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A FREE BOUNDARY VALUE PROBLEM IN POTENTIAL THEORY

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1. Introduction.

In this paper we shall describe the formulation and solution of a free boundary value problem in the framework of variational inequalities. For simplicity, we confine our attention to a problem in the plane which consists in finding a domain Ω and a function u defined in Ω satisfying there a given differential equation together with both assigned Dirichlet and Neumann data on the boundary Γ of Ω. Under appropriate hypotheses about the given data we prove that there is a unique solution pair Ω, u which resolves this problem and that Γ is a smooth curve.

Let $z = x_1 + ix_2 = \rho e^{i\theta}$, $0 \leqslant \theta < 2\pi$, denote a point in the z-plane. Let us suppose, for the moment, that $F(z)$ is a function in $C^2(R^2)$ which satisfies the conditions

$$
\begin{aligned}
&\rho^{-2}F(z) \in C^2(R^2)\\
&\inf_{R^2} \rho^{-2}F(z) > 0\\
(1.1)\qquad &F_\rho(z) \geqslant 0 \quad z \in R^2\\
&F(0) = F_\rho(0) = 0.
\end{aligned}
$$

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These conditions will be weakened. Our object is to solve, in some manner, this

Problem 1. — To find a bounded Ω *and a function* u *such that*

$$- \Delta u = \rho^{-1} F_\rho \quad in \quad \Omega \tag{1.2}$$

$$\begin{cases} u = 0 \\ \dfrac{\partial u}{\partial \nu} = - F \dfrac{d\theta}{ds} \end{cases} \quad on \quad \Gamma \tag{1.3}$$

$$u(0) = \gamma \tag{1.4}$$

where $\Gamma = \partial \Omega$, ν *is the outward directed normal vector and* s *the arc length on* Γ, F *satisfies* (1.1), *and* γ *is given.*

Supposing Ω, u to be a solution to *Problem 1*, the maximum principle for superharmonics implies that $u > 0$ in Ω since $-\Delta u \geqslant 0$ in Ω. We assume, consequently, that $\gamma > 0$ and that $u \in C(R^2)$ with $\Omega = \{z : u(z) > 0\}$. Further, if Ω is a domain with smooth boundary Γ and u satisfies (1.2) in Ω and (1.3) on Γ then

$$\frac{\partial u}{\partial \nu}(z) < 0 \quad \text{for} \quad z \in \Gamma$$

in view of Hopf's well known maximum principle. Therefore

$$\frac{d\theta}{ds}(z) = - \frac{1}{F(z)} \frac{\partial u}{\partial \nu}(z) > 0 \quad \text{for} \quad z \in \Gamma,$$

or the central angle θ is a strictly increasing function of the arc length parameter on Γ. Interpreting this situation geometrically, we conclude *if* Γ *is smooth and* u *satisfies* (1.2) *in* Ω *and* (1.3) *on* Γ, *then* Ω *is starshaped with respect to* $z = 0$.

We shall solve *Problem 1* by means of a variational inequality suggested by the properties of a function $g(z)$ which satisfies

$$g_\rho = - \rho^{-1} u \tag{1.5}$$

The idea of introducing a new unknown related to the original one through differentiation is due to C. Baiocchi [1] who studied a filtration problem. It has subsequently been employed by H. Brézis and G. Stampacchia [5], V. Benci [2], Duvaut [6], and also in [12].

A characteristic of the present work is the logarithmic nature of a function g defined by (1.5) at $z = 0$. This difficulty will be overcome by considering an unbounded obstacle.

In the following section we transform our problem to one concerning a variational inequality. In § 3 we solve the variational inequality. With the aid of [4] we are able to show in § 5 that Γ is a Jordan curve represented by a continuous function of the central angle θ. In § 6 we use a result of [8] to conclude the smoothness of Γ and the existence of a classical solution to *Problem 1.*

2.

In this section we introduce a variational inequality and determine its relationship to *Problem* 1. We begin with some notations. Set $B_r = \{z : |z| < r\}$, $r > 0$, and [1]

$$K_r = \{v \in H^1(B_r) : v \geqslant \log \rho \ \text{ in } B_r \ \text{ and } \ v = \log r \ \text{ on } \partial B_r\}.$$

Define the bilinear form

$$a(v, \zeta) = \int_{B_r} v_{x_i} \zeta_{x_i} \, dx = \int_{B_r} \left\{ v_\rho \zeta_\rho + \frac{1}{\rho^2} v_\theta \zeta_\theta \right\} \rho \, d\rho \, d\theta, \qquad v, \ \zeta \in H^1(B_r).$$

We always depress the dependence of $a(v, \zeta)$ on $r > 0$. Let

$$f \in L^p_{\text{loc}}(R^2) \quad \text{for some} \quad p > 2.$$

Problem (*). — *To find a pair* $r > 1$ *and* $w \in K_r$ *such that*

$$(2.1) \quad w \in K_r : a(w, v - w) \geqslant \int_{B_r} f(v - w) \, dx \qquad v \in K_r$$

(1) Usual notation is employed for function spaces.

and the function $\tilde{w}(z)$ *defined by*

$$(2.2) \qquad \tilde{w}(z) = \begin{cases} w(z) & z \in B_r \\ \log|z| & z \notin B_r \end{cases} \quad \textit{is in} \quad C^1(R^2)$$

The existence and other properties of a solution to *Problem* (*) will be investigated in the next paragraph. We note here that the restriction of $\tilde{w}$ to B_R for $R > r$ will be a solution of (2.1) in B_R. Since this means that (2.2) will be automatically satisfied, so that $R, \tilde{w}|_{B_R} \in K_R$ is also a solution to *Problem* (*), we shall not distinguish between w and $\tilde{w}$ in the sequel.

THEOREM 1. — *Let* Ω, u *be a solution of Problem 1 where* F *satisfies* (1.1) *and* $\gamma > 0$. *Suppose that* Γ *is a smooth curve. Then there exists a solution* $r, w \in K_r$ *of Problem* (*) *for*

$$f(z) = -\frac{1}{\gamma\rho^2} F(z)$$

such that

$$(2.3) \quad \Omega = \{z : w(z) > \log \rho\} \quad \textit{and} \quad u(z) = \gamma(1 - \rho w_\rho(z)).$$

The theorem is based on the lemma below which also explains the role of the normal derivative condition in (1.3).

