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CORNELIU CONSTANTINESCU

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# ON VECTOR MEASURES

by Corneliu CONSTANTINESCU

*Dédié à Monsieur M. Brelot à l'occasion de son 70e anniversaire.*

The aim of this paper is to prove some properties concerning the measures which take their values in Hausdorff locally convex spaces. $\delta$-rings of sets rather than $\sigma$-rings of sets will be used and a certain regularity of the measures will be assumed in order to include the Radon measures on Hausdorff topological spaces in these considerations.

A *ring of sets* is a set $\mathfrak{R}$ such that for any $A, B \in \mathfrak{R}$ we have $A \triangle B, A \cap B \in \mathfrak{R}$. A ring of sets is called a *$\sigma$-ring of sets* (resp *$\delta$-ring of sets*) if the union (resp. the intersection) of any countable family in $\mathfrak{R}$ belongs to $\mathfrak{R}$. Any $\sigma$-ring of sets is a $\delta$-ring of sets. Let G be Hausdorff topological additive group and let $\mathfrak{R}$ be a ring of sets. A *G-valued measure* on $\mathfrak{R}$ is a map $\mu$ of $\mathfrak{R}$ into G such that for any countable family $(A_\iota)_{\iota \in I}$ of pairwise disjoint sets of $\mathfrak{R}$ whose union belongs to $\mathfrak{R}$, the family $(\mu(A_\iota))_{\iota \in I}$ is summable and its sum is $\mu\left(\bigcup_{\iota \in I} A_\iota\right)$. Let $\mathfrak{K}$ be a set and let $\mathfrak{K}^u$ be the set of finite unions of sets of $\mathfrak{K}$ (then $\emptyset \in \mathfrak{K}^u$). For any $A \in \mathfrak{R}$ we denote by $\mathfrak{F}(A, \mathfrak{K})$ the filter on $\mathfrak{R}$ generated by the filter base

$$\{\{B \in \mathfrak{R} \mid K \subset B \subset A\} \mid K \in \mathfrak{K}^u, K \subset A\}.$$

A G-valued measure $\mu$ on $\mathfrak{R}$ will be called *$\mathfrak{K}$-regular* if for any $A \in \mathfrak{R}$, $\mu$ converges along $\mathfrak{F}(A, \mathfrak{K})$ to $\mu(A)$.

Any G-valued measure on $\mathfrak{R}$ is $\mathfrak{R}$-regular. A set $A \in \mathfrak{R}$ is called a *null set for* $\mu$ if $\mu(B) = 0$ for any $B \in \mathfrak{R}$ with $B \subset A$. Let $\mathfrak{R}$ be a ring of sets, let G, G′ be Hausdorff topological additive groups, and let $\mu$ (resp $\mu'$) be a G-valued (resp. G′valued) measure on $\mathfrak{R}$. We say that $\mu$ *is absolutely continuous with respect to* $\mu'$ (in symbols $\mu \ll \mu'$) if any null set for $\mu'$ is a null set for $\mu$. For any real valued measure $\mu$ on a $\sigma$-ring of sets $\mathfrak{R}$ we denote by $|\mu|$ the supremum of $\mu$ and $-\mu$ in the vector lattice of real valued measures on $\mathfrak{R}$. If $\mathfrak{K}$ is a set such that $\mu$ is $\mathfrak{K}$-regular then $|\mu|$ is $\mathfrak{K}$-regular.

PROPOSITION 1. — *Let* G *be a topological additive group whose one point sets are* $G_\delta$*-sets (*G *is therefore Hausdorff) and let* $(x_\iota)_{\iota \in I}$ *be a family in* G *such that any countable subfamily of it is summable. Then there exists a countable subset* J *of* I *such that* $x_\iota = 0$ *for any* $\iota \in I \setminus J$.

Let $(U_n)_{n \in \mathbf{N}}$ be a sequence of 0-neighbourhoods in G whose intersection is equal to $\{0\}$. The sets

$$J_n := \{\iota \in I \mid x_\iota \notin U_n\}$$

being finite for any $n \in \mathbf{N}$ the set $J := \bigcup_{n \in \mathbf{N}} J_n$ is countable. For any $\iota \in I \setminus J$ we get $x_\iota \in \bigcap_{n \in \mathbf{N}} U_n$ and therefore $x_\iota = 0$. ■

PROPOSITION 2. — *Let* G *be a topological additive group whose one point sets are* $G_\delta$*-sets, let* $\mathfrak{R}$ *be a* $\sigma$*-ring of sets, and let* $\mu$ *be a* G*-valued measure on* $\mathfrak{R}$. *Then there exists* $A \in \mathfrak{R}$ *such that* $\mu(B) = 0$ *for any* $B \in \mathfrak{R}$ *with* $B \cap A = \emptyset$.

Let us denote by $\Sigma$ the set of sets $\mathfrak{S}$ of pairwise disjoint sets of $\mathfrak{R}$ such that $\mu(S) \neq 0$ for any $S \in \mathfrak{S}$. It is obvious that $\Sigma$ is inductively ordered by the inclusion relation. By Zorn's theorem there exists a maximal element $\mathfrak{S}_0 \in \Sigma$. Then any countable subfamily of the family $(\mu(S))_{S \in \mathfrak{S}_0}$ is summable. By the preceding proposition $\mathfrak{S}_0$ is countable. We set

$$A := \bigcup_{S \in \mathfrak{S}_0} S.$$

Then $A \in \mathfrak{R}$. Let $B \in \mathfrak{R}$ with $B \cap A = \emptyset$. If $\mu(B) \neq 0$

then $\mathfrak{S}_0 \cup \{B\} \in \Sigma$ and this contradicts the maximality of $\mathfrak{S}_0$. ■

THEOREM 3. — *Let* T *be a Hausdorff topological space possessing a dense* $\sigma$*-compact set, let* E *be a locally convex space whose one point sets are* $G_\delta$*-sets, and let* $\mathscr{C}(T, E)$ *be the vector space of continuous maps of* T *into* E *endowed with the topology of pointwise convergence. Let further* $\mathfrak{R}$ *be a* $\sigma$*-ring of sets, let* $\mathfrak{K}$ *be a set, and let* $\mu$ *be a* $\mathfrak{K}$*-regular* $\mathscr{C}(T, E)$*-valued measure on* $\mathfrak{R}$*. Then there exists a positive* $\mathfrak{K}$*-regular real valued measure* $\nu$ *on* $\mathfrak{R}$ *such that* $\mu$ *is absolutely continuous with respect to* $\nu$.

Assume first $E = \mathbf{R}$ and let us denote by $\mathscr{C}_{\mathfrak{K}}(T)$ the vector space of continuous real functions on T endowed with the topology of compact convergence. Since T possesses a dense $\sigma$-compact set the one point sets of $\mathscr{C}_{\mathfrak{K}}(T)$ are $G_\delta$-sets.

Let us denote for any $t \in T$ by $\mu_t$ the map

$$A \longmapsto (\mu(A))(t) : \mathfrak{R} \to \mathbf{R}.$$

Then $\mu_t$ is a $\mathfrak{K}$-regular real valued measure on $\mathfrak{R}$ for any $t \in T$. Assume that for any countable subset M of T there exists $A \in \mathfrak{R}$ which is a null set for any $\mu_t$ with $t \in M$ and is not a null set for $\mu$. Let $\omega_1$ be the first uncountable ordinal number. We construct by transfinite induction a family $(t_\xi)_{\xi<\omega_1}$ in T and a decreasing family $(A_\xi)_{\xi<\omega_1}$ in $\mathfrak{R}$ such that we have for any $\xi < \omega_1$ :

*a*) $A_\xi$ is a null set for any $\mu_{t_\eta}$ with $\eta \leqslant \xi$;

*b*) any set $A \in \mathfrak{R}$ is a null set for $\mu$ if it is a null set for any $\mu_{t_\eta}$ with $\eta \leqslant \xi$ and if $A \cap A_\xi = \emptyset$;

*c*) $\bigcap_{\eta<\xi} A_\eta \setminus A_\xi$ is not a null set for $\mu$.

