CORNELIU CONSTANTINESCU

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Annales de l’institut Fourier, tome 25, n° 3-4 (1975), p. 139-161

<http://www.numdam.org/item?id=AIF_1975__25_3-4_139_0>
ON VECTOR MEASURES
by Corneliu CONSTANTINESCU

Dédié à Monsieur M. Brelot à l'occasion
de son 70e anniversaire.

The aim of this paper is to prove some properties concerning the measures which take their values in Hausdorff locally convex spaces. \( \delta \)-rings of sets rather than \( \sigma \)-rings of sets will be used and a certain regularity of the measures will be assumed in order to include the Radon measures on Hausdorff topological spaces in these considerations.

A *ring of sets* is a set \( \mathcal{R} \) such that for any \( A, B \in \mathcal{R} \) we have \( A \cap B, A \cup B \in \mathcal{R} \). A ring of sets is called a *\( \sigma \)-ring of sets* (resp. *\( \delta \)-ring of sets*) if the union (resp. the intersection) of any countable family in \( \mathcal{R} \) belongs to \( \mathcal{R} \). Any \( \sigma \)-ring of sets is a \( \delta \)-ring of sets. Let \( G \) be Hausdorff topological additive group and let \( \mathcal{R} \) be a ring of sets. A *\( G \)-valued measure* on \( \mathcal{R} \) is a map \( \mu \) of \( \mathcal{R} \) into \( G \) such that for any countable family \( (A_i)_{i \in I} \) of pairwise disjoint sets of \( \mathcal{R} \) whose union belongs to \( \mathcal{R} \), the family \( (\mu(A_i))_{i \in I} \) is summable and its sum is \( \mu\left( \bigcup_{i \in I} A_i \right) \). Let \( \mathcal{F} \) be a set and let \( \mathcal{F}^u \) be the set of finite unions of sets of \( \mathcal{F} \) (then \( \emptyset \in \mathcal{F}^u \)). For any \( A \in \mathcal{R} \) we denote by \( \mathcal{F}(A, \mathcal{F}) \) the filter on \( \mathcal{F} \) generated by the filter base

\[ \{ \{B \in \mathcal{R} | K \subset B \subset A\} | K \in \mathcal{F}^u, K \subset A \} \].

A \( G \)-valued measure \( \mu \) on \( \mathcal{R} \) will be called *\( \mathcal{F} \)-regular* if for any \( A \in \mathcal{R} \), \( \mu \) converges along \( \mathcal{F}(A, \mathcal{F}) \) to \( \mu(A) \).
Any $G$-valued measure on $\mathcal{R}$ is $\mathcal{R}$-regular. A set $A \in \mathcal{R}$ is called a null set for $\mu$ if $\mu(B) = 0$ for any $B \in \mathcal{R}$ with $B \subseteq A$. Let $\mathcal{R}$ be a ring of sets, let $G, G'$ be Hausdorff topological additive groups, and let $\mu$ (resp. $\mu'$) be a $G$-valued (resp. $G'$ valued) measure on $\mathcal{R}$. We say that $\mu$ is absolutely continuous with respect to $\mu'$ (in symbols $\mu \ll \mu'$) if any null set for $\mu'$ is a null set for $\mu$. For any real valued measure $\mu$ on a $\sigma$-ring of sets $\mathcal{R}$ we denote by $|\mu|$ the supremum of $\mu$ and $-\mu$ in the vector lattice of real valued measures on $\mathcal{R}$. If $\mathcal{R}$ is a set such that $\mu$ is $\mathcal{R}$-regular then $|\mu|$ is $\mathcal{R}$-regular.

**Proposition 1.** Let $G$ be a topological additive group whose one point sets are $G_0$-sets ($G$ is therefore Hausdorff) and let $(x_i)_{i \in I}$ be a family in $G$ such that any countable subfamily of it is summable. Then there exists a countable subset $J$ of $I$ such that $x_i = 0$ for any $i \in I \setminus J$.

Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of $0$-neighbourhoods in $G$ whose intersection is equal to $\{0\}$. The sets

$$J_n := \{ i \in I | x_i \notin U_n \}$$

being finite for any $n \in \mathbb{N}$ the set $J := \bigcup_{n \in \mathbb{N}} J_n$ is countable. For any $i \in I \setminus J$ we get $x_i \in \bigcap_{n \in \mathbb{N}} U_n$ and therefore $x_i = 0$. 

**Proposition 2.** Let $G$ be a topological additive group whose one point sets are $G_0$-sets, let $\mathcal{R}$ be a $\sigma$-ring of sets, and let $\mu$ be a $G$-valued measure on $\mathcal{R}$. Then there exists $A \in \mathcal{R}$ such that $\mu(B) = 0$ for any $B \in \mathcal{R}$ with $B \cap A = \emptyset$.

Let us denote by $\Sigma$ the set of sets $\mathcal{S}$ of pairwise disjoint sets of $\mathcal{R}$ such that $\mu(S) \neq 0$ for any $S \in \mathcal{S}$. It is obvious that $\Sigma$ is inductively ordered by the inclusion relation. By Zorn's theorem there exists a maximal element $\mathcal{S}_0 \in \Sigma$. Then any countable subfamily of the family $(\mu(S))_{S \in \mathcal{S}_0}$ is summable. By the preceding proposition $\mathcal{S}_0$ is countable. We set

$$A := \bigcup_{S \in \mathcal{S}_0} S.$$ 

Then $A \in \mathcal{R}$. Let $B \in \mathcal{R}$ with $B \cap A = \emptyset$. If $\mu(B) \neq 0$
then \( \mathcal{G}_0 \cup \{B\} \in \Sigma \) and this contradicts the maximality of \( \mathcal{G}_0 \). □

**Theorem 3.** — Let \( T \) be a Hausdorff topological space possessing a dense \( \sigma \)-compact set, let \( E \) be a locally convex space whose one point sets are \( G_\sigma \)-sets, and let \( C(T, E) \) be the vector space of continuous maps of \( T \) into \( E \) endowed with the topology of pointwise convergence. Let further \( \mathcal{R} \) be a \( \sigma \)-ring of sets, let \( \mathcal{S} \) be a set, and let \( \mu \) be a \( \mathcal{R} \)-regular \( C(T, E) \)-valued measure on \( \mathcal{R} \). Then there exists a positive \( \mathcal{R} \)-regular real valued measure \( \nu \) on \( \mathcal{R} \) such that \( \mu \) is absolutely continuous with respect to \( \nu \).

Assume first \( E = \mathbb{R} \) and let us denote by \( \mathscr{C}_g(T) \) the vector space of continuous real functions on \( T \) endowed with the topology of compact convergence. Since \( T \) possesses a dense \( \sigma \)-compact set the one point sets of \( \mathscr{C}_g(T) \) are \( G_\sigma \)-sets.

Let us denote for any \( t \in T \) by \( \mu_t \) the map

\[
A \mapsto (\mu(A))(t) : \mathbb{R} \rightarrow \mathbb{R}.
\]

Then \( \mu_t \) is a \( \mathcal{R} \)-regular real valued measure on \( \mathbb{R} \) for any \( t \in T \). Assume that for any countable subset \( M \) of \( T \) there exists \( A \in \mathbb{R} \) which is a null set for any \( \mu_t \) with \( t \in M \) and is not a null set for \( \mu \). Let \( \omega_1 \) be the first uncountable ordinal number. We construct by transfinite induction a family \( \{t_\xi\}_{\xi < \omega_1} \) in \( T \) and a decreasing family \( \{A_\xi\}_{\xi < \omega_1} \) in \( \mathbb{R} \) such that we have for any \( \xi < \omega_1 \):

a) \( A_\xi \) is a null set for any \( \mu_\eta \) with \( \eta \leq \xi \);

b) any set \( A \in \mathbb{R} \) is a null set for \( \mu \) if it is a null set for any \( \mu_\eta \) with \( \eta \leq \xi \) and if \( A \cap A_\xi = \emptyset \);

c) \( \bigcap_{\eta < \xi} A_\eta \backslash A_\xi \) is not a null set for \( \mu \).