LEMMA 2.1. — *Let* Ω *be a simply connected domain containing the origin and* $\Gamma' \subset \partial\Omega$ *a smooth arc. Let* $F \in C^2(R^2)$ *satisfy* (1.1). *Suppose that* u *satisfies*

$$-\Delta u = \rho^{-1} F_\rho \quad \textit{in} \quad \Omega$$

$$\begin{cases} u = 0 \\ \dfrac{\partial u}{\partial \nu} = -F\dfrac{d\theta}{ds}. \end{cases} \quad \textit{on} \quad \Gamma'$$

Let $g \in C^1(\overline{\Omega} - \{0\})$ *denote any function with the property*

$$g_\rho = -\rho^{-1}u \quad \textit{in} \quad \overline{\Omega} - \{0\} \quad \textit{and} \quad \Delta g \in C(\overline{\Omega} - \{0\}).$$

Let $\zeta \in C_0^\infty(R^2)$ *vanish in a neighborhood of* $\partial\Omega - \Gamma'$ *and* $z = 0$. *Then*

$$\int_{\Gamma'} \zeta\rho^2 \Delta g \, d\theta = \int_{\Gamma'} \zeta F \, d\theta - \int_{\Gamma'} g_\theta(\zeta_\rho \, d\rho + \zeta_\theta \, d\theta)$$

Proof. — First we compute Δg in Ω. For this, observe that

$$\begin{aligned} -F_\rho &= (\rho u_\rho)_\rho + \rho^{-1} u_{\theta\theta} \\ &= -(\rho(\rho g_\rho)_\rho)_\rho - g_{\rho\theta\theta} \\ &= -\frac{\partial}{\partial\rho}\{\rho(\rho g_\rho)_\rho + g_{\theta\theta}\} \\ &= -\frac{\partial}{\partial\rho}(\rho^2 \Delta g). \end{aligned}$$

Hence

$$\frac{\partial}{\partial\rho}(\rho^2\Delta g) = F_\rho \quad \text{in} \quad \Omega. \tag{2.4}$$

Let $\zeta \in C_0^\infty(B_r)$, where $\overline{\Omega} \subset B_r$, satisfy $\zeta = 0$ in a neighborhood of $\partial\Omega - \Gamma'$ and $z = 0$. Then observing that $-F\,d\theta = \frac{\partial u}{\partial\nu}\,ds = \rho u_\rho\,d\theta - \frac{1}{\rho}\,u_\theta\,d\rho$,

$$\begin{aligned} &-\int_{\Gamma'} F\zeta\,d\theta \\ &= \int_{\Gamma'} \zeta\left(\rho u_\rho\,d\theta - \frac{1}{\rho}\,u_\theta\,d\rho\right) \\ &= \int_\Omega \zeta\left((\rho u_\rho)_\rho + \frac{1}{\rho}\,u_{\theta\theta}\right)d\rho\,d\theta + \int_\Omega\left(\rho u_\rho\zeta_\rho + \frac{1}{\rho}\,u_\theta\zeta_\theta\right)d\rho\,d\theta \\ &= -\int_\Omega \zeta F_\rho\,d\rho\,d\theta - \int_\Omega\{\rho(\rho g_\rho)_\rho\zeta_\rho + g_{\rho\theta}\zeta_\theta\}\,d\rho\,d\theta \\ &= -\int_\Omega F_\rho\zeta\,d\rho\,d\theta - \int_\Omega\{(\rho^2\Delta g - g_{\theta\theta})\zeta_\rho + g_{\rho\theta}\zeta_\theta\}\,d\rho\,d\theta \\ &= -\int_\Omega\{\zeta F_\rho + \rho^2\Delta g\zeta_\rho\}\,d\rho\,d\theta + \int_\Omega\{g_{\theta\theta}\zeta_\rho - g_{\rho\theta}\zeta_\theta\}\,d\rho\,d\theta. \end{aligned}$$

We evaluate the first integral by (2.4). Hence

$$\begin{aligned} \int_\Omega\{F_\rho\zeta + \rho^2\Delta g\zeta_\rho\}\,d\rho\,d\theta &= \int_\Omega \frac{\partial}{\partial\rho}(\zeta\rho^2\Delta g)\,d\rho\,d\theta \\ &= \int_{\Gamma'} \zeta\rho^2\Delta g\,d\theta. \end{aligned}$$

Turning to the second integral, we compute that

$$\begin{aligned} \int_\Omega\{g_{\theta\theta}\zeta_\rho - g_{\rho\theta}\zeta_\theta\}\,d\rho\,d\theta &= \int_\Omega\{(g_\theta\zeta_\rho)_\theta - (g_\theta\zeta_\theta)_\rho\}\,d\rho\,d\theta \\ &= \int_{\Gamma'}(g_\theta\zeta_\rho\,d\rho + g_\theta\zeta_\theta\,d\theta). \end{aligned}$$

Finally, we obtain that

$$\int_{\Gamma'} F\zeta\,d\theta = \int_{\Gamma'} \rho^2\Delta g\zeta\,d\theta + \int_{\Gamma'} g_\theta(\zeta_\rho\,d\rho + \zeta_\theta\,d\theta). \quad \text{Q.E.D.}$$

LEMMA 2.2. — *Let* Ω, u *be a solution to Problem 1 and suppose that* $\Gamma = \partial\Omega$ *is smooth. Set* $u = 0$ *in* $R^2 - \Omega$. *Let* r *be large enough that* $\overline{\Omega} \subset B_r$ *and choose*

$$(2.5) \quad g(z) = \int_\rho^r t^{-1} u(t, \theta)\, dt, \quad |z| = \rho, \quad 0 \neq z \in B_r$$

Then

$$g \in C^1(\overline{\Omega} - \{0\}), \qquad \Delta g \in C(\overline{\Omega} - \{0\}),$$

and

$$\Omega = \{z : g(z) > 0\}$$

and moreover

$$\Delta g = \begin{cases} \rho^{-2} F & in \quad \overline{\Omega} - \{0\} \\ 0 & in \quad B_r - \Omega. \end{cases}$$