Assume that the families were constructed up to $\xi < \omega_1$. By the hypothesis of the proof there exists a set of $\mathfrak{R}$ which is a null set for any $\mu_{t_\eta}$ with $\eta < \xi$ and which is not a null set for $\mu$. Hence there exists $B \in \mathfrak{R}$ and $t_\xi \in T$ such that B is a null set for any $\mu_{t_\eta}$ with $\eta < \xi$ and such that

$$\mu_{t_\xi}(B) \neq 0.$$

Let $\mathfrak{R}'$ be the set of sets of $\mathfrak{R}$ which are null sets for any $\mu_{t_\eta}$ with $\eta \leqslant \xi$. Then $\mathfrak{R}'$ is a $\sigma$-ring of sets and by [7] Theorem II.4 (*) the map $\mathfrak{R}' \to \mathscr{C}_{\mathfrak{K}}(T)$ induced by $\mu$ is a measure. By the preceding proposition there exists $C \in \mathfrak{R}'$ such that any $D \in \mathfrak{R}'$ with $C \cap D = \emptyset$ is a null set for $\mu$. We set

$$A_\xi : = C \cap \Big(\bigcap_{\eta<\xi} A_\eta\Big).$$

*a*) is obviously fulfilled. Let $A \in \mathfrak{R}'$ with $A \cap A_\xi = \emptyset$. Then $A \setminus C \in \mathfrak{R}'$ and it is therefore a null set for $\mu$. For any $\eta < \xi$ the set $A \setminus A_\eta$ is a null set for $\mu$ by the hypothesis of the induction. Hence $A$ is a null set for $\mu$ and *b*) is fulfilled. Since $B \cap C$ is a null set for $\mu_{t_\xi}$ we get

$$\mu_{t_\xi}(B \setminus C) \neq 0.$$

For any $\eta < \xi$ the set $(B\setminus C)\setminus A_\eta$ is a null set for $\mu_{t_\zeta}$ for any $\zeta \leqslant \eta$ and by the hypothesis of the induction

$$(B\setminus C)\setminus A_\eta$$

is a null set for $\mu$. It follows that $(B\setminus C)\setminus \bigcap_{\eta<\xi} A_\eta$ is a null set for $\mu$ and therefore

$$\mu_{t_\xi}\Big((B\setminus C) \cap \Big(\bigcap_{\eta<\xi} A_\eta \setminus A_\xi\Big)\Big) = \mu_{t_\xi}\Big((B\setminus C) \cap \Big(\bigcap_{\eta<\xi} A_\eta\Big)\Big) \neq 0.$$

We deduce that $\bigcap_{\eta<\xi} A_\eta \setminus A_\xi$ is not a null set for $\mu$ which proves *c*).

Again by [7] Theorem II 4 any countable subfamily of the family $\Big(\mu\Big(\bigcap_{\eta<\xi} A_\eta \setminus A_\xi\Big)\Big)_{\xi<\omega_1}$ is summable in $\mathscr{C}_{\mathfrak{K}}(T)$ and this contradicts Proposition 1. Hence there exists a sequence $(t_n)_{n\in\mathbf{N}}$ in $T$ such that any set of $\mathfrak{R}$ is a null set for $\mu$ if it is a null set for any $\mu_{t_n}$ with $n \in \mathbf{N}$. We set

$$\alpha_n : = \sup_{A\in\mathfrak{R}} |\mu_{t_n}|(A) < \infty$$

(*) Or [8] Theorem 7.

([1], III 4.5). The map

$$A \longmapsto \sum_{n\in\mathbf{N}} \frac{1}{2^n} |\mu_{t_n}|(A) : \mathfrak{R} \to \mathbf{R}$$

is a positive $\mathfrak{K}$-regular real valued measure on $\mathfrak{R}$ and $\mu$ is absolutely continuous with respect to it.

Let us treat now the general case. Let $E'$ be the dual of $E$ endowed with the $\sigma(E', E)$-topology and let $(U_n)_{n\in\mathbf{N}}$ be a sequence of closed convex 0-neighbourhoods in $E$ whose intersection is equal to $\{0\}$ and sucht hat

$$U_{n+1} \subset \frac{1}{2} U_n \quad \text{for any} \quad n \in \mathbf{N}.$$

For any $n \in \mathbf{N}$ let $U_n^0$ be the polar set of $U_n$ in $E'$. Then, for any $n \in \mathbf{N}$, $U_n^0$ is a compact set of $E'$ and $\bigcup_{n\in\mathbf{N}} U_n^0$ is a dense set in $E'$. Let $T'$ be the topological (disjoint) sum of the sequence $(T \times U_n^0)_{n\in\mathbf{N}}$ of topological spaces. Then $T'$ is a Hausdorff topological space possessing a dense $\sigma$-compact set. Let $\mathscr{C}(T')$ be the vector space of continuous real functions on $T'$ endowed with the topology of pointwise convergence. For any $A \in \mathfrak{R}$ let us denote by $\lambda(A)$ the real function on $T'$ equal to

$$(t, x') \longmapsto \langle(\mu(A))(t), x'\rangle : T \times U_n^0 \to \mathbf{R}$$

on $T \times U_n^0$. It is easy to see that $\lambda(A) \in \mathscr{C}(T')$ and that $\lambda$ is a $\mathfrak{K}$-regular measure on $\mathfrak{R}$ with values in $\mathscr{C}(T')$. Let $A \in \mathfrak{R}$ be a null set for $\lambda$ and let $t \in T$. Since $(\mu(A))(t)$ vanishes on $\bigcup_{n\in\mathbf{N}} U_n^0$ and since this set is dense in $E'$ we deduce $(\mu(A))(t) = 0$. The point $t$ being arbitrary $\mu(A)$ vanishes. Hence $\mu$ is absolutely continuous with respect to $\lambda$. By the first part of the proof there exists a positive $\mathfrak{K}$-regular real valued measure $\nu$ on $\mathfrak{R}$ such that $\lambda$ is absolutely continuous with respect to $\nu$. Then $\mu$ is absolutely continuous with respect to $\nu$. ■

*Remark.* For $\mathfrak{K} = \mathfrak{R}$ this result could be deduced from [4] Theorem 2.2 and [3] Theorem 2.5. A simpler proof can be given by using [9] Theorem 2.3 or [10] Theorem 2.

2. Let $\mathfrak{R}$ be a $\delta$-ring of sets, let $\mathfrak{K}$ be a set, let E be a Hausdorff locally convex space, and let $\mathcal{M}$ be the set of $\mathfrak{K}$-regular E-valued measures on $\mathfrak{R}$. Then $\mathcal{M}$ is a subspace of the vector space $E^{\mathfrak{R}}$. For any continuous semi-norm $p$ on E and for any $\sigma$-ring of sets $\mathfrak{R}'$ contained in $\mathfrak{R}$ the map

$$\mu \longmapsto \sup_{A \in \mathfrak{R}'} p(\mu(A)) : \mathcal{M} \to \mathbf{R}_+$$

([1], III 4.5) is a semi-norm on $\mathcal{M}$. We shall call the topology on $\mathcal{M}$ generated by these semi-norms the *semi-norm topology* of $\mathcal{M}$. If $\mathfrak{R}$ is a $\sigma$-ring and E is $\mathbf{R}$ then the semi-norm topology on $\mathcal{M}$ is defined by the lattice norm

$$\mu \to \sup_{A \in \mathfrak{R}} |\mu|(A) : \mathcal{M} \to \mathbf{R}_+$$

and $\mathcal{M}$ endowed with this norm is an order complete Banach lattice.