Assume that the families were constructed up to \( \xi < \omega_1 \). By the hypothesis of the proof there exists a set of \( \mathbb{R} \) which is a null set for any \( \mu_\eta \) with \( \eta < \xi \) and which is not a null set for \( \mu \). Hence there exists \( B \in \mathbb{R} \) and \( t_\xi \in T \) such that \( B \) is a null set for any \( \mu_\eta \) with \( \eta < \xi \) and such that

\[
\mu_{t_\xi}(B) \neq 0.
\]
Let $\mathcal{R}'$ be the set of sets of $\mathcal{R}$ which are null sets for any $\mu_\eta$ with $\eta \leq \xi$. Then $\mathcal{R}'$ is a $\sigma$-ring of sets and by [7] Theorem II.4 (*) the map $\mathcal{R}' \to \mathcal{C}_X(T)$ induced by $\mu$ is a measure. By the preceding proposition there exists $C \in \mathcal{R}'$ such that any $D \in \mathcal{R}'$ with $C \cap D = \emptyset$ is a null set for $\mu$. We set

$$A_\xi := C \cap \left(\bigcap_{\eta < \xi} A_\eta\right).$$

a) is obviously fulfilled. Let $A \in \mathcal{R}'$ with $A \cap A_\xi = \emptyset$. Then $A \setminus C \in \mathcal{R}'$ and it is therefore a null set for $\mu$. For any $\eta < \xi$ the set $A \setminus A_\eta$ is a null set for $\mu$ by the hypothesis of the induction. Hence $A$ is a null set for $\mu$ and $b)$ is fulfilled. Since $B \cap C$ is a null set for $\mu_\eta$ we get

$$\mu_\xi(B \setminus C) \neq 0.$$ 

For any $\eta < \xi$ the set $(B \setminus C) \setminus A_\eta$ is a null set for $\mu_\xi$ for any $\zeta \leq \eta$ and by the hypothesis of the induction

$$(B \setminus C) \setminus A_\eta$$

is a null set for $\mu$. It follows that $(B \setminus C) \bigcap_{\eta < \xi} A_\eta$ is a null set for $\mu$ and therefore

$$\mu_\xi\left((B \setminus C) \cap \left(\bigcap_{\eta < \xi} A_\eta \setminus A_\xi\right)\right) = \mu_\xi\left((B \setminus C) \cap \left(\bigcap_{\eta < \xi} A_\eta\right)\right) \neq 0.$$ 

We deduce that $\bigcap_{\eta < \xi} A_\eta \setminus A_\xi$ is not a null set for $\mu$ which proves $c)$. Again by [7] Theorem II.4 any countable subfamily of the family $\left(\mu_\eta\left(\bigcap_{\eta < \xi} A_\eta \setminus A_\xi\right)\right)_{\xi \leq \omega_1}$ is summable in $\mathcal{C}_X(T)$ and this contradicts Proposition 1. Hence there exists a sequence $(\xi_n)_{\eta \in \mathbb{N}}$ in $T$ such that any set of $\mathcal{R}$ is a null set for $\mu$ if it is a null set for any $\mu_{\xi_n}$ with $n \in \mathbb{N}$. We set

$$\alpha_n := \sup_{A \in \mathcal{R}} |\mu_{\xi_n}(A)| < \infty$$

The map
\[ A \mapsto \sum_{n \in \mathbb{N}} \frac{1}{2^n} |\mu_n|(A) : \mathcal{R} \to \mathbb{R} \]
is a positive \( \mathcal{R} \)-regular real valued measure on \( \mathcal{R} \) and \( \mu \) is absolutely continuous with respect to it.