Proof. — As we remarked in the introduction, smoothness of Γ implies that Ω is starshaped with respect to $z = 0$. Hence if $g(z) = 0$ for $z = \rho e^{i\theta}$, then the non-negative continuous integrand in (2.5) vanishes for $te^{i\theta}$, $t > \rho$, so that $g(te^{i\theta}) = 0$, $t > \rho$. Therefore, since $u > 0$ in Ω, we see that $g(z) > 0$ in $\Omega - \{0\}$ and $g(z) = 0$ in $B_r - \Omega \supset \Gamma$. Because u is smooth in Ω it is easy to derive that $g \in C^1(B_r - \{0\})$. On the other hand g attains its minimum on $B_r - \Omega$ whence

$$(2.6) \qquad g_\rho = 0 = g_\theta \quad \text{on} \quad B_r - \Omega.$$

Since $g_\rho = -\rho^{-1} u$ in Ω, by (2.4),

$$(2.7) \qquad \frac{\partial}{\partial \rho}(\rho^2 \Delta g) = F_\rho \quad \text{in} \quad \Omega.$$

We may integrate (2.7) in Ω since Ω is starshaped to obtain

$$\rho^2 \Delta g(z) = F(z) + \psi(\theta), \qquad z = \rho e^{i\theta} \in \Omega,$$

where ψ is a function of the central angle θ only. Now by Lemma 2.1

$$\int_\Gamma \zeta F(z)\, d\theta + \int_\Gamma \psi(\theta)\zeta\, d\theta = \int_\Gamma F\zeta\, d\theta - \int_\Gamma g_\theta(\zeta_\rho\, d\rho + \zeta_\theta\, d\theta)$$

for $\zeta \in C_0^\infty(B_r - \{0\})$. Since $g_\theta = 0$ on $\Gamma \subset B_r - \Omega$ (cf. 2.6),

$$\int_\Gamma \psi(\theta)\zeta\, d\theta = 0 \qquad \zeta \in C_0^\infty(B_r - \{0\})$$

or

$$\psi(\theta) = 0, \quad 0 \leqslant \theta < 2\pi. \qquad \text{Q.E.D.}$$

Proof of Theorem 1. — As we have observed, Ω is star-shaped with respect to $z = 0$ so the function $g(z)$ defined by (2.5) satisfies the conclusions of Lemma 2.2. Let r be so large that $\overline{\Omega} \subset B_r$ and define

$$\begin{aligned} w^*(z) &= \frac{1}{\gamma} g(z) + \log \rho \quad 0 \neq z \in B_r \\ &= \frac{1}{\gamma} \int_\rho^r t^{-1}(u(t, \theta) - \gamma) dt + \log r \end{aligned}$$

where $\gamma = u(0) > 0$. We shall show that $r, w^* \in K_r$ is a solution to. *Problem* (*). Clearly w^* is bounded in B_r and satisfies

$$(2.8) \qquad -\Delta w^* = \begin{cases} f & \text{in } \Omega - \{0\} \\ 0 & \text{in } B_r - \Omega \end{cases} \quad \text{a.e.}$$

by *Lemma 2.2* where $f(z) = -\dfrac{1}{\gamma\rho^2} F(z)$. Since $f \in C^2(R^2)$, cf. (1.1), it follows from Riemann's Theorem on removable singularities that w^* is smooth in Ω. We observe that

$$w^*(z) \geqslant \log \rho \qquad \text{since} \qquad g(z) \geqslant 0$$

and $\Omega = \{z : w^*(z) > \log \rho\}$. Further, $\overline{\Omega} \subset B_r$ implies that, for $|z| = r$,

$$\begin{aligned} &w^*(z) = \log r \\ &w^*_\rho(z) = 1/r \qquad \text{and} \qquad w^*_\theta(z) = 0 \end{aligned}$$

Therefore, $w^* \in K_r$ and the function

$$\tilde{w}^*(z) = \begin{cases} w^*(z) & z \in B_r \\ \log \rho & z \notin B_r \end{cases}$$

is a $C^1(R^2)$ function. Hence (2.2) holds.

It is easy to verify (2.1). Let $v \in K_r$. Then

$$a(w^*, v - w^*) = \int_\Omega f(v - w^*)\, dx$$

by (2.8) and an integration by parts, valid since $w^* \in C^1(\overline{\Omega})$. Indeed, $w^* \in C^1(R^2)$, as noted above. Hence

$$a(w^*, v - w^*) - \int_{B_r} f(v - w^*)\, dx = -\int_{B_r - \Omega} f(v - w^*)\, dx.$$

Since $f \leqslant 0$ in B_r and $v \in K_r$ implies

$$0 \leqslant v - \log \rho = v - w^* \quad \text{in} \quad B_r - \Omega,$$

the last integral is non-negative so that

$$a(w^*, \ v - w^*) \geqslant \int_{B_r} f(v - w^*)\, dx \qquad v \in K_r \qquad \text{Q.E.D.}$$

3.

This paragraph is devoted to the solution of the variational inequality *Problem* (*). According to a well known theorem [11], there is a solution to (2.1) for each $r > 0$. To establish its smoothness in B_r, we shall prove that it is bounded. For once this is known, the obstacle $\log \rho$ may be replaced by a smooth obstacle ψ which equals $\log \rho$ when

$$\log \rho > - \|w\|_{L^\infty(B_r)}$$

and (2.1) may be solved in the convex K_ψ of $H^1(B_r)$ functions which exceed ψ in B_r and satisfy the boundary condition $v(z) = \log r$, $|z| = r$. The solution to this latter problem is known to be suitable smooth (cf. [10]) and is easily shown to be the solution of (2.1).

LEMMA 3.1. — *Let* $f \in L^p(B_r)$ *for some* $p > 2$ *and satisfy*

$$f \leqslant 0 \quad \text{in} \quad B_r.$$

Then the solution w *of* (2.1) *for* f *satisfies*

$$\log r - c\|f\|_{L^p(B_r)} \leqslant w(z) \leqslant \log r \quad \textit{in} \quad B_r,$$

where $c = c(r, p) > 0$.

