geq 1.$$

Case 2. $p \in \partial\Omega$. In this case p is irregular. Let $q \in \partial\Omega$ and $G(z, q) > 0$. Then by definition

$$G(z, q) = \lim_i G(z, q_i),$$

where $q_i \in \Omega$, $q_i \rightarrow q$, $G(q_i, t_0) \rightarrow \delta(q) = \overline{\lim_{z \rightarrow q}} G(z, t_0)$. Hence $G(z, q) = \delta(q)K(z, q)$: $K(t_0, q) = 1$. Since the greatest subharmonic minorant of Green's potential (in ordinary sense) $= 0$, we have by

$$\begin{aligned} G(z, p) &\leq \int_{\partial\Omega} G(z, q) d\mu(q) + \int_{\Omega} G(z, q) d\mu(q) \\ G(z, p) &\leq \int_{\partial\Omega} G(z, q) d\mu(q) \text{ i.e.} \\ \delta(p)K(z, p) &\leq \int_{\Delta_1} \delta(q)K(z, q) d\mu(q). \end{aligned}$$

Now $\delta(q)\mu(q)$ is a canonical representation, hence

$$\delta(p) \leq \delta(q)\mu(q)$$

and $\delta(p) \leq \int_{\nu_n(p)} d(\delta(q)\mu(q))$. On the other hand, $\delta(q)$ is upper semicontinuous. Hence $\lim_n \int_{\nu_n(p)} d\mu(q) \geq 1$.

Suppose an analytic function in a lacunary end G' :

$$G' = G - F$$

of a Riemann surface R with null boundary such that

$$\omega = f(z) : z \in G'$$

falls on the ω -sphere. We investigate the behaviour of $f(z)$ and the structure of Δ_1 . Then

THEOREM 2. — Suppose $f(G')$ does not cover a set E of positive capacity. Then:

1) Let $\{z_i\}$ be a sequence in G' such that $z_i \xrightarrow{M} p \in \Delta_1$ and $\varliminf_i G(z_i, q_0) > 0$. Then $f(z_i) \rightarrow$ a point not depending on the choice of $\{z_i\}$. We denote it by $f(p)$.

2) If $\Delta_1 \cap \nabla(p)$ consists of at most countably infinite number of points $\{p_i\}$ and $\delta(p_i) \geq \delta > 0$ for any i . Then $f(p_i) = f(p_j)$ for two kindred points p_i and p_j in $\Delta_1 \cap \nabla(p)$.

Proof of 1). — The complementary set of E relative to the ω -sphere consists of domains. Let Ω be the one containing $f(G')$. Let $G^w(\omega, \omega_0)$ be a Green's function of Ω . Then $G^w(f(z), f(z_i)) \geq G'(z, z_i)$. Consider a sequence $\{z_i\}$ such that $z_i \xrightarrow{M} p$ and $\varliminf_i G'(z_i, q_0) > 0$. Assume $f(z_i)$ has two limiting points t_l : $l = 1, 2$, $t_1 \neq t_2$ on $\Omega + \partial\Omega$. Then we can find two subsequences $\{z'_i\}$ of $\{z_i\}$ such that $z'_i \xrightarrow{M} p$, $f(z'_i) \rightarrow t_l$, $G'(z, z'_i) \rightarrow$ a harmonic function $G''(z)$ in G' and $G^w(\omega, f(z'_i)) \rightarrow$ a positive harmonic function $G(\omega, \{f(z'_i)\})$ in $\Omega - t_l$. Then by Lemma 4

$$(1) \quad 0 < G''(z) \leq G^w(\omega, \{f(z'_i)\}) \leq G^w(\omega, t_l) : \omega = f(z),$$

where $G^w(\omega, t_l)$ is the function defined in Lemma 4. Now by Lemma 3.2, $G''(z) = \alpha_l i_G^{R_0} G(z, p) : 0 < \alpha_l < \infty$, whence

$$(2) \quad G^1(z) = bG^{2'}(z) : 0 < b < \infty.$$

Let $\nu(t_l)$ be a neighbourhood of t_l such that

$$\overline{\nu(t_1)} \cap \overline{\nu(t_2)} = 0.$$

Then by Lemma 4 $G^w(\omega, t_l)$ is bounded in $C\nu(t_l)$. Whence by (2) and (1) $G^1(z)$ and $G^{2'}(z)$ are bounded in G' and $G^1(z) = G^{2'}(z) = 0$. This is a contradiction. Hence we have 1). Because the following fact is well known: Let ω be a domain

in a Riemann surface with null boundary and let $U(z)$ be a bounded harmonic function in ω with $U(z) = 0$ on $\partial\omega$, then $U(z) = 0$.

For the proof of 2) we use following:

PROPOSITION 1. — Let $\{z_i\}$ be a sequence in G' such that $z_i \xrightarrow{s} p$, $f(z_i) \rightarrow \omega^* \in \Omega + \partial\Omega$ satisfying.

a) $G'(z, z_i) \rightarrow$ a positive harmonic function $G'(z, \{z_i\})$ in G' .

b) $G^w(\omega, f(z_i)) \rightarrow$ a positive harmonic function $G^w(\omega, \{f(z_i)\})$ in $\Omega - \omega^*$. Let $\dot{E}G'(z, \{z_i\})$ be the least positive superharmonic function in Ω larger than $G'(z, \{z_i\})$. Then

$$\dot{E}G'(z, \{z_i\})$$

is minimal in $\Omega - \omega^*$ and $\dot{E}G'(z, \{z_i\}) = \alpha G^w(\omega, \omega^*)$: $0 < \alpha \leq 1$.