Let us treat now the general case. Let \( E' \) be the dual of \( E \) endowed with the \( \sigma(E', E) \)-topology and let \( (U_n)_{n \in \mathbb{N}} \) be a sequence of closed convex 0-neighbourhoods in \( E \) whose intersection is equal to \{0\} and such that
\[ U_{n+1} \subseteq \frac{1}{2} U_n \quad \text{for any } n \in \mathbb{N}. \]

For any \( n \in \mathbb{N} \) let \( U_n^0 \) be the polar set of \( U_n \) in \( E' \). Then, for any \( n \in \mathbb{N} \), \( U_n^0 \) is a compact set of \( E' \) and \( \bigcup_{n \in \mathbb{N}} U_n^0 \) is a dense set in \( E' \). Let \( T' \) be the topological (disjoint) sum of the sequence \((T \times U_n^0)_{n \in \mathbb{N}}\) of topological spaces. Then \( T' \) is a Hausdorff topological space possessing a dense \( \sigma \)-compact set. Let \( \mathcal{C}(T') \) be the vector space of continuous real functions on \( T' \) endowed with the topology of pointwise convergence. For any \( A \in \mathcal{R} \) let us denote by \( \lambda(A) \) the real function on \( T' \) equal to
\[ (t, x') \mapsto \langle (\mu(A))(t), x' \rangle : T \times U_n^0 \to \mathbb{R} \]
on \( T \times U_n^0 \). It is easy to see that \( \lambda(A) \in \mathcal{C}(T') \) and that \( \lambda \) is a \( \mathcal{R} \)-regular measure on \( \mathcal{R} \) with values in \( \mathcal{C}(T') \). Let \( A \in \mathcal{R} \) be a null set for \( \lambda \) and let \( t \in T \). Since \( (\mu(A))(t) \) vanishes on \( \bigcup_{n \in \mathbb{N}} U_n^0 \) and since this set is dense in \( E' \) we deduce \( (\mu(A))(t) = 0 \). The point \( t \) being arbitrary \( \mu(A) \) vanishes. Hence \( \mu \) is absolutely continuous with respect to \( \lambda \). By the first part of the proof there exists a positive \( \mathcal{R} \)-regular real valued measure \( \nu \) on \( \mathcal{R} \) such that \( \lambda \) is absolutely continuous with respect to \( \nu \). Then \( \mu \) is absolutely continuous with respect to \( \nu \). \( \blacksquare \)

**Remark.** For \( \mathcal{R} = \mathcal{R} \) this result could be deduced from \[4\] Theorem 2.2 and \[3\] Theorem 2.5. A simpler proof can be given by using \[9\] Theorem 2.3 or \[10\] Theorem 2.
2. Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathcal{K}$ be a set, let $E$ be a Hausdorff locally convex space, and let $\mathcal{M}$ be the set of $\mathcal{R}$-regular $E$-valued measures on $\mathcal{R}$. Then $\mathcal{M}$ is a subspace of the vector space $E^{\mathcal{K}}$. For any continuous semi-norm $p$ on $E$ and for any $\sigma$-ring of sets $\mathcal{K}'$ contained in $\mathcal{R}$ the map

$$
\mu \mapsto \sup_{A \in \mathcal{K}'} p(\mu(A)) : \mathcal{M} \to \mathbb{R}_+
$$

([1], III 4.5) is a semi-norm on $\mathcal{M}$. We shall call the topology on $\mathcal{M}$ generated by these semi-norms the semi-norm topology of $\mathcal{M}$. If $\mathcal{R}$ is a $\sigma$-ring and $E$ is $\mathbb{R}$ then the semi-norm topology on $\mathcal{M}$ is defined by the lattice norm

$$
\mu \mapsto \sup_{A \in \mathcal{K}} |\mu|(A) : \mathcal{M} \to \mathbb{R}_+
$$

and $\mathcal{M}$ endowed with this norm is an order complete Banach lattice.