Proof. — Let w_0 denote the solution to the Dirichlet problem

$$\begin{aligned} -\Delta w_0 &= f \quad \text{in} \quad B_r \\ w_0 &= 0 \quad \text{on} \quad \partial B_r. \end{aligned}$$

We know that $w_0 \in H^{2,p}(B_r)$ and

$$(3.1) \qquad \|w_0\|_{L^\infty(B_r)} \leqslant c\|f\|_{L^p(B_r)}, \qquad c = c(r, p) > 0.$$

Consequently, for any $\zeta \in H_0^1(B_r)$,

$$a(w - w_0, \zeta) = a(w, \zeta) - \int_{B_r} f\zeta \, dx.$$

We define $v = \max(w, w_0 + \log r) \in K_r$ so by (2.1)

$$a(w - w_0, v - w) \geqslant 0$$

Further, computing explicitly, we find

$$\begin{aligned} a(w - w_0, v - w) &= \int_{B_r} (w - w_0)_{x_i}(v - w)_{x_i} \, dx \\ &= - \int_{\{v > w\}} (w - w_0)_{x_i}^2 \, dx \leqslant 0. \end{aligned}$$

Hence meas $\{v > w\} = 0$ or $\log r + w_0 \leqslant w$ a.e. This proves the lower bound in view of (3.1). The same argument may be employed to prove the upper bound, with

$$v = \min(w, \log r),$$

using that $f \leqslant 0$ in B_r. Q.E.D.

For general f, we observe that an upper bound for the solution of (2.1) is

$$\log r + c(r, p) \, \|\max(0, f)\|_{L^p(B_r)}.$$

Corollary 3.2. — *Let $f \in L^p(B_r)$ for some $p > 2$, $f \leqslant 0$ in B_r, and let w denote the solution to (2.1) for f. Then $w \in H^{2,p}(B_r)$. If $f \in C^1(\overline{B}_r)$, then $w \in H^{2,\infty}_{loc}(B_r)$.*

Proof. — This is clear from the remarks preceding the proof of the lemma. In particular, that $w \in H^{2,\infty}_{loc}(B_r)$ follows by a result of Frehse [7] (cf. also [4]).

Lemma 3.3. — *Let $g \in H^1(B_r)$ satisfy*

$$g \geqslant \log \rho \quad in \quad B_r$$

and

$$a(g, \zeta) - \int_{B_r} f\zeta \, dx \geqslant 0 \quad for \quad 0 \leqslant \zeta \in H_0^1(B_r).$$

Let w denote the solution of Problem (*) for $f \in L^p(B_r)$, for some $p > 2$. Then $w \leqslant g$ in B_r.

Proof. — This is a familiar property of supersolutions. cf. [10], [11].

THEOREM 2. — *Let* $f \in L^p_{loc}(R^2)$ *for a* $p > 2$ *satisfy*

$$\sup_{R^2} f < 0.$$

Then there exists a solution r, $w \in K_r$ *to Problem* (*). *In addition,* $w \in H^{2,p}(B_r)$.

Proof. — We shall construct a supersolution $g(z) = h(\rho)$ to the form

$$a(w, \zeta) - \int_{B_r} f\zeta \, dx,$$

for some $r > 1$, which satisfies

$$h \in K_r \tag{3.2}$$

$$h_\rho(r) = \frac{1}{r}. \tag{3.3}$$

Indeed, suppose that

$$0 < \beta \leqslant -\sup_{R^2} f \quad \text{and} \quad \beta < 2e^{-1},$$

and define

$$h(\rho) = \alpha + \frac{1}{4}\beta\rho^2.$$

Then

$$-\Delta h = -\frac{1}{\rho}(\rho h_\rho)_\rho = -\beta \geqslant \sup f$$

Assume for the moment that (3.2) and (3.3) are fulfilled. Then

$$w \leqslant h \quad \text{in} \quad B_r$$

by the previous lemma. Moreover, since $\log\rho \leqslant w \leqslant h$ we conclude from (3.3) that

$$w_\rho(z) = \frac{1}{r} \quad \text{for} \quad |z| = r$$

and, since $w = \log r$ on $|z| = r$,

$$w_\theta(z) = 0 \quad \text{for} \quad |z| = r.$$

Therefore $\tilde{w}$ defined by (2.2) is in $C^1(R^2)$.

It remains to find α and r from the conditions (3.2), (3.3). One discovers that

$$r = \left(\frac{2}{\beta}\right)^{1/2} \geqslant 1$$

and

$$\alpha = \log r - \frac{1}{2} = \frac{1}{2}\left(\log \frac{2}{\beta} - 1\right) > 0.$$

To verify that $h \in K_r$, i.e., to verify that $h(\rho) \geqslant \log \rho$ knowing that $h(r) = \log r$, note that $h(\rho) - \log \rho$ is strictly convex and attains its (unique) minimum at the ρ where $h_\rho = \frac{1}{\rho} = 0$. This $\rho = r$. Q.E.D.

We wish to point out here that ideas similar to those in the proof of Theorem 2 were also studies by H. Brezis [3].

COROLLARY 3.4. — *Let* $f \in L^p_{loc}(R^2)$ *for a* $p > 2$ *satisfy* $\sup_{R^2} f < 0$. *Let* r, $w \in K_r$ *denote the solution to Problem* (*) *for* f. *Then for* $R > r$, *the pair* R, $\tilde{w} \in K_R$, *where* $\tilde{w}$ *is defined by* (2.2) *is a solution to Problem* (*).

In view of this Corollary, we shall not distinguish between w and $\tilde{w}$ in the sequel. Furthermore, we recall that $w \in H^{2,\infty}_{loc}(R^2)$ whenever $f \in C^1(R^2)$.

Proof. — We need only verify (2.1) in B_R. Let $\zeta \in C_0^\infty(B_R)$. Then

$$\begin{aligned} a(\tilde{w}, \zeta) &= \int_{B_r} w_{x_i}\zeta_{x_i}\,dx + \int_{B_R - B_r} \frac{\partial}{\partial x_i} \log \rho\, \zeta_{x_i}\,dx \\ &= -\int_{B_r} \Delta w \zeta\,dx + \int_{|z|=r} w_\rho \zeta r\,d\theta + \int_{B_R - B_r} \Delta \log \rho \zeta\,dx \\ &\quad - \int_{|z|=r} \frac{1}{r}\zeta r\,d\theta \end{aligned}$$

since ζ has support in B_R. Now $\tilde{w} \in C^1(B_R)$ implies, in particular, that $w_\rho(z) = \frac{1}{r}$ for $|z| = r$ and the two integrals over $|z| = r$ cancel. Hence

$$\begin{aligned} a(\tilde{w}, \zeta) &= -\int_{B_r} \Delta w \zeta\,dx \\ &= \int_{\Omega_r} f\zeta\,dx, \qquad \Omega = \{z : w(z) > \log \rho\}. \end{aligned}$$

Now given $\varphi \in K_R$,

$$a(\tilde{w}, \varphi - \tilde{w}) - \int_{B_R} f(\varphi - \tilde{w})\, dx = - \int_{B_R - \Omega} f(\varphi - \tilde{w})\, dx \geqslant 0$$

where the last integral is non-negative because $\tilde{w} = \log \rho$ in $B_R - \Omega$ and $f < 0$. This verifies (2.1). Q.E.D.