In fact by Lemma 4

$$G'(z, \{z_i\}) \leq G^w(f(z), \{f(z_i)\}) \leq G^w(f(z), \omega^*)$$

and by the minimality of $G^w(\omega, \omega^*)$ $\dot{E}G'(z, \{z_i\}) = \alpha G^w(\omega, \omega^*)$ and $\dot{E}G'(z, \{z_i\})$ is minimal in $\Omega - \omega^*$.

PROPOSITION 2. — Let $\{z_i\}$ be a sequence $z_i \xrightarrow{M} p \in G' + \Delta_1$ such that $G'(z_i, q_0) \rightarrow \delta(p) > 0$. Then by Lemma 3

$$G'(z, z_i) \rightarrow G'(z, p), \quad G(z, z_i) \rightarrow G(z, p).$$

By 1) of Theorem 2 $f(z_i) \rightarrow f(p)$ and by Lemma 4

$$G'(z, p) \leq \overline{\lim}_i G^w(\omega, f(z_i)) \leq G^w(\omega, f(p)):$$

$$p \in \Delta_1 + G', \quad \delta(p) > 0, \quad f(z) = \omega.$$

Proof of 2) — By 1) of Theorem 2 $f(p_i)$: $p_i \in \Delta_1 \cap \nabla(p)$ is defined. Let p_1 and p_2 be two points chained. Assume $f(p_1) \neq f(p_2)$. We can find a circle

$$C = \{\omega : |\omega - f(p_1)| < \frac{1}{2} |f(p_1) - f(p_2)|\}$$

and $C \cap \sum_{i=1}^{\infty} f(p_i) = 0$. By the definition there exists a sequence of curves γ_n with endpoints z_n^1 and z_n^2 such that

$z_n^1 \xrightarrow{M} p_1$, $z_n^2 \xrightarrow{M} p_2$ and $G'(z, q_0) \geq \delta_0 > 0$ on γ_n . Whence $f(z_n^1) \rightarrow f(p_1)$, $f(z_n^2) \rightarrow f(p_2)$. Consider $f(\gamma_n)$. Then $f(\gamma_n)$ intersects C for $n > n_0$. Let ω_n be one of intersecting points with $f(z_n) = \omega_n$ and $z_n \in \gamma_n$. Since C is compact, there exists a limiting point ω^* of $\{\omega_n\}$:

$$\omega^* \in \Omega + \partial\Omega, \quad \omega^* \cap \sum_{i=1} f(p_i) = 0.$$

Hence we can find a subsequence $\{z_m\}$ of $\{z_n\}$ such that $z_m \subset \gamma_m$, $z_m \xrightarrow{s} p$, $f(z_m) \rightarrow \omega^*$ and $G'(z, z_m) \rightarrow$ a positive harmonic function $G'(z) \leq G^w(f(z), \omega^*)$ by Proposition 1) and $G(z, \{z_m\}) \rightarrow$ a positive harmonic function $G(z)$ in $R - R_0$ by $\int_{\partial R_0} \frac{\partial}{\partial n} G(z) ds = 2\pi$.

Suppose Martin's topology is defined over $R - R_0 + \Delta$ by $K(z, p)$: $K(p_0, p) = 1$ and $p_0 \in R_1 - R_0$. Then $G(z)$ is represented by a canonical distribution over $\Delta_1 \cap \nabla(p)$, i.e. $G(z) = \sum_i a_i K(z, p_i)$, where $\sum_i a_i = G(p_0) < \infty$. By Lemma 3 $K(z, p_i) = G(z, p_i)/\alpha(p_i)$ and

$$G'(z) \leq G(z) = \sum_i (a_i/\alpha(p_i))G(z, p_i) \leq N'^{-1} \sum_i a_i G(z, p_i),$$

where $N' \leq \alpha(p_i)$ and $N' = \min_{z \in \partial R_1} G(z, p_0)$. Put

$$U_n(z) = \sum_{i=1}^n (a_i/\alpha(p_i))G(z, p_i).$$

Then $U_n(z) \nearrow G(z)$. By proposition 2

$$\begin{aligned} 0 < G'(z) &\leq rG(z) \leq \sum_i \frac{a_i}{N'} rG(z, p_i) \\ &\leq (M/N') \sum_i (a_i/\delta(p_i))G'(z, p_i) \leq (M/N'\delta) \sum_i a_i G'(z, p_i) \\ &\leq (M/N'\delta) \sum_i a_i G^w(\omega, f(p_i)), \end{aligned}$$

where $M = \max_{z \in \partial R_{n_0+i}} G(z, q_0) : \omega = f(z)$.