Let $\mathfrak{R}$ be a $\sigma$-ring of sets and let $T(\mathfrak{R}) : = \bigcup_{A \in \mathfrak{R}} A$. A real function $f$ on $T(\mathfrak{R})$ is called $\mathfrak{R}$-*measurable* if for any positive real number $\alpha$ the sets $\{x | f(x) > \alpha\}$, $\{x | f(x) < -\alpha\}$ belong to $\mathfrak{R}$. Let $\mu$ be a real valued measure on $\mathfrak{R}$. $\mathcal{L}^1(\mu)$ will denote the set of $\mathfrak{R}$-measurable $\mu$-integrable real functions on $T(\mathfrak{R})$. Let $f$ be a subset of $\mathcal{L}^1(\mu)$ such that $f' = f''$ $\mu$-almost everywhere and therefore

$$\int f' \, d\mu = \int f'' \, d\mu$$

for any $f', f'' \in f$. We set

$$\int f \, d\mu : = \int f' \quad \mu,$$

where $f'$ is an arbitrary function of $f$. $L^1(\mu)$ and $L^\infty(\mu)$ will denote the usual Banach lattices and $\| \quad \|_\mu^1$, $\| \quad \|_\mu^\infty$ will denote their norms respectively. Any element of $L^\infty(\mu)$ is a subset of $\mathcal{L}^1(\mu)$ ([1], III 4.5).

Proposition 4. — *Let $\mathfrak{R}$ be a $\sigma$-ring of sets, let $\mathfrak{K}$ be a set, let $\mathcal{M}$ be the Banach lattice of $\mathfrak{K}$-regular real valued measures on $\mathfrak{R}$ and let*

$$\mathcal{F} : = \left\{ f \in \prod_{\mu \in \mathcal{M}} L^\infty(\mu) | \mu \ll \nu \Longrightarrow f_\nu \subset f_\mu \right\}.$$

*Then* $\mathscr{F}$ *is a subvector lattice of* $\prod_{\mu \in \mathfrak{M}} L^\infty(\mu)$ *such that for any subset of* $\mathscr{F}$ *which possesses a supremum in* $\prod_{\mu \in \mathfrak{M}} L^\infty(\mu)$ *this supremum belongs to* $\mathscr{F}$. *For any* $f \in \mathscr{F}$ *we have*

$$\|f\| := \sup \|f_\mu\|_\mu^\infty < \infty$$

*and the map*

$$f \longmapsto \|f\| : \mathscr{F} \to \mathbf{R}_+$$

*is a lattice norm.* $\mathscr{F}$ *endowed with it is a Banach lattice. For any* $f \in \mathscr{F}$ *we denote by* $\varphi(f)$ *the map*

$$\mu \longmapsto \int f_\mu \, d\mu : \mathscr{M} \to \mathbf{R}.$$

*Then* $\varphi(f)$ *belongs to the dual of* $\mathscr{M}$ *for any* $f \in \mathscr{F}$ *and* $\varphi$ *is an isomorphism of Banach lattices of* $\mathscr{F}$ *onto the dual of* $\mathscr{M}$.

Let $f, g \in \mathscr{F}$, let $\alpha \in \mathbf{R}$, and let $\mu, \nu \in \mathscr{M}$ such that $\mu \ll \nu$. Then $f_\nu \subset f_\mu$, $g_\nu \subset g_\mu$ and therefore

$$(f+g)_\nu = f_\nu + g_\nu \subset f_\mu + g_\mu = (f+g)_\mu,$$
$$(\alpha f)_\nu = \alpha f_\nu \subset \alpha f_\mu = (\alpha f)_\mu.$$

This shows that $\mathscr{F}$ is a vector subspace of $\prod_{\mu \in \mathfrak{M}} L^\infty(\mu)$.

Let $\mathscr{G}$ be a subset of $\mathscr{F}$ possessing a supremum $f$ in $\prod_{\mu \in \mathfrak{M}} L^\infty(\mu)$ and let $\mu, \nu \in \mathscr{M}$ such that $\mu \ll \nu$. Then for any $g \in \mathscr{G}$ we have $g_\nu \subset g_\mu$ and therefore

$$f_\nu = \sup_{g \in \mathscr{G}} g_\nu \subset \sup_{g \in \mathscr{G}} g_\mu = f_\mu.$$

Hence $\mathscr{F}$ is a subvector lattice of $\prod_{\mu \in \mathfrak{M}} L^\infty(\mu)$ such that for any subset of $\mathscr{F}$, which possesses a supremum in

$$\prod_{\mu \in \mathfrak{M}} L^\infty(\mu),$$

this supremum belongs to $\mathscr{F}$.

Let $f \in \mathscr{F}$. Assume

$$\sup_{\mu \in \mathfrak{M}} \|f_\mu\|_\mu^\infty = \infty.$$

Then there exists a sequence $(\mu_n)_{n\in\mathbf{N}}$ in $\mathscr{M}$ such that

$$\lim_{n\to\infty} \|f_{\mu_n}\|_{\mu_n}^{\infty} = \infty .$$

We set

$$\mu := \sum_{n\in\mathbf{N}} \frac{1}{2^n\|\mu_n\|} |\mu_n| .$$

Then $\mu_n \ll \mu$ for any $n \in \mathbf{N}$ and therefore $f_\mu \subset f_{\mu_n}$. We get

$$\|f_{\mu_n}\|_{\mu_n}^{\infty} \leqslant \|f_\mu\|_\mu^\infty,$$

and this leads to the contradictory relation

$$\infty = \lim_{n\to\infty} \|f_{\mu_n}\|_{\mu_n}^{\infty} \leqslant \|f_\mu\|_\mu^\infty < \infty .$$

Let $f, g \in \mathscr{F}$, and let $\alpha \in \mathbf{R}$. We have

$$\|f + g\| = \sup_{\mu\in\mathscr{M}} \|f_\mu + g_\mu\|_\mu^\infty \leqslant \sup_{\mu\in\mathscr{M}} (\|f_\mu\|_\mu^\infty + \|g_\mu\|_\mu^\infty) \leqslant \|f\| + \|g\|,$$

$$\|\alpha f\| = \sup_{\mu\in\mathscr{M}} \|\alpha f_\mu\|_\mu^\infty = \sup_{\mu\in\mathscr{M}} |\alpha| \|f_\mu\|_\mu^\infty = |\alpha| \|f\| ,$$

$$f = 0 \Longleftrightarrow (\mu \in \mathscr{M} \Longrightarrow \|f_\mu\|_\mu^\infty = 0) \Longleftrightarrow \|f\| = 0,$$

$$|f| \leqslant |g| \Longrightarrow \|f\| = \sup_{\mu\in\mathscr{M}} \|f_\mu\|_\mu^\infty \leqslant \sup_{\mu\in\mathscr{M}} \|g_\mu\|_\mu^\infty = \|g\|$$

Hence

$$f \longmapsto \|f\| : \mathscr{F} \to \mathbf{R}_+$$

is a lattice norm.

Let $f \in \mathscr{F}$, let $\mu, \nu \in \mathscr{M}$, and let $\alpha \in \mathbf{R}$. Then

$$f_{|\mu|+|\nu|} \subset f_\mu \cap f_\nu \subset f_{\mu+\nu}, \quad f_\mu \subset f_{\alpha\mu},$$

and therefore

$$\begin{aligned}(\varphi(f))(\mu + \nu) &= \int f_{|\mu|+|\nu|}\, d(\mu + \nu)\\ &= \int f_{|\mu|+|\nu|}\, d\mu + \int f_{|\mu|+|\nu|}\, d\nu = (\varphi(f))(\mu) + (\varphi(f))(\nu),\end{aligned}$$

$$(\varphi(f))(\alpha\mu) = \int f_\mu\, d(\alpha\mu) = \alpha \int f_\mu\, d\mu = \alpha(\varphi(f))(\mu).$$

This shows that $\varphi(f)$ is linear. From

$$|(\varphi(f))(\mu)| = \left|\int f_\mu\, d\mu\right| \leqslant \|f_\mu\|_\mu^\infty \|\mu\| \leqslant \|f\| \|\mu\|$$

we get $\|\varphi(f)\| \leqslant \|f\|$. Hence $\varphi(f)$ belongs to the dual of $\mathscr{M}$. It is obvious that $\varphi$ is an injection and that $\varphi$ maps the positive elements of $\mathscr{F}$ into positive linear forms on $\mathscr{M}$.