4.

Here we show that the set where the solution to *Problem* (*) exceeds $\log \rho$ is starshaped under an assumption about f. First we prove a lemma which is useful also in the succeeding sections. It is a form of converse to *Lemma 2.1* with an analogous proof.

LEMMA 4.1. — *Let* $f \in L^p_{loc}(R^2)$ *for some* $p > 2$ *satisfy* $\sup_{R^2} f < 0$. *Let* r, $w \in K_r$ *denote the solution to Problem* (*) *for* f *and define*

$$u(z) = 1 - \rho w_\rho(z) \quad z \in B_r$$

and

$$\Omega = \{z \in B_r : w(z) > \log \rho\}.$$

i) *Then* $u \in H^{1,p}(B_r)$.

ii) *Let* $\omega \subset B_r$ *be open and suppose that* $-\Delta w = f$ *in* ω. *Then*

$$(4.1) \qquad -\Delta u = -\rho^{-1}(\rho^2 f)_\rho \quad \text{in} \quad \omega$$

iii) *Suppose that* $f \in C^1(\overline{B}_r)$ *and that* Γ' *is a smooth* (*open*) *arc in* $\partial\Omega$. *Then*

$$(4.2) \qquad \frac{\partial u}{\partial \nu} = \rho^2 f \frac{d\theta}{ds} \quad \text{on} \quad \Gamma'$$

where ν *denotes the outward directed normal vector on* Γ'.

Proof. — Since $f \in L^p_{loc}(R^2)$, $p > 2$, $w \in H^{2,p}(B_r)$, so $u = 1 - \Sigma x_i w_{x_i} \in H^{1,p}(B_r)$. The statement (4.1) will be understood in the sense of distributions.

Let $\zeta \in C_0^\infty(\omega)$. Then

$$\begin{aligned}\int_\omega u_{x_i}\zeta_{x_i}\,dx &= \int_\omega \left(\rho u_\rho \zeta_\rho + \frac{1}{\rho} u_\theta \zeta_\theta\right) d\rho\, d\theta \\ &= \int_\omega \left\{\rho(1-\rho w_\rho)_\rho \zeta_\rho + \frac{1}{\rho}(1-\rho w_\rho)_\theta \zeta_\theta\right\} d\rho\, d\theta \\ &= -\int_\omega \{\rho(\rho w_\rho)_\rho \zeta_\rho + w_{\rho\theta}\zeta_\theta\}\, d\rho\, d\theta.\end{aligned}$$

We integrate by parts in the last term, first with respect to ρ and then with respect to θ, to obtain

$$\begin{aligned}\int_\omega u_{x_i}\zeta_{x_i}\,dx &= -\int_\omega \{\rho(\rho w_\rho)_\rho \zeta_\rho + w_{\theta\theta}\zeta_\rho\}\, d\rho\, d\theta \\ &= -\int_\omega \rho^2 \Delta w \zeta_\rho\, d\rho\, d\theta \\ &= \int_\omega \rho^2 f \zeta_\rho\, d\rho\, d\theta\end{aligned}$$

since $-\Delta w = f$ in ω by hypothesis. Hence

$$\int_\omega u_{x_i}\zeta_{x_i}\,dx = -\int_\omega \frac{1}{\rho}(\rho^2 f)_\rho \zeta\rho\, d\rho\, d\theta.$$

We turn now to the proof of iii). Suppose that Γ' has a Hölder continuous tangent vector as a function of the arc-length parameter. In Ω, that $w(z) > \log \rho$ implies

$$-\Delta w = f,$$

whence

$$-\Delta u = -\frac{1}{\rho}(\rho^2 f)_\rho \quad \text{in} \quad \Omega.$$

Moreover, $w_\rho(z) = \dfrac{1}{\rho}$ for $z \in \partial\Omega$ so $u = 0$ on $\Gamma' \subset \partial\Omega$. From this and the fact $f \in C^1(\overline{B}_r)$ we may conclude that $u \in C^{1,\lambda}(\Omega \cup \Gamma')$ for some $\lambda > 0$. Let $\zeta \in C_0^\infty(B_r)$ with $\operatorname{supp} \zeta \cap (\partial\Omega - \Gamma') = \emptyset$. Then

$$(4.3)\quad \int_{\Gamma'} u_\nu \zeta\, ds = \int_{\Gamma'} \zeta\left(\rho u_\rho\, d\theta - \frac{1}{\rho} u_\theta\, d\rho\right)$$

$$\begin{aligned}&= \int_\Omega \zeta\left((\rho u_\rho)_\rho + \frac{1}{\rho} u_{\theta\theta}\right) d\rho\, d\theta + \int_\Omega \left(\rho\zeta_\rho u_\rho + \frac{1}{\rho} u_\theta \zeta_\theta\right) d\rho\, d\theta \\ &= \int_\Omega \zeta(\rho^2 f)_\rho\, d\rho\, d\theta \\ &\quad - \int_\Omega \{\rho(\rho w_\rho)_\rho \zeta_\rho + w_{\theta\theta}\zeta_\rho - w_{\theta\theta}\zeta_\rho + w_{\rho\theta}\zeta_\theta\}\, d\rho\, d\theta \\ &= \int_\Omega (\zeta(\rho^2 f)_\rho - \rho^2 \Delta w \zeta_\rho)\, d\rho\, d\theta + \int_\Omega (w_{\theta\theta}\zeta_\rho - w_{\rho\theta}\zeta_\theta)\, d\rho\, d\theta.\end{aligned}$$