By proposition 1

$$\overset{*}{E}G'(z) = \alpha G^w(\omega, \omega^*) \leq (M/N'\delta) \sum a_i G^w(z, f(p_i)) < \infty.$$

Now $\omega^* \in C$ and $C \cap \sum_i f(p_i) = 0$. $G^w(\omega, \omega^*)$ has mass at

ω^* , on the other hand, the term on the right hand has no mass at ω^* . This contradicts 3) of Lemma 4. Hence

$$f(p_1) = f(p_2) \quad \text{and} \quad f(p_i) = f(p_j)$$

for kindred points p_i and p_j . Thus we have 2).

THEOREM 3. — *If the spherical area of $f(G')$ $< \infty$, then:*

1) *Let $\{z_i\}$ be a sequence in G' such that $z_i \xrightarrow{M} p$ and $\lim G'(z_i, q_0) > 0$. Then $f(z_i) \rightarrow a$ point not depending on the choice of $\{z_i\}$. We denote it by $f(p)$.*

2) *Let p^1 and p^2 be two chained points. Then $f(p^1) = f(p^2)$. Hence $f(p^i) = f(p^j)$ for two kindred points p^i and p^j .*

At first we define $\tilde{R}$ and $\tilde{G}$ as follows. Since spherical area of $f(G') < \infty$, we can find a number n_0 such that spherical area of $f(G' \cap (R - R_{n_0})) < \frac{\pi}{4}$ and $\partial R_{n_0} \cap F = 0$.

Evidently $f(G' \cap (R - R_{n_0}))$ does not cover a set E of positive capacity. Now $f(z)$ is analytic on ∂R_{n_0} ,

$$G' \cap (R - R_{n_0})$$

consists of a finite number of components $G'_1, \dots, G'_k$ and ∂R_{n_0} consists of $\partial R_{n_0}^1, \dots, \partial R_{n_0}^k$. We can find an arc Γ_j on $\partial R_{n_0}^j$ such that $f(\Gamma_j)$ is a simple arc,

$$f(\Gamma_i) \cap f(\Gamma_j) = 0 : i \neq j$$

and $\sum_j f(\Gamma_j) \cap E = 0$. Let $\mathcal{F}$ be the whole ω -sphere. Put $\mathcal{F}' = \mathcal{F} - \sum f(\Gamma_j)$. Connect $\mathcal{F}'$ with $G'_1, \dots, G'_k$ at an adequate side of $f(\Gamma_j)$ with Γ_j of G'_j so that

$$\mathcal{F}' + \sum^k f(G'_j)$$

may be a connected covering surface. By deforming R_{n_0}

$$\mathcal{F}' + \sum^k G'_j$$

can be considered a domain and $\mathcal{F}' + G \cap (R - R_{n_0})$ can be considered an end $\tilde{G}$ of another Riemann surface $\tilde{R}$ with null boundary. Now $\partial \tilde{G}$ consists of $\partial R_{n_0} - \sum^k \Gamma_j$

and the other side of $f(\Gamma_j)$ ($j = 1, 2, \dots, k$) where $\mathcal{F}'$ and G'_j are not connected. Put $\tilde{G}' = \mathcal{F}' + (G' \cap (R - R_{n_0}))$. Then $f(z)$ can be continued analytically into $\mathcal{F}'$ by putting $f(z) =$ projection of z over w -sphere, which we also denote by $f(z) : z \in \tilde{G}'$. Let $\tilde{p}_0$ and $\tilde{q}_0$ be points in $G' \cap (R - R_{n_0})$. Then Martin's topology M will be defined over $\tilde{G}$. Then $\tilde{M}$ -top. and M -top (given originally on $R - R_0 + \Delta$) are isomorphic on $(R - R_{n_0}) + \Delta$ and the minimality does not change. Also let $\tilde{G}'(z, q_0)$ be a Green's function of $\tilde{G}' : \partial R_n \cap F = 0$, there exists a const. K such that

$$G'(z, q_0)/K \leq \tilde{G}'(z, \tilde{q}_0) < KG'(z, q_0)$$

in $G' \cap (R - R_{n_0})$ and $k^{-1}\delta(p) < \tilde{\delta}(p) < K\delta(p)$ for $p \in \Delta_1$, where $\tilde{\delta}(p)$ is defined in $\tilde{G}'$ relative to $\tilde{q}_0$. Put

$$*\tilde{G}' = \tilde{G}' - E_{\tilde{\mathcal{F}}},$$

then $f(*\tilde{G}')$ does not cover a set E of positive capacity, where $E_{\tilde{\mathcal{F}}}$ is the set of $\mathcal{F}$ over E .

Proof of 1). — So long as we investigate $f(z)$ in a neighbourhood of the ideal boundary of R , we can consider $*\tilde{G}'$ instead of G' . Then we have at once 1) by 1) of Theorem 2.

Proof of 2). — For the purpose we consider only

$$G' \cap (R - R_{n_0})$$

such that spherical area of $f(G' \cap (R - R_{n_0})) < \frac{\pi}{4}$.

$$G \cap (R - R_{n_0})$$

consists of a finite number of ends. Let $*G$ be one of them and put $*G' = *G - F$. Let $*G'(z, q'_0)$ be Greens function of $*G'$. Then there exists a const. K such that

$$(3) \quad \frac{1}{K} G'(z, q_0) \leq *G'(z, q'_0) \leq KG'(z, q_0)$$

in $G' \cap (R - R_{n_0})$, where q_0 and $q'_0 \in G' \cap R_{m_0-1}$.