Let $\mathcal{R}$ be a $\sigma$-ring of sets and let $T(\mathcal{R}) := \bigcup_{\Lambda \in \mathcal{R}} \Lambda$. A real function $f$ on $T(\mathcal{R})$ is called $\mathcal{R}$-measurable if for any positive real number $\alpha$ the sets $\{x \mid f(x) > \alpha\}$, $\{x \mid f(x) < -\alpha\}$ belong to $\mathcal{R}$. Let $\mu$ be a real valued measure on $\mathcal{R}$. $L^1(\mu)$ will denote the set of $\mathcal{R}$-measurable $\mu$-integrable real functions on $T(\mathcal{R})$. Let $f$ be a subset of $L^1(\mu)$ such that $f' = f''$ $\mu$-almost everywhere and therefore

$$
\int f' \, d\mu = \int f'' \, d\mu
$$

for any $f', f'' \in f$. We set

$$
\int f \, d\mu := \int f' \, \mu,
$$

where $f'$ is an arbitrary function of $f$. $L^1(\mu)$ and $L^\infty(\mu)$ will denote the usual Banach lattices and $\| \|_1$, $\| \|_{\infty}$ will denote their norms respectively. Any element of $L^\infty(\mu)$ is a subset of $L^1(\mu)$ ([1], III 4.5).

**Proposition 4.** — Let $\mathcal{R}$ be a $\sigma$-ring of sets, let $\mathcal{K}$ be a set, let $\mathcal{M}$ be the Banach lattice of $\mathcal{R}$-regular real valued measures on $\mathcal{R}$ and let

$$
\mathcal{F} := \left\{ f \in \prod_{\mu \in \mathcal{M}} L^\infty(\mu) \mid \mu \ll \nu \implies f_\nu \in f_\mu \right\}.
$$
Then $\mathcal{F}$ is a subvector lattice of $\prod_{\mu \in A_b} L^\infty(\mu)$ such that for any subset of $\mathcal{F}$ which possesses a supremum in $\prod_{\mu \in A_b} L^\infty(\mu)$ this supremum belongs to $\mathcal{F}$. For any $f \in \mathcal{F}$ we have

$$\|f\| := \sup_{\mu \in A_b} \|f_\mu\|_\mu < \infty$$

and the map

$$f \mapsto \|f\| : \mathcal{F} \to \mathbb{R}_+$$

is a lattice norm. $\mathcal{F}$ endowed with it is a Banach lattice. For any $f \in \mathcal{F}$ we denote by $\varphi(f)$ the map

$$\mu \mapsto \int f_\mu \, d\mu : \mathcal{M} \to \mathbb{R}.$$

Then $\varphi(f)$ belongs to the dual of $\mathcal{M}$ for any $f \in \mathcal{F}$ and $\varphi$ is an isomorphism of Banach lattices of $\mathcal{F}$ onto the dual of $\mathcal{M}$.

Let $f, g \in \mathcal{F}$, let $\alpha \in \mathbb{R}$, and let $\mu, \nu \in \mathcal{M}$ such that $\mu \ll \nu$. Then $f_\nu \leq f_\mu$, $g_\nu \leq g_\mu$ and therefore

$$(f + g)_\nu = f_\nu + g_\nu \leq f_\mu + g_\mu = (f + g)_\mu,$$

$$(\alpha f)_\nu = \alpha f_\nu \leq \alpha f_\mu = (\alpha f)_\mu.$$}

This shows that $\mathcal{F}$ is a vector subspace of $\prod_{\mu \in A_b} L^\infty(\mu)$.

Let $\mathcal{G}$ be a subset of $\mathcal{F}$ possessing a supremum $f$ in $\prod_{\mu \in A_b} L^\infty(\mu)$ and let $\mu, \nu \in \mathcal{M}$ such that $\mu \ll \nu$. Then for any $g \in \mathcal{G}$ we have $g_\nu \leq g_\mu$ and therefore

$$f_\nu = \sup_{g \in \mathcal{G}} g_\nu \leq \sup_{g \in \mathcal{G}} g_\mu = f_\mu.$$}

Hence $\mathcal{F}$ is a subvector lattice of $\prod_{\mu \in A_b} L^\infty(\mu)$ such that for any subset of $\mathcal{F}$, which possesses a supremum in

$$\prod_{\mu \in A_b} L^\infty(\mu),$$

this supremum belongs to $\mathcal{F}$.