Since $-\Delta w = f$ in Ω, we evaluate the first integral to yield

$$(4.4)\qquad \int_\Omega ((\rho^2 f)_\rho \zeta - \rho^2 \Delta w \zeta_\rho)\, d\rho\, d\theta = \int_{\Gamma'} \zeta \rho^2 f\, d\theta .$$

On the other hand, $w_\theta = 0$ on $\Gamma' \subset B_r - \Omega$, therefore

$$\int_\Omega (w_{\theta\theta}\zeta_\rho - w_{\rho\theta}\zeta_\theta)\, d\rho\, d\theta = \int_\Omega \{(w_\theta \zeta_\rho)_\theta - (w_\theta \zeta_\theta)_\rho\}\, d\rho\, d\theta$$
$$= - \int_{\Gamma'} w_\theta(\zeta_\rho\, d\rho + \zeta_\theta\, d\theta) = 0.$$

Finally, from (4.3) and (4.4) we obtain that

$$\int_{\Gamma'} u_\nu \zeta\, ds = \int_{\Gamma'} \rho^2 f \zeta\, ds, \quad \zeta \in C_0^\infty(B_r), \ \text{supp}\ \zeta \cap (\partial\Omega - \Gamma') = \emptyset.$$

THEOREM 3. — *Let* $f \in L^p_{\text{loc}}(R^2)$ *satisfy* $\sup_{R^2} f < 0$ *and* $\rho^{-1}(\rho^2 f)_\rho \leqslant 0$. *Let* r, $w \in K_r$ *denote the solution of Problem* (*) *for* f *and set*

$$\Omega = \{z : \ w(z) > \log \rho\}$$

Then Ω *is starshaped with respect to* $z = 0$.

Proof. — Consider, as in the preceding proposition,

$$u(z) = 1 - \rho w_\rho(z), \quad z \in B_r,$$

and note that $u \in C^{0,1-\frac{2}{p}}(B_r)$ and $u = 0$ on $\Gamma \subset B_r - \Omega$, $\Gamma = \partial\Omega$. By the hypothesis on f and (4.1),

$$\int_\Omega u_{x_i}\zeta_{x_i}\, dx = - \int_\Omega \rho^{-1}(\rho^2 f)_\rho \zeta\, dx \geqslant 0 \quad \text{for} \quad 0 \leqslant \zeta \in C_0^\infty(\Omega).$$

The maximum principle may now be applied to conclude that

$$u(z) \geqslant \min_\Gamma u = 0 \quad \text{for} \quad z \in \Omega.$$

Hence the function

$$g(z) = -\log \rho + w(z), \quad 0 \neq z \in B_r$$

is decreasing on each ray $\rho e^{i\theta}$, $0 < \rho < r$, because it has derivative

$$g_\rho(z) = -\frac{1}{\rho}(1 - \rho w_\rho(z)) = -\frac{1}{\rho} u(z) \leqslant 0, \quad z \in B_r, \quad z \neq 0.$$

Therefore, given $z = \rho e^{i\theta}$ with $w(z) > \log \rho$, then

$$w(te^{i\theta}) > \log t \quad \text{for} \quad t \leqslant \rho.$$

This proves that Ω is starshaped. Q.E.D.

5.

In this paragraph we initiate the study of the free boundary determined by a solution to *Problem* (*). To begin, we fix an $f \in C^1(R^2)$ which satisfies

$$(5.1) \qquad \sup_{R^2} f < 0 \quad \textit{and} \quad (\rho^2 f)_\rho \leqslant 0 \quad \textit{in} \quad R^2$$

and let r, $w \in K_r$ denote the solution to *Problem* (*) for f. As before, set

$$\Omega = \{z : w(z) > \log \rho\}$$

and let

$$E = \overline{B}_r - \Omega.$$

Observe that, by Theorem 3, E is starshaped with respect to the point at ∞ in the sense that

$$z \in E, t \geqslant 1 \text{ and } |tz| \leqslant r \text{ implies } tz \in E.$$

Define

$$(5.2) \qquad \mu(\theta) = \inf\{\rho : z = \rho e^{i\theta} \in E\}, 0 \leqslant \theta < 2\pi,$$

Note that $\mu(\theta)$ is lower semicontinuous since E is closed. For given $z_n = \rho_n e^{i\theta_n}$, $\rho_n = \mu(\theta_n)$, and $z_n \to z = \rho e^{i\theta}$, we conclude that $z \in E$ and hence $\rho \geqslant \mu(\theta)$. In addition

$$(5.3) \qquad E = \{z = \rho e^{i\theta} : \mu(\theta) \leqslant \rho \leqslant r\}$$

by the starshaped quality of E and Ω. In the next lemma, we utilize that the characteristic function of E, φ_E, is of bounded variation in R^2 which follows from [4] (*Corollary 2.1*).

LEMMA 5.1. — *Let f satisfy* (5.1). *Then* $\mu(\theta)$ *defined by* (5.2) *is a lower semi-continuous function of bounded variation.*

Proof. — The characteristic function of E, $\varphi_E \in BV(R^2)$ as we have noted. This means that

$$\left|\int_{R^2} \varphi_E \zeta_{x_i}\, dx\right| \leqslant C \sup_{R^2} |\zeta|, \quad \zeta \in H_0^{1,\infty}(R^2)$$

for $i = 1, 2$ and some $C > 0$. Hence by Fubini's Theorem and (5.3)

$$\begin{aligned}\int_0^{2\pi}\int_{\mu(\theta)}^r \zeta_{x_i}\rho\, d\rho\, d\theta &= \int_0^{2\pi}\int_0^r \varphi_E \zeta_{x_i}\rho\, d\rho\, d\theta \\ &= \int_{R^2} \varphi_E \zeta_{x_i}\rho\, d\rho\, d\theta \\ &\leqslant C\|\zeta\|_{L^\infty(R^2)} \quad \text{for} \quad \zeta \in H_0^{1,\infty}(R^2).\end{aligned}$$