Hence $\overline{\lim}_n \min_{z \in \gamma_n} {}^\bullet G'(z, q'_0) > 0$ for $\{\gamma_n\}$ defining chainedness of points. Hence for simplicity we denote ${}^\bullet G$, ${}^\bullet G'$, ${}^\bullet G'(z, q'_0)$ by G , G' and $G'(z, q'_0)$.

By Evans's [5] theorem there exists a positive harmonic function $U(z)$ in $G' = G - F$ such that

$$1) \ U(z) = 0 \text{ on } \partial G + F, \ D(\min(M, U(z))) = 2\pi M,$$

$$\int_{\partial \Omega_L} \frac{\partial}{\partial n} U(z) \, ds = 2\pi$$

for almost $L < \infty$, where $\Omega_L = \{z \in G' : U(z) > L\}$.

$$2) \ U(z) \rightarrow \infty \text{ as } z \rightarrow B \text{ in any}$$

$$G_\delta = \{z \in G' : G'(z, q_0) > \delta\} :$$

$\delta > 0$. Ω_L consists of at most countably number of domains. Let Ω'_L be one component of Ω_L . Then Ω'_L is a domain in a surface with null boundary, whence $\sup_{z \in \Omega'_L} U(z) = \infty$. Since spherical area of $f(G') < \frac{\pi}{4}$, by 1) we see by length and area's method there exists a sequence $L_i : i = 1, 2, \dots$ such that $L_i \nearrow \infty$ and spherical length of

$$f(\partial \Omega_{L_i}) = \varepsilon_i \rightarrow 0 \text{ as } i \rightarrow \infty.$$

Let $\{\gamma_n\}$ be a sequence of curves defining the chainedness of p_1 and p_2 . Then $\overline{\lim}_n \min_{z \in \gamma_n} G'(z, q_0) > 0$, and there exists a subsequence $\{\gamma_m\}$ of $\{\gamma_n\}$ such that

$$\min_{z \in \gamma_m} G'(z, q_0) > \delta > 0.$$

Since $\gamma_m \rightarrow$ boundary of R , by 2) for any given L_i , there exists a number $m(L_i)$ such that $U(z) > L_i$ on

$$\gamma_m : m > m(L_i).$$

Hence for any L_i there exists $m(L_i)$ such that

$$\Omega_{L_i} \supset \gamma_m : m > m(L_i)$$

and there exists only one component $\Omega'_i(\gamma_m)$ of Ω_{L_i} such that $\Omega'_i(\gamma_m) \supset \gamma_m$ where $\Omega'_i(\gamma_m)$ depends on γ_m .

By Evans's theorem there exists a harmonic function $V(z)$ in G such that

$$1) V(z) = 0 \text{ on } \partial G, D(\min(M, V(z))) = 2\pi M,$$

$$\int_{\partial D_M} \frac{\partial}{\partial n} V(z) ds = 2\pi$$

for M , where $D_M = \{z \in G : V(z) < M\}$.

2) $V(z) \rightarrow \infty$ as $z \rightarrow$ boundary of R . Similarly as $U(z)$, there exists a sequence M_j such that spherical length of

$$f(\partial D_{M_j} \cap G) = \varepsilon_j \rightarrow 0$$

as $j \rightarrow \infty$. Since $\Omega'_i(\gamma_m) = \lim_j \Omega'_i(\gamma_m) \cap D_{M_j}$, there exists a number M_j such that $\Omega'_i(\gamma_m) \cap D_{M_j} \supset \gamma_m$. Put

$$\Omega'_{i,j}(\gamma_m) = \Omega'_i(\gamma_m) \cap D_{M_j}.$$

Since $\Omega'_{i,j}(\gamma_m)$ is compact boundary of

$$f(\Omega'_{ij}(\gamma_m)) \subset f(\partial \Omega'_{ij}(\gamma_m))$$

and the spherical length of $f(\partial \Omega'_{ij}(\gamma_m)) < \varepsilon_i + \varepsilon_j$. $f(\partial \Omega'_{ij}(\gamma_m))$ divides the ω -sphere into a number of domains $G_1^w, G_2^w, \dots$.

Since the spherical length of $f(\partial \Omega'_{ij}(\gamma_m)) < \varepsilon_i + \varepsilon_j < \frac{1}{4}$, there exists only one domain with spherical area

$$\geq 4\pi - (\varepsilon_i + \varepsilon_j)^2.$$

We denote such domain by $\tilde{G}$. Then since spherical area of $f(\Omega'_{ij}(\gamma_m)) < \frac{\pi}{4}$, $f(\Omega'_{ij}(\gamma_m)) \cap \tilde{G} = 0$ and $f(\Omega'_{ij}(\gamma_m))$ is contained in a semisphere and the spherical diameter of $f(\gamma_m) \leq$ spherical diameter of $f(\Omega'_{ij}(\gamma_m)) < \varepsilon_i + \varepsilon_j$.

Let $j \rightarrow \infty$.