Let us prove now that $\varphi$ is a surjection. Let $\theta$ be a continuous linear form on $\mathscr{M}$ and let $\mu \in \mathscr{M}$. For any $g \in L^1(\mu)$ we denote by $g.\mu$ the map $A \longmapsto \int_A g\, d\mu : \mathfrak{R} \to \mathbf{R}$. Then $g.\mu \in \mathscr{M}$ and the map $g \longmapsto \theta(g.\mu) : L^1(\mu) \to \mathbf{R}$ is a continuous linear form on $L^1(\mu)$. Hence there exists $f_\mu \in L^\infty(\mu)$ such that $\|f_\mu\|_\mu^\infty \leqslant \|\theta\|$ and

$$\theta(g.\mu) = \int f_\mu g\, d\mu$$

for any $g \in L^1(\mu)$. Let $\mu, \nu \in \mathscr{M}$ such that $\mu \ll \nu$. By Lebesgue-Radon-Nikodym theorem there exists $h \in L^1(\nu)$ such that $\mu = h.\nu$. We get for any $g \in L^1(\mu)$, $gh \in L^1(\nu)$ and

$$\int f_\mu g\, d\mu = \theta(g.\mu) = \theta(gh.\nu) = \int f_\nu gh\, d\nu = \int f_\nu g\, d\mu.$$

This shows that $f_\nu \subset f_\mu$. Hence $f := (f_\mu)_{\mu \in \mathfrak{M}} \in \mathscr{F}$ and it is clear that $\varphi(f) = \theta$. Moreover

$$\|f\| = \sup_{\mu \in \mathfrak{M}} \|f_\mu\|_\mu^\infty \leqslant \|\theta\|.$$

Hence $\varphi$ is an isomorphism of normed vector lattices. We deduce that $\mathscr{F}$ is a Banach lattice. ■

PROPOSITION 5. — *Let $\mathfrak{R}$ be a $\delta$-ring of sets and let $\mathfrak{R}_1$, $\mathfrak{R}_2$ be $\sigma$-ring of sets contained in $\mathfrak{R}$. Then there exists a $\sigma$-ring of sets $\mathfrak{R}_0$ contained in $\mathfrak{R}$ and containing $\mathfrak{R}_1 \cup \mathfrak{R}_2$ and such that any set of $\mathfrak{R}$ which is contained in a set of $\mathfrak{R}_0$ belongs to $\mathfrak{R}_0$.*

Let us denote by $\mathfrak{R}_0$ the set of $A \in \mathfrak{R}$ for which there exists $(B, C) \in \mathfrak{R}_1 \times \mathfrak{R}_2$ such that $A \subset B \cup C$. It is easy to check that $\mathfrak{R}_0$ possesses the required properties. ■

PROPOSITION 6. — *Let $\mathfrak{R}$ be a $\delta$-ring of sets, let $\mathfrak{K}$ be a set, and let $\mathfrak{R}'$ be a $\sigma$-ring of sets contained in $\mathfrak{R}$ and such that any set of $\mathfrak{R}$ contained in a set of $\mathfrak{R}'$ belongs to $\mathfrak{R}'$. Let further E be a Hausdorff locally convex space, let $\mathscr{M}$ (resp. $\mathscr{M}_0$) be the vector space of $\mathfrak{K}$-regular E-valued measures on $\mathfrak{R}$ (resp. $\mathfrak{R}'$) endowed with the semi-norm topology, and let $\mathscr{M}'$ (resp. $\mathscr{M}_0'$) be its dual. For any $\mu \in \mathscr{M}$ we have $\mu|\mathfrak{R}' \in \mathscr{M}_0$ and the map $\varphi$*

$$\mu \longmapsto \mu|\mathfrak{R}' : \mathscr{M} \to \mathscr{M}_0$$

*is linear and continuous. Let* $p$ *be a continuous semi-norm on* E, *let* $\mathcal{N}$ *(resp.* $\mathcal{N}_0$*) be the set of* $\mu \in \mathcal{M}$ *(resp.* $\mu \in \mathcal{M}_0$*) such that*

$$\sup_{A \in \mathfrak{R}'} p(\mu(A)) \leqslant 1,$$

*let* $\mathcal{N}^0$ *(resp.* $\mathcal{N}_0^0$*) be its polar set in* $\mathcal{M}'$ *(resp.* $\mathcal{M}_0'$*) and let* $\varphi' : \mathcal{M}_0' \to \mathcal{M}'$ *be the adjoint map of* $\varphi$. *Then* $\varphi'(\mathcal{N}_0^0) = \mathcal{N}^0$.

It is obvious that $\mu \in \mathcal{M}$ implies $\mu|\mathfrak{R}' \in \mathcal{M}_0$, that $\varphi$ is linear and continuous, and that $\varphi(\mathcal{N}) \subset \mathcal{N}_0$. Hence

$$\varphi'(\mathcal{N}_0^0) \subset \mathcal{N}^0.$$

Let $\theta \in \mathcal{N}^0$ and let $\nu \in \mathcal{M}_0$. For any $A \in \mathfrak{R}'$ we denote by $\nu_A$ the map

$$B \longmapsto \nu(A \cap B) : \mathfrak{R} \to E.$$

It is immediate that $\nu_A \in \mathcal{M}$. Let F be the quotient locally convex space $E/p^{-1}(0)$ and let $u$ be the canonical map $E \to F$. Then the one point sets of F are $G_\delta$-sets and $u \circ \nu$ is an F-valued measure on $\mathfrak{R}'$. By Proposition 2 there exists $A \in \mathfrak{R}'$ such that any $B \in \mathfrak{R}'$ with $B \cap A = \varnothing$ is a null set for $u \circ \nu$. Let $A' \in \mathfrak{R}'$, $A \subset A'$. For any $B \in \mathfrak{R}$ the set $A' \cap B \setminus A \cap B$ is a null set for $u \circ \nu$ and therefore

$$p(\nu_{A'}(B) - \nu_A(B)) = 0.$$

Hence $\nu_{A'} - \nu_A \in \varepsilon\mathcal{N}$ for any $\varepsilon > 0$. We get $\theta(\nu_{A'}) = \theta(\nu_A)$. Hence if $\mathfrak{F}$ denotes the section filter of $\mathfrak{R}'$ ordered by the inclusion relation then the map

$$A \longmapsto \theta(\nu_A) : \mathfrak{R}' \to \mathbf{R}$$

converges along $\mathfrak{F}$.

Let $\theta \in \mathcal{N}^0$. With the above notations we set for any $\nu \in \mathcal{M}_0$

$$\theta_0(\nu) := \lim_{A, \mathfrak{F}} \theta(\nu_A).$$

It is easy to see that $\theta_0$ is a linear form on $\mathcal{M}_0$. If $\nu \in \mathcal{N}_0$ then $\nu_A \in \mathcal{N}$ for any $A \in \mathfrak{R}'$ and therefore $|\theta_0(\nu)| \leqslant 1$. It follows $\theta_0 \in \mathcal{N}_0^0$. Let $\mu \in \mathcal{M}$. We set $\nu := \varphi(\mu)$. Let A be a set of $\mathfrak{R}'$ such that any $B \in \mathfrak{R}'$ with $B \cap A = \varnothing$

is a null set for $u \circ \nu$. Then $\theta_0(\nu) = \theta(\nu_A)$. For any $B \in \mathfrak{R}'$ we have

$$p(\mu(B) - \nu_A(B)) = p(\mu(B - A \cap B)) = 0.$$

Hence $\mu - \nu_A \in \varepsilon\mathcal{N}$ for any $\varepsilon > 0$ and therefore

$$\theta(\mu) = \theta(\nu_A).$$

We get

$$\langle \mu, \varphi'(\theta_0) \rangle = \langle \varphi(\mu), \theta_0 \rangle = \langle \nu, \theta_0 \rangle = \langle \nu_A, \theta \rangle = \langle \mu, \theta \rangle.$$