In particular, we choose $\zeta = \zeta(\theta) \in C^1(0,2\pi)$, periodic of period 2π, and $\eta(\rho)$ a function vanishing identically in a neighborhood of 0 in Ω, identically one in a neighborhood of E, and vanishing outside, say, B_{2r}. Applying the above to the product $\zeta(\theta)\eta(\rho)$ we see that

$$\begin{aligned}\int_0^{2\pi}\int_{\mu(\theta)}^r \left(\frac{1}{\rho}\zeta'\right)\rho\, d\rho\, d\theta &= -\int_0^{2\pi} \zeta'(\theta)(r - \mu(\theta))\, d\theta \\ &= \int_0^{2\pi} \mu(\theta)\zeta'(\theta)\, d\theta\end{aligned}$$

and hence, by the foregoing,

$$\left|\int_0^{2\pi} \mu(\theta)\zeta'(\theta)\, d\theta\right| \leqslant C \sup_{0\leqslant\theta\leqslant 2\pi} |\zeta|, \qquad \zeta \in C^1(0,2\pi).$$

We may invoke the Riesz Representation Theorem to the functional

$$\zeta \to \int_0^{2\pi} \zeta'(\theta)\mu(\theta)\, d\theta$$

defined and uniformly bounded on the dense subset $C^1(0,2\pi)$ of $C^0(0,2\pi)$ to infer the existence of

$$g(\theta) \in BV(0,2\pi)$$

with the properties

$$\int_0^{2\pi} \zeta'(\theta)\mu(\theta)\, d\theta = -\int_0^{2\pi} \zeta(\theta)\, dg(\theta) = \int_0^{2\pi} \zeta'(\theta)g(\theta)\, d\theta.$$

In particular, $\mu(\theta) - g(\theta) = \text{const.}$ a.e., which we may take to be zero, so that

$$\mu(\theta) = g(\theta) \quad \text{a.e. in} \quad [0,2\pi]. \tag{5.4}$$

We proceed to show that $\mu(\theta) = g(\theta)$ everywhere. We may assume that g is lower semicontinuous. Let us agree to further modify g so that

$$g(\theta) = \liminf_{t \to \theta} g(t) \tag{5.5}$$

It follows that $\mu(\theta) \leqslant g(\theta)$. For suppose that $g(\theta) < \mu(\theta)$ and select $\theta_k \to \theta$ such that $g(\theta) = \lim_{k \to \infty} g(\theta_k)$. Since μ is lower semi-continuous given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mu(\theta) - \varepsilon < \mu(t) \quad \text{for} \quad |t - \theta| < \delta.$$

Hence for k so large that

$$|g(\theta_k) - g(\theta)| < \varepsilon$$

we may find a neighborhood $I_k = (\theta_k - \delta_k, \ \theta_k + \delta_k)$,

$$I_k \cap I_h = \varnothing \quad \text{for} \quad h \neq k,$$

of θ_k with the property

$$\begin{aligned} \operatorname*{Var}_{I_k} g &\geqslant \max_{I_k} g - \min_{I_k} g \\ &\geqslant g(t) - (g(\theta) - \varepsilon) \quad \text{for any} \quad t \in I_k \\ &\geqslant \mu(t) - (g(\theta) - \varepsilon) \quad \text{for almost all} \quad t \in I_k \end{aligned}$$

by (5.4). Hence, by our choice of ε,

$$\operatorname*{Var}_{I_k} g \geqslant \mu(\theta) - g(\theta) - 2\varepsilon > 0$$

Consequently, $\operatorname{Var} g = +\infty$, a contradiction. Therefore once (5.5) is assumed, $\mu(\theta) \leqslant g(\theta)$ in $[0,2\pi)$. Observe that g satisfying (5.5) has no inessential discontinuities.

Consider the set

$$F = \{z : \rho e^{i\theta} : g(\theta) \leqslant \rho \leqslant r\} \subset E \quad \text{since} \quad \mu \leqslant g.$$

Since the points θ in $[0,2\pi]$ for which $g \neq \mu$ have measure zero,

$$N = E - F = \{z = \rho e^{i\theta} : \mu(\theta) \leqslant \rho < g(\theta)\}$$

satisfies meas $N = 0$. Furthermore F is closed by lower semi-continuity of g so $\overline{B}_r - F$ is open, $\Omega \subset \overline{B}_r - F$, and

$$\overline{B}_r - F = \Omega \cup N.$$

Recall here that $w \in H^{2,\infty}(B_r)$ since $f \in C^1(B_r)$ by *Corollary 3.2*. Inasmuch as $-\Delta w = f$ in Ω, we see that $-\Delta w = f$ a.e. in $\Omega \cup N$. Since $\Omega \cup N$ is open, we may deduce that

$$-\Delta w = f \quad \text{in} \quad \Omega \cup N$$

and

$$w \in C^{2,\lambda}(\Omega \cup N) \quad \text{for} \quad 0 < \lambda < 1.$$

Now consider $u(z) = 1 - \rho w_\rho(z)$, $z \in B_r$, which satisfies

$$\int_{\Omega \cup N} u_{x_i} \zeta_{x_i}\, dx = -\int_{\Omega \cup N} \frac{1}{\rho} (\rho^2 f)_\rho \zeta\, dx, \quad \zeta \in C_0^1(\Omega \cup N)$$

by *Lemma 4.2* (ii). Hence $u \in C^1(\Omega \cup N)$ and

$$\int_{N \cup \Omega} u_{x_i} \zeta_{x_i}\, dx \geqslant 0 \quad \text{when} \quad 0 \leqslant \zeta \in C_0^1(\Omega \cup N)$$

so that by the strong maximum principle

$$u(z) > \min_{\partial(\Omega \cup N)} u = 0$$

because $\partial(\Omega \cup N) \subset B_r - \Omega$ where $w_\rho = \dfrac{1}{\rho}$ and $w_\theta = 0$. In particular, $u(z) = 0$ for $z \in \partial(\Omega \cup N)$. However, if $z \in N$

$$w_\rho(z) = \frac{1}{\rho} \quad \text{and} \quad w_\theta(z) = 0$$

so that

$$u(z) = 1 - \rho w_\rho(z) = 0,$$

a contradiction. Therefore $N = \varnothing$, and

$$\mu(\theta) = g(\theta), \qquad 0 \leqslant \theta \leqslant 2\pi. \qquad \text{Q.E.D.}$$