Then spherical diameter of $f(\gamma_m) < \varepsilon_i$. Let z_m^1 and z_m^2 be endpoints of γ_m . Then $fz_m^1 \rightarrow f(p_1)$ and $fz_m^2 \rightarrow f(p_2)$ as $m \rightarrow \infty$. Let $m \rightarrow \infty$ and then $i \rightarrow \infty$. Then $f(p_1) = f(p_2)$ for chained points p_1 and p_2 . Thus we have 2).

THEOREM 4. — Let $\tilde{G} \supset G$ be two ends of a Riemann surface R with null boundary and let $G' = G - F$ be a

lacunary domain. Let E_z be a compact set of positive capacity in $\tilde{G} - G$. Suppose an analytic function $f(z)$ in $\tilde{G} - F$ and there exists a neighbourhood $\nu(E_z)$ of E_z with the property: $f(z)$ is univalent in $\nu(E_z)$, $f(\tilde{G} - F - \nu(E_z))$ does not cover $E = f(E_z)$ (clearly E is of positive capacity). Let $u(z)$ be a harmonic measure of E_z with respect to $\tilde{G} - F - E_z$. Suppose Martin's topology M is defined over $R = R_0 + \Delta$. Let $G'(z, q_0)$ be a Green's function of G' and let

$$\delta(p) : p \in G' + \Delta_1 = \overline{\lim_{z \xrightarrow{M} p}} G'(z, q_0) : q_0 \in G' \cap R_{n_0}.$$

Then by theorem 2 $f(z_i) \rightarrow f(p)$ for $z_i \xrightarrow{M} p$ and

$$\underline{\lim} G'(z_i, q_0) > 0.$$

Then

1) Let $\{z_i\}$ be a sequence such that $z_i \xrightarrow{M} p \in \Delta_1$ and $\underline{\lim} G'(z_i, q_0) > 0$. Then $f(z_i) \rightarrow f(p)$ and there exists a uniquely determined connected piece ω over $|\omega - f(p)| < r$ such that $f(z_i) \in \omega$ for $i \geq i_0$ and $f(z_i) \rightarrow f(p)$.

2) Let $u(p) = \overline{\lim_{z \xrightarrow{M} p}} u(z)$ $p \in \Delta_1$. Let $\{z_i\}$ be a sequence such that $z_i \xrightarrow{M} p$ and $u(z_i) \rightarrow u(p) > 0$. Let $G^\omega(z, z_i)$ be a Green's function of ω . Then $G^\omega(z, z_i) \rightarrow$ a unique positive minimal harmonic function $G^\omega(z, p)$ and

$$u(p) = \int_{\partial^1 \omega} u(\zeta) \frac{\partial}{\partial n} G^\omega(\zeta, p) ds,$$

where $\partial^1 \omega$ is the part of $\partial \omega$ such that

$$f(\partial^1 \omega) \subset \{\omega : |\omega - f(p)| = r\}.$$

3) Let ω be a point and let $p_i \in \Delta_1$ with $\delta(p_i) > 0$ and $f(p_i) = \omega$ and let $q_j \in \tilde{G}' = \tilde{G} - F - E_z$ with $f(q_j) = \omega$. Then

$$\Sigma u(p_i) + \Sigma u(q_j) \leq 1 \quad \text{for any } \omega.$$

Case 1. $\omega_0 \in E$. Let $0 < r < \text{dist}(\omega_0, f(\partial \nu(E_z)))$ (> 0 by the univalence of $f(z)$ in $\nu(E_z)$). The part of $\tilde{G} - F - E_z$ over $|\omega - \omega_0| < r$ consists of a most countably infinite

number of domains (connected pieces). Let $\{\omega'\}$ be the set of connected pieces contained in $\nu(E_z)$ and let $\omega_i: i = 1, 2, \dots$ be pieces except $\{\omega'\}$. Then $\omega_i \cap \nu(E_z) = 0$. By the assumption, there exists no point z in $\tilde{G} - E_z - F$ such that $f(z) = \omega_0$. Further let $p \in \Delta_1$, then for any sequence $\{z_i\}$ with $z_i \rightarrow p$, there exists a number i_0 such that $z_i \notin \nu(E_z)$ for $i > i_0$. If there exists a point $p \in \Delta_1$ such that $\delta(p) > 0$, $f(p) = \omega_0$, there exists a certain ω_j containing z_i (in this case clearly ω_0 is an irregular point of the domain $= \omega$ -sphere $- f(\tilde{G} - F - E_z)$). Let ω be one of $\{\omega_i\}$. Then by $\nu(E_z) \cap f^{-1}(\omega) = 0$ it is proved similarly as Lemma 1 $U^\omega(\omega) = \Sigma u(z_i): f(z_i) = \omega, z_i \in \bar{\omega}$.