Let $f \in \mathcal{F}$. Assume

$$\sup_{\mu \in A_b} \|f_\mu\|_\mu^\infty = \infty.$$
Then there exists a sequence \((\mu_n)_{n \in \mathbb{N}}\) in \(\mathcal{M}\) such that
\[\lim_{n \to \infty} \|f_{\mu_n}\|_{\mathcal{P}_n} = \infty.\]
We set
\[\mu := \sum_{n \in \mathbb{N}} \frac{1}{2^n \|\mu\|} |\mu_n|.
Then \(\mu_n \ll \mu\) for any \(n \in \mathbb{N}\) and therefore \(f_\mu \leq f_{\mu_n}\). We get
\[\|f_{\mu_n}\|_{\mathcal{P}_n} \leq \|f_\mu\|_{\mathcal{P}},\]
and this leads to the contradictory relation
\[\infty = \lim_{n \to \infty} \|f_{\mu_n}\|_{\mathcal{P}_n} \leq \|f_\mu\|_{\mathcal{P}} < \infty.\]

Let \(f, g \in \mathcal{F}\), and let \(\alpha \in \mathbb{R}\). We have
\[\|f + g\| = \sup_{\mu \in \mathcal{M}} \|f_\mu + g_\mu\| \leq \sup_{\mu \in \mathcal{M}} (\|f_\mu\| + \|g_\mu\|) \leq \|f\| + \|g\|,\]
\[\|\alpha f\| = \sup_{\mu \in \mathcal{M}} \|\alpha f_\mu\| = \sup_{\mu \in \mathcal{M}} |\alpha| \|f_\mu\| = |\alpha| \|f\|,\]
\[f = 0 \iff (\mu \in \mathcal{M} \implies \|f_\mu\| = 0) \iff \|f\| = 0,\]
\[|f| \leq \|g\| \iff \|f\| = \sup_{\mu \in \mathcal{M}} \|f_\mu\| \leq \sup_{\mu \in \mathcal{M}} \|g_\mu\| = \|g\|\]
Hence
\[f \mapsto \|f\| : \mathcal{F} \to \mathbb{R}_+\]
is a lattice norm.

Let \(f \in \mathcal{F}\), let \(\mu, \nu \in \mathcal{M}\), and let \(\alpha \in \mathbb{R}\). Then
\[f_{|\mu|+|\nu|} \leq f_\mu \cap f_\nu \leq f_{\mu+\nu},\]
and therefore
\[(\varphi(f))(\mu + \nu) = \int f_{|\mu|+|\nu|} d(\mu + \nu)
= \int f_{|\mu|+|\nu|} d\mu + \int f_{|\mu|+|\nu|} d\nu = (\varphi(f))(\mu) + (\varphi(f))(\nu),\]
\[(\varphi(f))(\alpha \mu) = \int f_\mu d(\alpha \mu) = \alpha \int f_\mu d\mu = \alpha (\varphi(f))(\mu).\]
This shows that \(\varphi(f)\) is linear. From
\[|(\varphi(f))(\mu)| = \left| \int f_\mu d\mu \right| \leq \|f_\mu\| \|\mu\| \leq \|f\| \|\mu\|\]
we get \(\|\varphi(f)\| \leq \|f\|\). Hence \(\varphi(f)\) belongs to the dual of \(\mathcal{M}\).

It is obvious that \(\varphi\) is an injection and that \(\varphi\) maps the positive elements of \(\mathcal{F}\) into positive linear forms on \(\mathcal{M}\).