THEOREM 4. — *Let* $f \in C^1(R^2)$ *satisfy* (5.1) *and let* $r, w \in K_r$ *denote the solution to Problem* (*) *for* f. *Let*

$$\Omega = \{z : w(z) > \log \rho\}.$$

Then the boundary Γ *of* Ω *has the representation*

$$\Gamma : \rho = \mu(\theta), \qquad 0 \leqslant \theta \leqslant 2\pi$$

where μ *is a continuous function of bounded variation.*

Proof. — Let $\mu(\theta)$ be defined by (5.2) so that the conclusion of *Lemma 5.1* holds. Suppose that $\theta = 0$ is a discontinuity of μ. Then $\theta = 0$ is a jump discontinuity so that

$$\lim_{\theta \to 0^-} \mu(\theta) = L > \lim_{\theta \to 0^+} \mu(\theta) = \mu(0)$$

without any loss in generality. For $\varepsilon > 0$ sufficiently small, there is a $\delta > 0$ so that the segments

$$\{z = \rho e^{i\theta} : 0 \leqslant \rho \leqslant L - \varepsilon\} \subset \Omega \quad \text{for} \quad -\delta < \theta < 0$$

and

$$\{z = \rho e^{i\theta} : \mu(0) + \varepsilon \leqslant \rho \leqslant r\} \subset E.$$

Hence we may find a disc $B_\eta(z_0)$, $z_0 = \frac{1}{2}(L + \mu(0))$, such that

$$B_\eta(z_0) \cap \Omega = \{z \in B_\eta(z_0) : \text{Im}\, z < 0\}$$

Let $\sigma = \{z : \text{Im}\, z = 0,\ z_0 - \eta < \text{Re}\, z < z_0 + \eta\}$ and set

$$u = 1 - \rho w_\rho.$$

It follows that $u \in C^1(\sigma \cup \Omega \cap B_\eta(z_0))$ and u attains its minimum value zero at each point of σ by Hopf's maximum principle and *Lemma 4.1* (ii). Therefore

$$\frac{\partial u}{\partial \nu}(z) < 0 \quad \text{for} \quad z \in \sigma.$$

But according to *Lemma 4.1.* (iii) with $\Gamma' = \sigma$

$$\frac{\partial u}{\partial \nu}(z) = \rho^2 f(z) \frac{d\theta}{ds}(z) = 0 \quad \text{for} \quad z \in \sigma$$

since $\theta = 0$ on σ. This is a contradiction. Q.E.D.

6.

In this paragraph we show that Γ has a smooth parameterization and that a solution to *Problem 1* exists in the classical sense. For this, we employ the results of [8]. In the case where f is real analytic, these questions may be treated by the results of H. Lewy [9].

THEOREM 5. — *Let* $f \in C^1(R^2)$ *satisfy* $\sup f < 0$ *and* $(\rho^2 f)_\rho \leqslant 0$ *in* R^2. *Let* r, $w \in K_r$ *denote the solution to Problem* (*) *for* f *and* Γ *the boundary of* $\Omega = \{z : w(z) > \log \rho\}$. *Then* Γ *has a* $C^{1,\tau}$ *parameterization,* $0 < \tau < 1$.

Proof. — From *Theorem 4* it is known that Γ is a Jordan curve. We now apply [8] (Theorem 1). Let $z_0 \in \Gamma$ and set $\omega = B_\varepsilon(z_0) \cap \Omega$, $\varepsilon < |z_0|$, and consider

$$g(z) = -\frac{1}{z} + \frac{1}{2}(w_{x_1}(z) - iw_{x_2}(z)) \quad z \in \overline{\Omega} - \{0\}.$$

From the known regularity of w, $g \in H^{1,\infty}(\omega)$. Furthermore

$$g_{\bar z}(z) = \frac{1}{4}\Delta w(z) = -\frac{1}{4} f(z), \quad z \in \omega$$
$$g(z) = 0 \qquad z \in \Gamma \cap \overline{\omega}$$

Since $-\frac{1}{4} f(z) > 0$ in $B_\varepsilon(z_0)$, we may conclude that a conformal mapping φ of $G = \{|t| < 1, \operatorname{Im} t > 0\}$ onto ω which maps $-1 < t < 1$ onto $\Gamma \cap \overline{\omega}$ has boundary values in $C^{1,\tau}$ for every τ, $0 < \tau < 1$.

THEOREM 6. — *Let* $F \in C^1(R^2)$ *satisfy* $\rho^{-2}F \in C^1(R^2)$ *and*

$$\inf \rho^{-2}F > 0$$
$$F_\rho \geqslant 0$$
$$F(0) = F_\rho(0) = 0.$$

Then there exists a domain Ω *and a function* $u \in H^{1,\infty}_{loc}(R^2)$ *such that*

$$-\Delta u = \rho^{-1}F_\rho \quad \text{in } \Omega \tag{6.1}$$
$$u = 0 \quad \text{on } \Gamma \tag{6.2}$$
$$u_\nu = -F\frac{d\theta}{ds} \text{ a.e.} \quad \text{on } \Gamma \tag{6.3}$$
$$u(0) = \gamma \tag{6.4}$$

where ν *is the outward directed normal vector and* s *is the arclength of* Γ *and* $\gamma > 0$ *is given.*

Proof. — Given F, define $f(z) = -\frac{1}{\gamma p^2} F(z)$ and observe that $\sup f < 0$ and $(\rho^2 f)_\rho \leqslant 0$ in R^2. Denote by $r, w \in K_r$ the solution to *Problem* (*) for f and define

$$u(z) = \gamma(1 - \rho w_\rho(z)) \quad z \in R^2.$$

Then, in view of *Corollary 3.2*, $u \in H^{1,\infty}_{loc}(R^2)$ and satisfies (6.1) (by *Lemma 4.1*), (6.2), and (6.4). Moreover,

$$\Omega = \{z : u(z) > 0\}.$$

According to *Theorem 5*, Γ has a $C^{1,\tau}$ parameterization $t \to \varphi(t)$, t real, where we may assume that

$$\varphi : \{t : \operatorname{Im} t > 0\} \to \Omega$$

is a conformal mapping. It is known that $\varphi'(t) \neq 0$ a.e., $-\infty < t < \infty$. In a neighborhood of any t_0 for which $\varphi'(t_0) \neq 0$, the tangent angle to Γ is of class $C^{0,\tau}$. From this one checks that u_ν is continuous in a neighborhood of $\varphi(t_0)$ in $\overline{\Omega}$, e.g., by use of conformal mapping. Now *Lemma 4.1* (iii) may by applied to verify (7.3) on this neighborhood of $\varphi(t_0)$ in Γ.

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