Since $\mu$ is arbitrary it follows $\varphi'(\theta_0) = \theta$. Hence

$$\varphi'(\mathcal{N}_0^0) = \mathcal{N}^0. \blacksquare$$

PROPOSITION 7. — *Let $\mathfrak{R}$ be a $\delta$-ring of sets, let $\mathfrak{K}$ be a set, let $\Gamma$ be the set of $\sigma$-rings of sets $\mathfrak{R}'$ contained in $\mathfrak{R}$ and such that any set of $\mathfrak{R}$ contained in a set of $\mathfrak{R}'$ belongs to $\mathfrak{R}'$, and let E be a Hausdorff locally convex space. For any $\mathfrak{R}' \in \Gamma \cup \{\mathfrak{R}\}$ let $\mathcal{M}(\mathfrak{R}')$ be the vector space of $\mathfrak{K}$-regular E-valued measures on $\mathfrak{R}'$ endowed with the semi-norm topology, let $\mathcal{M}(\mathfrak{R}')'$ be its dual, let $\varphi_{\mathfrak{R}'}$ be the map*

$$\mu \longmapsto \mu|\mathfrak{R}' : \mathcal{M}(\mathfrak{R}) \to \mathcal{M}(\mathfrak{R}')$$

*(Proposition 6), and let $\varphi'_{\mathfrak{R}'} : \mathcal{M}(\mathfrak{R}')' \to \mathcal{M}(\mathfrak{R})'$ be its adjoint map. Then*

$$\mathcal{M}(\mathfrak{R})' = \bigcup_{\mathfrak{R}' \in \Gamma} \varphi'_{\mathfrak{R}'}(\mathcal{M}(\mathfrak{R}')').$$

Let $\theta \in \mathcal{M}(\mathfrak{R})'$. By Proposition 5 there exists $\mathfrak{R}' \in \Gamma$ and a continuous semi-norm $p$ on E such that $|\theta(\mu)| \leqslant 1$ for any $\mu \in \mathcal{M}(\mathfrak{R})$ with

$$\sup_{A \in \mathfrak{R}'} p(\mu(A)) \leqslant 1.$$

By Proposition 6 there exists $\theta_0 \in \mathcal{M}(\mathfrak{R}')'$ such that

$$\varphi'_{\mathfrak{R}'}(\theta_0) = \theta. \blacksquare$$

3. Let $\mathfrak{R}$ be a $\delta$-ring of sets, let $\mathfrak{K}$ be a set, let $\mathcal{M}$ be the vector space of $\mathfrak{K}$-regular real valued measures on $\mathfrak{R}$ endowed with the semi-norm topology, and let $\mathcal{M}'$ be its dual. Let further E be a Hausdorff locally convex space, let E′ be its dual, and let $\mu$ be a $\mathfrak{K}$-regular E-valued

measure on $\mathfrak{R}$. Then for any $x' \in E'$, $x' \circ \mu$ belongs to $\mathcal{M}$. If $\theta \in \mathcal{M}'$ then

$$x' \longmapsto \langle x' \circ \mu, \theta \rangle : E' \to \mathbf{R}$$

is a linear form on $E'$. If there exists $x \in E$ such that

$$\langle x' \circ \mu, \theta \rangle = \langle x, x' \rangle$$

for any $x' \in E'$ we say that $\theta$ is *$\mu$-integrable*. Then $x$ is uniquely defined by the above relation and we shall denote it by $\int \theta \, d\mu$. Any $A \in \mathfrak{R}$ may be considered as an element of $\mathcal{M}'$ namely as the linear form $\theta_A$ on $\mathcal{M}$

$$\nu \longmapsto \nu(A) : \mathcal{M} \to \mathbf{R}.$$

It is easy to see that

$$A \longmapsto \theta_A : \mathfrak{R} \to \mathcal{M}'$$

is an injection, that $\theta_A$ is $\mu$-integrable and

$$\int \theta_A \, d\mu = \mu(A).$$

If any $\theta \in \mathcal{M}'$ is $\mu$-integrable we say that the measure $\mu$ is *normal*. It will be shown in Theorem 10 that if $E$ is quasi-complete then any $E$-valued measure is normal. If $\mathfrak{R}$ is a $\sigma$-ring of sets then any bounded $\mathfrak{R}$-measurable real function $f$ may be considered as a map $\theta_f$

$$\nu \longmapsto \int f \, d\nu : \mathcal{M} \to \mathbf{R}$$

which obviously belongs to $\mathcal{M}'$. For any normal measure $\mu$ we shall write

$$\int f \, d\mu := \int \theta_f \; \mu.$$

If $\mu$ is a normal measure then it may be regarded as a map

$$\theta \longmapsto \int \theta \, d\mu : \mathcal{M}' \to E$$

and, identifying $\mathfrak{R}$ with a subset of $\mathcal{M}'$ via the above injection, this map is an extension of $\mu$ to $\mathcal{M}'$. If $\mathcal{N}$ is a set of normal $\mathfrak{K}$-regular E-valued measures on $\mathfrak{R}$ then, taking into account the above extensions of the normal measures, it may be regarded as a set of maps of $\mathcal{M}'$ into $E$ and so we may speak of the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{M}'$.

We want to make still another remark. If $F$ is another Hausdorff locally convex space and if $u : E \to F$ is a continuous linear map then for any $\mathfrak{K}$-regular $E$-valued measure $\mu$ on $\mathfrak{R}$ the map $u \circ \mu$ is a $\mathfrak{K}$-regular $F$-valued measure on $\mathfrak{R}$. Moreover any $\mu$-integral $\theta \in \mathscr{M}'$ is $u \circ \mu$-integral and

$$\int \theta \, d(u \circ \mu) = u\left(\int \theta \, d\mu\right).$$

Proposition 8. — *Let $\mathfrak{R}$ be a $\delta$-ring of sets, let $\mathfrak{K}$ be a set, let $\mathscr{M}$ be the vector space of $\mathfrak{K}$-regular real valued measures on $\mathfrak{R}$ endowed with the semi-norm topology, and let $\mathscr{M}'$ be its dual. Let further $E$ be a Hausdorff locally convex space, let $\mathscr{M}(E)$ be the vector space of $\mathfrak{K}$-regular $E$-valued measures on $\mathfrak{R}$ endowed with the topology of pointwise convergence in $\mathfrak{R}$, and let $\mathscr{N}$ be a compact set of $\mathscr{M}(E)$ such that any measure of $\mathscr{N}$ is normal. Then the topologies on $\mathscr{N}$ of pointwise convergence in $\mathfrak{R}$ or in $\mathscr{M}'$ coincide.*

Since $\mathfrak{R}$ may be identified with a subset of $\mathscr{M}'$ we have only to show that the topology on $\mathscr{N}$ of pointwise convergence in $\mathfrak{R}$ is finer than the topology on $\mathscr{N}$ of pointwise convergence in $\mathscr{M}'$. By Proposition 7 we may assume that $\mathfrak{R}$ is a $\sigma$-ring of sets. Let $\theta \in \mathscr{M}'$ and let $p$ be a continuous semi-norm on $E$. We denote by $E_p$ the normed quotient space $E/p^{-1}(0)$, by $u_p$ the canonical map $E \to E_p$, and by $\mathscr{C}(\mathscr{N}, E_p)$ the vector space of continuous maps of $\mathscr{N}$ (endowed with the topology of pointwise convergence in $\mathfrak{R}$) into $E_p$ endowed with the topology of pointwise convergence. For any $A \in \mathfrak{R}$ let $\lambda(A)$ be the map

$$\mu \longmapsto u_p \circ \mu(A) : \mathscr{N} \to E_p.$$

Then $\lambda(A) \in \mathscr{C}(\mathscr{N}, E_p)$ and it is obvious that $\lambda$ is a $\mathfrak{K}$-regular measure on $\mathfrak{R}$ with values in $\mathscr{C}(\mathscr{N}, E_p)$. By theorem 3 there exists a $\mathfrak{K}$-regular real valued measure $\nu$ on $\mathfrak{R}$ such that $\lambda$ is absolutely continuous with respect to $\nu$. By Proposition 4 there exists a bounded $\mathfrak{R}$-measurable real function $f$ on $\bigcup_{A \in \mathfrak{R}} A$ such that

$$\theta(\rho) = \int f \, d\rho$$

for any $\mathfrak{K}$-regular real valued measure $\rho$ on $\mathfrak{R}$ which is absolutely continuous with respect to $\nu$. Let $E'_p$ be the dual of $E_p$. Then for any $x' \in E'_p$ and for any $\mu \in \mathcal{N}$ the map $x' \circ u_p \circ \mu$ is a $\mathfrak{K}$-regular real valued measure on $\mathfrak{R}$ absolutely continuous with respect to $\nu$. Hence