Case 2. $\omega_0 \notin E$. The part of $\tilde{G} - E_z - F$ over

$$|\omega - \omega_0| < \text{dist}(E, \omega_0)$$

consists of connected pieces $\omega_i (i = 1, 2, \dots)$. In this case ω_i does not tend to E_z by the univalence of $f(z)$ in $\nu(E_z)$. In both cases it is sufficient to consider only $\omega_i: i = 1, 2, \dots$. Let ω be one of $\{\omega_i\}$. Then ω is compact or non compact in $\tilde{G} - E_z - F$ and $\partial\omega$ consists of $\partial^1\omega$ and $\partial^2\omega$ such that $f(\partial^1\omega) \subset \{\omega: |\omega - \omega_0| = r\}$ and $\partial^2\omega = \partial\omega - \partial^1\omega$. Then $u(z)$ is harmonic on $\partial^1\omega$ and > 0 on $\partial^1\omega - F$ and $u(z) = 0$ on $\partial^2\omega$ and $U^\omega(\omega)$ is quasisubharmonic in $|\omega - \omega_0| < r$ and by Lemma 1.

$$\sum_i U_i^\omega(\omega) \leq U(\omega) \leq 1.$$

Proof of 1). — There exists a const. K such that

$$KG'(z, q_0) > u(z) > \frac{1}{K} G'(z, q_0)$$

in $G' \cap (R - R_l): R_l \rightarrow q_0$.

Hence without loss of generality we can suppose

$$u(z_i) \geq \delta > 0$$

and $|f(z_i) - \omega_0| < \frac{r}{2}$. Suppose a connected piece $\omega \rightarrow z_i$.

Then by Lemma 2

$$u(z_i) = \frac{1}{2\pi} \int_{\partial^1 \omega} u(\zeta) \frac{\partial}{\partial n} G^\omega(\zeta, z_i) ds,$$

where $G^\omega(\zeta, z)$ is a Green's function of ω .

Let $G^w(\varpi, \eta)$ be Green's function of $|\varpi - \varpi_0| < r$. Then by $G^w(f(z), f(z_i)) \geq G^\omega(z, z_i)$

$$(4) \quad \frac{\partial}{\partial n} G^w(f(\zeta), f(z_i)) \geq \frac{\partial}{\partial n} G^\omega(\zeta, z_i) \geq 0$$

on $\partial^1 \omega$. Let $\omega_n = \omega \cap R_n$, then $\omega_n \nearrow \omega$. Hence by considering ω_n we have similarly as Lemma 2

$$\begin{aligned} \int_{\partial^1 \omega} u(\zeta) \frac{\partial}{\partial n} G^w(f(\zeta), f(z_i)) ds \\ = \int_{|w - w_0| = r} U^\omega(\eta) \frac{\partial}{\partial n} G^w(\eta, f(z_i)) ds. \end{aligned}$$

On the other hand, there exists a const. K' such that

$$(5) \quad \frac{\partial}{\partial n} G^w(\eta, \varpi) \leq K' \frac{\partial}{\partial n} G^w(\eta, \varpi_0)$$

on

$$|\eta - \varpi_0| = r \quad \text{for} \quad |\varpi - \varpi_0| < \frac{r}{2}.$$

Hence

$$\begin{aligned} (6) \quad \delta < u(z_i) &= \frac{1}{2\pi} \int_{\omega \setminus \partial} u(\zeta) \frac{\partial}{\partial n} G^\omega(\zeta, z_i) ds \\ &\leq \frac{K'}{2\pi} \int U^\omega(\varpi_0 + re^{i\theta}) d\theta \leq K' \end{aligned}$$

Assume there exist $m \left(> \frac{K'}{\delta} \right)$ number of connected pieces $\omega_i: i = 1, 2, \dots, m$ containing at least one z_i of $\{z_i\}$. Then by (6) and $1 \geq \sum_i U^{\omega_i}(\varpi)$

$$m\delta \leq \frac{1}{2\pi} K' \sum_i \int U^{\omega_i}(\varpi_0 + re^{i\theta}) d\theta \leq K'.$$

This is a contradiction. Hence there exists at least one ω containing a subsequence $\{z'_i\}$ of $\{z_i\}$. Let $\{z''_i\}$ be a

subsequence of $\{z'_i\}$ such that $u(z''_i) \rightarrow a(>0)$, $G^\omega(z, z''_i) \rightarrow a$ harmonic function $G^\omega(z, \{z''_i\})$.

Then by (4), (5), (6) and by Lebesgue's theorem

$$(7) \quad 0 < \delta \leq \lim_i u(z''_i) = \lim_i \frac{1}{2\pi} \int_{\partial\omega} u(\zeta) \frac{\partial}{\partial n} G^\omega(\zeta, z''_i) ds \\ = \frac{1}{2\pi} \int_{\partial\omega} u(\zeta) \frac{\partial}{\partial n} G(\zeta, \{z''_i\}) ds$$

and

$$G^\omega(z, \{z''_i\}) > 0.$$

Put $\omega' = \omega \cap (R - R_0)$. Then $\omega' \subset \omega$. Since $\omega - \omega'$ is compact and $G^\omega(z, z''_i) \leq \tilde{G}(z, z''_i)$ is uniformly bounded on $\omega - \omega'$ for $i \geq i_0$, the convergence of $\{G^\omega(z, z''_i)\}$ implies $G^{\omega'}(z, z''_i) \rightarrow$ a positive harmonic function $G^{\omega'}(z, \{z''_i\})$ and

$$G^\omega(z, \{z''_i\}) = \sum_{\omega'}^{\omega} G^{\omega'}(z, \{z''_i\}) > 0,$$

where $G^\omega(z, z_i)$ and $\tilde{G}(z, z_i)$ are Green's function of ω' and $\tilde{G}$ respectively. We suppose Martin's is defined over $\bar{R} - R_0 \supset G$. Now $\omega' \subset R - R_0$ and by Lemma 4