Let us prove now that \(\varphi\) is a surjection. Let \(\theta\) be a conti-
nuous linear form on $\mathcal{M}$ and let $\mu \in \mathcal{M}$. For any $g \in L^1(\mu)$ we denote by $g \cdot \mu$ the map $\Lambda \mapsto \int_A g \, d\mu : \mathbb{R} \to \mathbb{R}$. Then $g \cdot \mu \in \mathcal{M}$ and the map $g \mapsto \theta(g \cdot \mu) : L^1(\mu) \to \mathbb{R}$ is a continuous linear form on $L^1(\mu)$. Hence there exists $f_\mu \in L^\infty(\mu)$ such that $\|f_\mu\|_\infty \leq \|\theta\|$ and

$$\theta(g \cdot \mu) = \int f_\mu g \, d\mu$$

for any $g \in L^1(\mu)$. Let $\mu, \nu \in \mathcal{M}$ such that $\mu \ll \nu$. By Lebesgue-Radon-Nikodym theorem there exists $h \in L^1(\nu)$ such that $\mu = h \cdot \nu$. We get for any $g \in L^1(\mu)$, $gh \in L^1(\nu)$ and

$$\int f_\mu gh \, d\nu = \theta(gh \cdot \nu) = \int f_\mu gh \, d\nu = \int f_\mu g \, d\mu.$$

This shows that $f_\nu \subset f_\mu$. Hence $f : = (f_\mu)_{\mu \in \mathcal{M}} \in \mathcal{F}$ and it is clear that $\varphi(f) = \theta$. Moreover

$$\|f\| = \sup_{\mu \in \mathcal{M}} \|f_\mu\|_\infty \leq \|\theta\|.$$  

Hence $\varphi$ is an isomorphism of normed vector lattices. We deduce that $\mathcal{F}$ is a Banach lattice. 

**Proposition 5.** Let $\mathcal{R}$ be a $\delta$-ring of sets and let $\mathcal{R}_1$, $\mathcal{R}_2$ be $\sigma$-ring of sets contained in $\mathcal{R}$. Then there exists a $\sigma$-ring of sets $\mathcal{R}_0$ contained in $\mathcal{R}$ and containing $\mathcal{R}_1 \cup \mathcal{R}_2$ and such that any set of $\mathcal{R}$ which is contained in a set of $\mathcal{R}_0$ belongs to $\mathcal{R}_0$.

Let us denote by $\mathcal{R}_0$ the set of $A \in \mathcal{R}$ for which there exists $(B, C) \in \mathcal{R}_1 \times \mathcal{R}_2$ such that $A \subset B \cup C$. It is easy to check that $\mathcal{R}_0$ possesses the required properties.

**Proposition 6.** Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathcal{R}$ be a set, and let $\mathcal{R}'$ be a $\sigma$-ring of sets contained in $\mathcal{R}$ and such that any set of $\mathcal{R}$ contained in a set of $\mathcal{R}'$ belongs to $\mathcal{R}'$. Let further $E$ be a Hausdorff locally convex space, let $\mathcal{M}$ (resp. $\mathcal{M}_0$) be the vector space of $\delta$-regular $E$-valued measures on $\mathcal{R}$ (resp. $\mathcal{R}'$) endowed with the semi-norm topology, and let $\mathcal{M}'$ (resp. $\mathcal{M}_0'$) be its dual. For any $\mu \in \mathcal{M}$ we have $\mu|\mathcal{R}' \in \mathcal{M}_0'$ and the map $\varphi$

$$\mu \mapsto \mu|\mathcal{R}' : \mathcal{M} \to \mathcal{M}_0$$
is linear and continuous. Let \( p \) be a continuous semi-norm on \( E \), let \( \mathcal{N} \) (resp. \( \mathcal{N}_0 \)) be the set of \( \mu \in \mathcal{M} \) (resp. \( \mu \in \mathcal{M}_0 \)) such that
\[
\sup_{A \in \mathcal{R}} p(\mu(A)) \leq 1,
\]
let \( \mathcal{N}^0 \) (resp. \( \mathcal{N}_0^0 \)) be its polar set in \( \mathcal{M}' \) (resp. \( \mathcal{M}_0' \)) and let \( \varphi' : \mathcal{M}_0' \to \mathcal{M}' \) be the adjoint map of \( \varphi \). Then \( \varphi'(\mathcal{N}_0^0) = \mathcal{N}^0 \).