$$\langle x' \circ u_p \circ \mu, \theta \rangle = \int f \, d(x' \circ u_p \circ \mu)$$

for any $\mu \in \mathcal{N}$ and for any $x' \in E'_p$. We get

$$u_p\left(\int \theta \, d\mu\right) = \int \theta \, d(u_p \circ \mu) = \int f \, d(u_p \circ \mu)$$

for any $\mu \in \mathcal{N}$. Let $(f_n)_{n \in \mathbf{N}}$ be a sequence of step functions with respect to $\mathfrak{R}$ converging uniformly to $f$. Since $\mathcal{N}$ is compact the set $\{\mu(A) | \mu \in \mathcal{N}\} \subset E$ is bounded for any $A \in \mathfrak{R}$. We deduce that the set $\{\mu(A) | \mu \in \mathcal{N}, A \in \mathfrak{R}\}$ is bounded ([5], Corollary 6). Hence the sequence

$$\left(\mu \longmapsto \int f_n \, d\mu : \mathcal{N} \to E\right)_{n \in \mathbf{N}}$$

of functions on $\mathcal{N}$ converges uniformly to the function

$$\mu \longmapsto \int f \, d\mu : \mathcal{N} \to E.$$

The functions of the sequence being continuous with respect to the topology on $\mathcal{N}$ of pointwise convergence in $\mathfrak{R}$ we deduce that the last function is continuous with respect to this topology. We deduce further that the map

$$\mu \longmapsto u_p\left(\int \theta \, d\mu\right) : \mathcal{N} \to E_p$$

is continuous with respect to the topology on $\mathcal{N}$ of pointwise convergence in $\mathfrak{R}$. Since $p$ is arbitrary it follows that the map

$$\mu \longmapsto \int \theta \, d\mu : \mathcal{N} \to E$$

is continuous with respect to this topology. Since $\theta$ is arbitrary the topology on $\mathcal{N}$ of pointwise convergence in $\mathfrak{R}$ is finer than the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{M}'$. ■

Corollary. — *Let $\mathfrak{R}$ be a $\sigma$-ring of sets, let $\mathfrak{K}$ be a set, and let $\mathcal{N}$ be a set of $\mathfrak{K}$-regular real valued measures on $\mathfrak{R}$*

*compact with respect to the topology of pointwise convergence in* $\mathfrak{R}$. *Then any sequence in* $\mathcal{N}$ *possesses a convergent subsequence with respect to this topology.*

Let $\mathcal{M}$ be the vector space of $\mathfrak{K}$-regular real valued measures on $\mathfrak{R}$ endowed with the semi-norm topology. By the proposition, $\mathcal{N}$ is weakly compact in $\mathcal{M}$ and the assertion follows from Šumlian theorem. ■

Let X be an ordered set and let Y be a topological space. We say that a map $f \colon X \to Y$ is *order continuous* if for any upper directed subset A of X possessing a supremum $x \in X$ the map $f$ converges along the section filter of A to $f(x)$. An ordered set X is called *order* $\sigma$*-complete* if any upper bounded increasing sequence in X possesses a supremum.

Theorem 9. — *Let* E *be an order* $\sigma$*-complete vector lattice, let* F *be a locally convex space, and let* $u$ *be a linear map of* E *into* F. *If* $u$ *is order continuous with respect to the weak topology of* F *then it is order continuous with respect to the initial topology of* F.

Let U be a 0-neighbourhood in F, let $U^0$ be its polar set in the dual $F'$ of F endowed with the induced $\sigma(F', F)$-topology, let $\mathcal{C}(U^0)$ (resp. $\mathcal{C}_u(U^0)$) be the vector space of continuous real functions on $U^0$ endowed with the topology of pointwise convergence (resp. with the topology of uniform convergence), and let us denote for any $x \in E$ by $f(x)$ the map

$$y' \longmapsto \langle u(x), y' \rangle : U^0 \to \mathbf{R}$$

which obviously belongs to $\mathcal{C}(U^0)$.

Let $(x_n)_{n \in \mathbf{N}}$ be an increasing sequence in E with supremum $x \in E$. Then for any $M \subset \mathbf{N}$ $\left(\sum_{\substack{n \in M \\ n \leqslant m}} (x_{n+1} - x_n)\right)_{m \in \mathbf{N}}$ is an upper bounded increasing sequence in E and possesses therefore a supremum. Since $u$ is order continuous with respect to the weak topology of E it follows that

$$(f(x_{n+1} - x_n))_{n \in M}$$

is summable in $\mathcal{C}(U^0)$. The space $U^0$ being compact we deduce by [7] Theorem II 4 that $(f(x_{n+1} - x_n))_{n \in \mathbf{N}}$ is summable in $\mathscr{C}_u(U^0)$. Its sum has to be $f(x - x_0)$. Hence

$$(f(x_n))_{n \in \mathbf{N}}$$

converges uniformly to $f(x)$.

Let now A be an upper directed subset of E with supremum $x \in E$ and let $\mathfrak{F}$ be its section filter. If $f$ does not map $\mathfrak{F}$ into a Cauchy filter on $\mathscr{C}_u(U^0)$ then it is easy to construct an increasing sequence $(x_n)_{n \in \mathbf{N}}$ in A such that $(f(x_n))_{n \in \mathbf{N}}$ is not a Cauchy sequence in $\mathscr{C}_u(U^0)$. Since E is order $\sigma$-complete and $(x_n)_{n \in \mathbf{N}}$ is upper bounded by $x$ it possesses a supremum and this contradicts the above considerations. Hence $f$ maps $\mathfrak{F}$ into a Cauchy filter on $\mathscr{C}_u(U^0)$ and therefore, by the completeness of $\mathscr{C}_u(U^0)$ into a convergent filter on $\mathscr{C}_u(U^0)$. Using again the hypothesis that $u$ is order continuous with respect to the weak topology of F we deduce that $f(\mathfrak{F})$ converges to $f(x)$ in $\mathscr{C}(U^0)$ and therefore in $\mathscr{C}_u(U^0)$. Since U is arbitrary it follows that $u$ converges along $\mathfrak{F}$ to $u(x)$ in the initial topology of F which shows that $u$ is order continuous with respect to this topology. ■

Let E be a locally convex space, let E′ be its dual endowed with the $\sigma(E', E)$-topology, and let $\hat{E}$ be the set of linear forms $y$ on E′ such that for any $\sigma$-compact set A of E′ there exists $x \in E$ such that $x$ and $y$ coincide on $\overline{A}$. We say that E is *δ-complete* if $\hat{E} = E$.

LEMMA. — *Any quasicomplete locally convex space is δ-complete.*

Let E be a quasicomplete locally convex space and let $y \in \hat{E}$ (with the above notations). Let $\mathfrak{U}$ be the neighbourhood filter of 0 in E and for any $U \in \mathfrak{U}$ let $U^0$ be its polar set in the dual of E and let $A_U$ be the set of $x \in E$ such that $x$ and $y$ coincide on $\bigcup_{n \in \mathbf{N}} nU^0$. It is obvious that there exists $\alpha_U \in \mathbf{R}$ such that $A_U \subset \alpha_U U$. Let $\mathfrak{F}$ be the filter on E generated by the filter base $\{A_U | U \in \mathfrak{U}\}$. Then $\mathfrak{F}$ is a Cauchy filter on E containing the bounded set $\bigcap_{U \in \mathfrak{U}} \alpha_U U$ and converging to $y$ uniformly on the sets $U^0 (U \in \mathfrak{U})$.

Since E is quasicomplete $y \in E$ and therefore E is $\delta$-complete. ■

*Remark.* — $l^1$ endowed with its weak topology is sequentially complete and $\delta$-complete but it is not quasicomplete.