$$G^{\omega'}(z, \{z''_i\}) = \alpha rK(z, p): 0 < \alpha < 1.$$

Hence $0 < G^\omega(z, \{z''_i\}) = \alpha srK(z, p)$ and $rK(z, p)$ and $G^\omega(z, \{z''_i\})$ is minimal (where r is relative to $R - R_0, \omega'$; s relative to ω, ω') (8). Assume there exists another connected piece ω^* containing a subsequence $\{z_j\}$ of $\{z_i\}$. Then as above we can find a subsequence $\{z'_j\}$ of $\{z_j\}$ such that (for r, s relative to $R - R_0, \omega^{*'}, \omega^*$, with

$$\omega^{*'} = \omega^* \cap (R - R_0))$$

$$(9) \quad 0 < G^{\omega^*}(z, \{z'_j\}) = srK(z, p) \text{ and } rK(z, p) > 0,$$

It is well known for minimal function $V(z)$ in $R - R_0$ if $rV > 0$ (relative to $R - R_0$ and D) $rV(z) = 0$ (relative to $R - R_0$ and CD for any domain D in $R - R_0$). Hence (8) contradicts (9). Thus there exists only one connected piece ω contains z_i for $i \geq i_0$.

Proof of 2). — Let z_i be a sequence such that $z_i \xrightarrow{M} p$, $\lim_i u(z_i) = u(p)$. Then $\lim_i G(z_i, q_0) \geq \frac{u(p)}{K} > 0$. Hence by

(1) of this theorem z_i is contained in the only one connected piece ω and by (8)

$$(10) \quad u(p) = \frac{1}{2\pi} \int_{\delta' \omega} u(\zeta) \frac{\partial}{\partial n} G^w(z, \{z_i\}) ds$$

and $G^w(z, \{z_i\})$ is the function when the value

$$G^w(z, \{z_i\})/srK(z, p),$$

(with r relative to $R - R_0$, ω' and s relative to ω , ω') attains the maximal value and the function $G'(z, \{z_i\})$ is uniquely determined. We denote it by $G^w(z, p)$. Thus we have 2).

Proof of 3. — For $p \in \Delta_1$ and $u(p) > 0$, there exists a uniquely determined connected piece ω (over $|\omega - f(p)| < r$) containing a sequence $z_i \xrightarrow{M} p$ and $\lim u(z_i) > 0$. In this case we say ω contains p .

Case 2. $\omega_0 \notin E$. Let ω be a connected piece over

$$|\omega - \omega_0| < \frac{1}{2} \text{dist}(\omega_0, E).$$

Let $p_i \in \Delta_1$: $f(p_i) = \omega_0$ be a point contained in ω . Then $G^w(z, p_i)$ is minimal and $\leq G^w(f(z), \omega_0) = \lim_i G^w(f(z), f(z_i))$.

Let q_j be a point in ω such that $f(q_j) = \omega_0$. Then $G^w(z, q_j)$ is minimal in ω and $\leq G^w(f(z), \omega_0)$. Hence

$$G^w(f(\zeta), \omega_0) \geq \sum_i G^w(\zeta, p_i) + \sum_j G^w(\zeta, q_j).$$

Clearly $u(q_j) = \frac{1}{2\pi} \int u(\zeta) \frac{\partial}{\partial n} G^w(\zeta, q_j) ds$. Hence by (10)

$$(11) \quad \int_{|\omega - \omega_0| = r} U^w(\zeta) \frac{\partial}{\partial n} G^w(\zeta, \omega_0) ds \geq \sum_i u(p_i) + \sum_j u(q_j).$$

Summing up over all connected pieces over $|\omega - \omega_0| < r$. Then by $\sum_i U^w i(\omega) \leq U(\omega) \leq 1$ we have

$$(12) \quad 1 \geq \sum_i u(p_i) + \sum_j u(q_j).$$

Case 1. $\omega_0 \in E$. In this case there exists no point q_j in ω such that $f(q_j) = \omega_0$. It is sufficient to consider $\omega_1, \omega_2, \dots$, remarked at the top of Theorem 4. Hence similarly as case 2) we have

$$(12) \quad 1 \geq \sum_i u(p_i).$$

Thus we have 3).

THEOREM 5. — *Let G be an end of a Riemann surface with null boundary. Let G' be a lacunary end: $G' = G - F$. Let $f(z)$ be an analytic function in G' and on ∂G . If $f(G')$ does not cover a set E of positive capacity, or spherical area of $f(G') < \infty$, then there exists a const. K not depending on ω such that*

$$\Sigma \delta(p_i) \leq K,$$

where $f(p_i) = \omega$ and $p_i \in \Delta_1$.