THEOREM 10. — *Let* $\mathfrak{R}$ *be a* $\delta$*-ring of sets, let* $\mathfrak{K}$ *be a set, let* $\mathscr{M}$ *be the vector space of* $\mathfrak{K}$*-regular real valued measures on* $\mathfrak{R}$ *endowed with the semi-norm topology, and let* $\mathscr{M}'$ *be its dual endowed with the Mackey* $\tau(\mathscr{M}', \mathscr{M})$*-topology. Let further* E *be a Hausdorff sequentially complete* $\delta$*-complete locally convex space, let* E′ *be its dual, let* $\mathscr{L}$ *be the vector space of continuous linear maps of* $\mathscr{M}'$ *into* E *endowed with the topology of uniform convergence on the equicontinuous sets of* $\mathscr{M}'$*, and let* $\mathscr{M}(E)$ *be the vector space of* $\mathfrak{K}$*-regular* E*-valued measures on* $\mathfrak{R}$ *endowed with the semi-norm topology. Then for any* $\theta \in \mathscr{M}'$ *and for any* $\mu \in \mathscr{M}(E)$ *there exists a unique element* $\int \theta \, d\mu$ *of* E *such that*

$$\langle x' \circ \mu, \theta \rangle = \left\langle \int \theta \, d\mu, x' \right\rangle$$

*for any* $x' \in E'$. *For any* $\mu \in \mathscr{M}(E)$ *the map* $\psi(\mu)$

$$\theta \longmapsto \int \theta \, d\mu : \mathscr{M}' \to E$$

*belongs to* $\mathscr{L}$ *and it is order continuous.* $\psi$ *is a linear injection of* $\mathscr{M}(E)$ *into* $\mathscr{L}$ *which induces a homeomorphism of* $\mathscr{M}(E)$ *onto the subspace* $\psi(\mathscr{M}(E))$ *of* $\mathscr{L}$*. For any* $\sigma$*-ring of sets* $\mathfrak{R}'$ *contained in* $\mathfrak{R}$ *and for any* $\mu \in \mathscr{M}(E)$ *the closed convex circled hull of* $\{\mu(A) \mid A \in \mathfrak{R}'\}$ *is weakly compact in* E.

In order to prove the existence of $\int \theta \, d\mu$ we may assume by Proposition 7 that $\mathfrak{R}$ is a $\sigma$-ring of sets. Let $\mathscr{F}$ be the Banach space of bounded $\mathfrak{R}$-measurable real functions on $\bigcup_{A \in \mathfrak{R}} A$ with the supremum norm. Since E is sequentially complete we may define in the usual way $\int f \, d\mu \in E$ for any $f \in \mathscr{F}$. Let A be a subset of E′ $\sigma$-compact with respect to the $\sigma(E', E)$-topology. By Theorem 3 there exists $\nu \in \mathscr{M}$ such that $x' \circ \mu \ll \nu$ for any $x' \in \overline{A}$. By Proposition 4 there exists $f \in \mathscr{F}$ such that

$$\langle x' \circ \mu, \theta \rangle = \int f d(x' \circ \mu) = \left\langle \int f\, d\mu, x' \right\rangle$$

for any $x' \in \overline{A}$. Since E is $\delta$-complete there exists

$$\int \theta\, d\mu \in E$$

such that

$$\langle x' \circ \mu, \ \theta \rangle = \left\langle \int \theta\, d\mu, \ x' \right\rangle$$

for any $x' \in E'$.

Let $\mu \in \mathscr{M}(E)$. It is obvious that $\psi(\mu)$ is linear and from the relation defining it, it follows that it is continuous with respect to the $\sigma(\mathscr{M}', \mathscr{M})$ and $\sigma(E, E')$ topologies. We deduce that $\psi(\mu)$ belongs to $\mathscr{L}$. From Proposition 4 or from the theory of Banach lattices we deduce that $\psi(\mu)$ is order continuous with respect to the weak topology of E. By the preceding theorem it is order continuous with respect to the initial topology of E.

It is obvious that $\psi$ is linear. Let $\mu \in \mathscr{M}(E)$ such that $\psi(\mu) = 0$. Let $A \in \mathfrak{R}$ and let $\theta$ be the map

$$\nu \longmapsto \nu(A) : \mathscr{M} \to \mathbf{R}.$$

Then $\theta \in \mathscr{M}'$ and we get

$$\mu(A) = \int \theta\, d\mu = (\psi(\mu))(\theta) = 0.$$

Since A is arbitrary we get $\mu = 0$. Hence $\psi$ is an injection.

Let $p$ be a continuous semi-norm on E and let $\mathscr{A}$ be an equicontinuous set of $\mathscr{M}'$. Then there exists a $\sigma$-ring of sets $\mathfrak{R}'$ contained in $\mathfrak{R}$ such that

$$\alpha := \sup_{\substack{\theta \in \mathscr{A} \\ \nu \in \mathscr{N}}} |\langle \nu, \theta \rangle| < \infty,$$

with

$$\mathscr{N} := \left\{ \nu \in \mathscr{M} \,\middle|\, \sup_{A \in \mathfrak{R}'} |\nu(A)| \leqslant 1 \right\}.$$

Let $\mu \in \mathscr{M}(E)$ such that

$$\sup_{A \in \mathfrak{R}'} p(\mu(A)) \leqslant \frac{1}{\alpha + 1}.$$

Let further $x' \in E'$ such that $\langle x, x' \rangle \leqslant 1$ for any $x \in E$ with $p(x) \leqslant 1$. We get

$$\sup_{A \in \mathfrak{R}'} |x' \circ \mu(A)| = \sup_{A \in \mathfrak{R}'} |\langle \mu(A), x' \rangle| \leqslant \frac{1}{\alpha + 1}$$

and therefore $x' \circ \mu \in \frac{1}{\alpha+1} \mathscr{N}$ and

$$|\langle (\psi(\mu))(\theta), x' \rangle| = \left| \left\langle \int \theta \, d\mu, x' \right\rangle \right| = |\langle x' \circ \mu, \theta \rangle| \leqslant 1$$

for any $\theta \in \mathscr{A}$. Since $x'$ is arbitrary it follows

$$p((\psi(\mu))(\theta)) \leqslant 1$$

for any $\theta \in \mathscr{A}$. Hence $\psi$ is a continuous map of $\mathscr{M}(E)$ into $\mathscr{L}$.

Let $p$ be a continuous semi-norm on $E$ and let $\mathfrak{R}'$ be a $\sigma$-ring of sets contained in $\mathfrak{R}$. Let us denote by $\mathscr{N}$ the set of $\nu \in \mathscr{M}$ such that

$$\sup_{A \in \mathfrak{R}'} |\nu(A)| \leqslant 1$$

and by $\mathscr{N}^0$ its polar set in $\mathscr{M}'$. Then $\mathscr{N}^0$ is an equicontinuous set of $\mathscr{M}'$. Let $\mu \in \mathscr{M}(E)$ such that

$$\sup_{\theta \in \mathscr{N}^0} p((\psi(\mu))(\theta)) \leqslant 1$$

and let $A \in \mathfrak{R}'$. We denote by $\theta$ the map

$$\nu \longmapsto \nu(A) : \mathscr{M} \to \mathbf{R}.$$

Then $\theta \in \mathscr{N}^0$ and therefore

$$p(\mu(A)) = p((\psi(\mu))(\theta)) \leqslant 1.$$

This shows that $\psi$ is an open map of $\mathscr{M}(E)$ onto the subspace $\psi(\mathscr{M}(E))$ of $\mathscr{L}$.

In order to prove the last assertion we may assume by Proposition 5 that any set of $\mathfrak{R}$ contained in a set of $\mathfrak{R}'$ belongs to $\mathfrak{R}'$. The map $\psi(\mu)$ is continuous if we endow $\mathscr{M}'$ with the $\sigma(\mathscr{M}', \mathscr{M})$-topology and $E$ with the weak topology. Let $\mathscr{N}$ be the set of $\mu \in \mathscr{M}$ such that

$$\sup_{A \in \mathfrak{R}'} |\mu(A)| \leqslant 1$$

and let $\mathcal{N}^0$ be its polar set in $\mathcal{M}'$. $\mathcal{N}^0$ is compact with respect to the $\sigma(\mathcal{M}', \mathcal{M})$-topology and therefore $(\psi(\mu))(\mathcal{N}^0)$ is weakly compact in E. Since $\mathcal{N}^0$ is circled and convex and since it contains the set $\{\mu(A) | A \in \mathfrak{R}'\}$ we infer that the closed convex hull of $\{\mu(A) | A \in \mathfrak{R}'\}$ is weakly compact. ■

*Remarks* 1. — J. Hoffmann-Jørgensen proved ([2] Theorem 7) that if E is quasicomplete and if $\mathfrak{R}$ is a $\sigma$-algebra then $\{\mu(A) | A \in \mathfrak{R}\}$ is weakly relatively compact in E, under weaker assumptions about $\mu$.

2. — In the proof we didn't use completely the hypothesis that E is sequentially complete but only the weaker assumptions that any sequence $(x_n)_{n \in \mathbf{N}}$ in E converges if there exists a bounded set A of E such that for any $\varepsilon > 0$ there exists $m \in \mathbf{N}$ with $x_n - x_m \in \varepsilon A$ for any $n \in \mathbf{N}$, $n \geqslant m$.

3. — Let F be another Hausdorff locally convex space, let $\mathcal{M}(F)$ be the vector space of $\mathfrak{K}$-regular F-valued measures on $\mathfrak{R}$ endowed with the seminorm topology, and let $u : E \to F$ be a continuous map. Then for any $\mu \in \mathcal{M}(E)$ we have $u \circ \mu \in \mathcal{M}(F)$, the map

$$\mu \longmapsto u \circ \mu : \mathcal{M}(E) \to \mathcal{M}(F)$$

is continuous, and for any $\theta \in \mathcal{M}'$ we have

$$\int \theta \, d(u \circ \mu) = u\left(\int \theta \, d\mu\right).$$

4. — The theorem doesn't hold any more if we drop the hypothesis that E is $\delta$-complete.

Theorem 11. — *Let $\mathfrak{R}$ be a $\delta$-ring of sets, let $\mathfrak{K}$ be a set, let* E *be a Hausdorff sequentially complete $\delta$-complete locally convex space such that for any convex weakly compact set* K *of* E *and for any equicontinuous set* A′ *of the dual* E′ *of* E *the map*

$$(x, x') \longmapsto \langle x, x' \rangle : K \times A' \to \mathbf{R}$$

*is continuous with respect to the $\sigma(E, E')$-topology on* K *and $\sigma(E', E)$-topology on* A′, *let $\mathcal{M}(E)$ be the vector space of $\mathfrak{K}$-regular* E-*valued measures on $\mathfrak{R}$, and let $(\mu_\iota)_{\iota \in I}$ be a family in $\mathcal{M}(E)$ such that for any $J \subset I$ the family $(\mu_\iota)_{\iota \in J}$*

*is summable in* $\mathscr{M}$ *with respect to the topology of pointwise convergence in* $\mathfrak{R}$. *Then for any* $J \subset I$ *the family* $(\mu_\iota)_{\iota \in J}$ *is summable in* $\mathscr{M}(E)$ *with respect to the semi-norm topology on* $\mathscr{M}(E)$.

Let $\mathfrak{P}(I)$ be the set of subsets of I. The map of $\mathfrak{P}(I)$ into $\{0, 1\}^I$ which associates to any subset of I its characteristic functions is a bijection. We endow $\{0, 1\}$ with the discrete topology, $\{0, 1\}^I$ with the product topology, and $\mathfrak{P}(I)$ with the topology for which the above bijection is an homeomorphism. Then $\mathfrak{P}(I)$ is a compact space. The assertion that any subfamily of a family $(x_\iota)_{\iota \in I}$ in a Hausdorff topological additive group is summable is equivalent with the assertion that there exists a continuous map $f$ of $\mathfrak{P}(I)$ into G such that $f(J) = \sum_{\iota \in J} x_\iota$ for any finite subset J of I ([6]). By the hypothesis there exists therefore a continuous map $f$ of $\mathfrak{P}(I)$ into $\mathscr{M}(E)$ endowed with the topology of pointwise convergence in $\mathfrak{R}$ such that $f(J) = \sum_{\iota \in J} \mu_\iota$ for any finite subset J of I.

Let $\mathscr{M}$ be the vector space of $\mathfrak{K}$-regular real valued measures on $\mathfrak{R}$ endowed with the semi-norm topology, and let $\mathscr{M}'$ be its dual. By Theorem 10 any measure of $\mathscr{M}(E)$ is normal and therefore $\mathscr{M}(E)$ may be considered as a set of maps of $\mathscr{M}'$ into E. By Proposition 8 the above map $f$ is continuous with respect to the topology on $\mathscr{M}(E)$ of pointwise convergence in $\mathscr{M}'$. It follows that for any $J \subset I$ the family $(\mu_\iota)_{\iota \in J}$ is summable in $\mathscr{M}(E)$ with respect to this last topology.

Let us endow $\mathscr{M}'$ with the Mackey $\tau(\mathscr{M}', \mathscr{M})$-topology, let $\mathscr{L}$ be the vector space of continuous linear maps of $\mathscr{M}'$ into E, and let $\psi$ be the injection $\mathscr{M}(E) \to \mathscr{L}$ defined in Theorem 10. It is obvious that $\psi$ is continuous with respect to the topology on $\mathscr{M}(E)$ and $\mathscr{L}$ of pointwise convergence in $\mathscr{M}'$. Hence for any $J \subset I$ the family $(\psi(\mu_\iota))_{\iota \in J}$ is summable in $\mathscr{L}$ with respect to the topology of pointwise convergence in $\mathscr{M}'$.

Let U be a closed convex 0-neighbourhood in E and let $U^0$ be its polar set in $E'$ endowed with the $\sigma(E', E)$-topology. Let $\mathfrak{R}'$ be a $\sigma$-ring of sets contained in $\mathfrak{R}$, let $\mathscr{N}$

be the set $\left\{\nu \in \mathcal{M} \mid \sup_{A \in \mathfrak{R}'} |\nu(A)| \leqslant 1\right\}$, and let $\mathcal{N}^0$ be its polar set in $\mathcal{M}'$ endowed with the $\sigma(\mathcal{M}', \mathcal{M})$-topology. For any $\mu \in \mathcal{M}(E)$ the map

$$\theta \longmapsto \int \theta \, d\mu : \mathcal{N}^0 \rightarrow E$$

is continuous with respect to the weak topology of E. It follows that the image of $\mathcal{N}^0$ through this map is a convex weakly compact set of E. By the hypothesis about E the map $\hat{\mu}$

$$(\theta, x') \longmapsto \left\langle \int \theta \, d\mu, x' \right\rangle : \mathcal{N}^0 \times U^0 \rightarrow \mathbf{R}$$

is continuous. Let $\mathscr{C}(\mathcal{N}^0 \times U^0)$ be the vector space of continuous real functions on $\mathcal{N}^0 \times U^0$. By the above proof for any $J \subset I$ the family $(\hat{\mu}_\iota)_{\iota \in J}$ is summable in $\mathscr{C}(\mathcal{N}^0 \times U^0)$ with respect to the topology of pointwise convergence. By [7] Theorem II 4 the same assertion holds with respect to the topology of uniform convergence. Let $J \subset I$. Then there exists a finite subset K of J such that

$$\left| \sum_{\iota \in L} \hat{\mu}_\iota(\theta, x') - \sum_{\iota \in J} \hat{\mu}_\iota(\theta, x') \right| \leqslant 1$$

for any finite subset L of J containing K and for any $(\theta, x') \in \mathcal{N}^0 \times U^0$. We get

$$\sum_{\iota \in L} \mu_\iota(A) - \sum_{\iota \in J} \mu_\iota(A) \in U$$

for any finite subset L of J containing K and for any $A \in \mathfrak{R}'$. Since $\mathfrak{R}$ and U are arbitrary this shows that the family $(\mu_\iota)_{\iota \in J}$ is summable in $\mathcal{M}(E)$ with respect to the seminorm topology. ■

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Corneliu Constantinescu,
ETH, Mathematisches Seminar
8006 Zürich, Switzerland.