Proof. — Suppose $f(G')$ does not cover a set E of positive capacity. Let $\mathcal{J}$ be ω -sphere. Let Γ be an arc on ∂G such that $f(\Gamma)$ is a simple arc and $f(\Gamma) \cap E = 0$. Put $\mathcal{J}' = \mathcal{J} - f(\Gamma)$. Then we can connect $\mathcal{J}'$ with G' at Γ (at adequate side of $f(\Gamma)$) so that we may have a prolonged surface $\tilde{G} = \mathcal{J}' + (G - F)$ and $G + \mathcal{J}'$ may be an end of another Riemann surface $\tilde{R}$ with null boundary. Let $E_{\mathcal{J}}$ be the set of $\mathcal{J}$ over E and put

$$\tilde{G}' = \tilde{G} - E_{\mathcal{J}}: E_{\mathcal{J}} \subset \mathcal{J}'.$$

For the case [spherical area of $f(G')]$ $< \infty$, we can define $\tilde{G}$, $\tilde{G}'$ and $\tilde{R}$ as above (see the proof of Theorem 3). Hence $f(z)$ can be continued analytically into $\tilde{G}$. Then since $f(z)$ is univalent in neighbourhood $\nu(E_{\mathcal{J}})$, $1 \geq U(\omega) = \sum_i u(z_i)$:

$$f(z_i) = \omega: z_i \in \tilde{G}',$$

where $u(z)$ is a harmonic measure of $E_{\mathcal{J}}$ relative to $\tilde{G}'$. Since $\partial R_n \cap F = 0$, there exists a const. K such that $\frac{1}{K} G'(z, q_0) \leq u(z) \leq K G'(z, q_0)$ in $(R - R_{m_0}) \cap G'$:

$$G' \cap R_{m_0} \rightarrow q_0,$$

where $G'(z, q_0)$ is a Green's function of $G' \subset \tilde{G}'$. Now $f(\tilde{G}')$ does not cover E . Hence by Theorem 4 we have Theorem 5.

COROLLARY 1. — Suppose spherical area of $f(G') < \infty$.

1) Let $p_1, p_2, \dots$ be kindred points of p_1 . Then there exists a const. K (defined in Theorem 5) such that $\sum_i \delta(p_i) \leq K$.

2) If $F = \Sigma F_i$ is completely thin at p , then $\Delta_1 \cap \nabla(p)$ consists of at most m points (with:

$$m \leq \frac{K}{\delta} : \delta = \overline{\lim}_n \min_{z \in \partial r_n(p)} G'(z, q_0)$$

on $\partial v_n(p) \leq K$).

1) Is evident by Theorem 3 and 4.

Proof of 2). — Let $p \in \Delta_1 \cap \nabla(p)$. Then there exists a path Γ tending to p . Γ must intersect $\partial r_n(p)$, where $r_n(p)$ is a determining sequence of p and $\partial r_n(p)$ is a dividing cut such that $\overline{\lim}_n \min_{z \in \partial r_n(p)} G'(z, q_0) > \delta_0 > 0$. Whence $\delta(p) \geq \delta_0$. Also any two points in $\Delta_1 \cap \nabla(p)$ are clearly chained. Hence by Theorem 3 and 1) of corollary 1 we have 2).

COROLLARY 2. — Suppose $f(G')$ does not cover set E of positive capacity and $\Delta_1 \cap \nabla(p)$ consists of at most countably infinite number of points p_i with $\delta(p_i) > \delta_0 > 0$.

1) Let $p_1, p_2, \dots, p_m$ be a set of kindred points. Then there exists a const. K such that $m \left(\leq \frac{K}{\delta_0} \right)$.

2) If F is completely thin at p , then $\Delta_1 \cap \nabla(p)$ consists of at most m points, where m is the integer given in Corollary 1.

By corollary 1 and 2, we have at once :

COROLLARY 3. — Let G be an end of a Riemann surface with null boundary. Suppose F is completely thin at p .

1) If the harmonic dimension of p is countably infinite (this is equivalent $\Delta_1 \cap \nabla(p)$ consists of countably infinite number of points) and $\delta(p_i) \geq \delta_0 > 0$, then there exist no

analytic functions in $G' = G - F$ such that $f(G')$ does not cover a set of positive capacity.

2) If the harmonic dimension of $\mathfrak{p}$ is infinite, then there exist no analytic functions in G' with spherical area of

$$f(G') < \infty.$$

Remark. — The ameliorations of this paper appear [6].

BIBLIOGRAPHY

- [1] Z. KURAMOUCHI, On minimal points of Riemann surfaces, II, *Hokkaido Math. J.*, II (1973), 139-175.
- [2] M. BRELOT, Sur le principe des singularités positives et la topologie de R. S. Martin., *Ann. Univ. Grenoble*, 23 (1948), 113-138.
- [3] Z. KURAMOUCHI, Mass distributions on the ideal boundaries of abstract Riemann surfaces, 1., *Osaka Math. J.*, 8 (1956), 119-137.
- [4] M. NAKAI, Green potential of Evans type on Royden's compactification of a Riemann surface, *Nagoya Math. J.*, 24 (1964), 205-239.
- [5] Z. KURAMOUCHI, On the existence of functions of Evans's type, *J. Fac. Sci. Hokkaido Univ.*, 19 (1965), 1-27.
- [6] Z. KURAMOUCHI. Analytic functions in a neighbourhood of boundary, *Proc. Japan Acad.*, 51 (1975), 320-327.
- [7] Z. KURAMOUCHI. Analytic fonctions in a neighbourhood of irregular boundary points, *Hokkado Math. J.*, V (1976).

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