It is obvious that \( \mu \in \mathcal{M} \) implies \( \mu|\mathcal{R}' \in \mathcal{M}_0 \), that \( \varphi \) is linear and continuous, and that \( \varphi(\mathcal{N}) \subset \mathcal{N}_0 \). Hence
\[
\varphi'(\mathcal{N}_0^0) \subset \mathcal{N}^0.
\]

Let \( \theta \in \mathcal{N}^0 \) and let \( \nu \in \mathcal{M}_0 \). For any \( A \in \mathcal{R}' \) we denote by \( \nu_A \) the map
\[
B \mapsto \nu(A \cap B) : \mathcal{R} \to E.
\]
It is immediate that \( \nu_A \in \mathcal{M} \). Let \( F \) be the quotient locally convex space \( E/p^{-1}(0) \) and let \( u \) be the canonical map \( E \to F \). Then the one point sets of \( F \) are \( G_\tau \)-sets and \( u \circ \nu \) is an \( F \)-valued measure on \( \mathcal{R}' \). By Proposition 2 there exists \( A \in \mathcal{R}' \) such that any \( B \in \mathcal{R}' \) with \( B \cap A = \emptyset \) is a null set for \( u \circ \nu \). Let \( A' \in \mathcal{R}' \), \( A \subseteq A' \). For any \( B \in \mathcal{R} \) the set \( A' \cap B \setminus A \cap B \) is a null set for \( u \circ \nu \) and therefore
\[
p(\nu_{A'}(B) - \nu_A(B)) = 0.
\]
Hence \( \nu_{A'} - \nu_A \in \varepsilon \mathcal{N} \) for any \( \varepsilon > 0 \). We get \( \theta(\nu_{A'}) = \theta(\nu_A) \). Hence if \( \mathcal{F} \) denotes the section filter of \( \mathcal{R}' \) ordered by the inclusion relation then the map
\[
A \mapsto \theta(\nu_A) : \mathcal{R}' \to \mathcal{R}
\]
converges along \( \mathcal{F} \).

Let \( \theta \in \mathcal{N}_0^0 \). With the above notations we set for any \( \nu \in \mathcal{M}_0 \)
\[
\theta_0(\nu) := \lim_{A \to \mathcal{F}} \theta(\nu_A).
\]
It is easy to see that \( \theta_0 \) is a linear form on \( \mathcal{M}_0 \). If \( \nu \in \mathcal{N}_0 \) then \( \nu_A \in \mathcal{N} \) for any \( A \in \mathcal{R}' \) and therefore \( |\theta_0(\nu)| \leq 1 \). It follows \( \theta_0 \in \mathcal{N}_0^0 \). Let \( \mu \in \mathcal{M} \). We set \( \nu := \varphi(\mu) \). Let \( A \) be a set of \( \mathcal{R}' \) such that any \( B \in \mathcal{R} \) with \( B \cap A = \emptyset \)
is a null set for \( u \circ v \). Then \( \theta_0(v) = \theta(v_A) \). For any \( B \in \mathcal{N}' \) we have
\[
p(\mu(B) - v_A(B)) = p(\mu(B - A \cap B)) = 0.
\]
Hence \( \mu - v_A \in \varepsilon \mathcal{N}' \) for any \( \varepsilon > 0 \) and therefore
\[
\theta(\mu) = \theta(v_A).
\]
We get
\[
\langle \mu, \varphi'(\theta_0) \rangle = \langle \varphi(\mu), \theta_0 \rangle = \langle v, \theta_0 \rangle = \langle v_A, \theta \rangle = \langle \mu, \theta \rangle.
\]
Since \( \mu \) is arbitrary it follows \( \varphi'(\theta_0) = \theta \). Hence
\[
\varphi'(\mathcal{N}_0) = \mathcal{N}_0.
\]

**Proposition 7.** — Let \( \mathcal{R} \) be a \( \delta \)-ring of sets, let \( \mathcal{K} \) be a set, let \( \Gamma \) be the set of \( \sigma \)-rings of sets \( \mathcal{R}' \) contained in \( \mathcal{R} \) and such that any set of \( \mathcal{R} \) contained in a set of \( \mathcal{R}' \) belongs to \( \mathcal{R}' \), and let \( E \) be a Hausdorff locally convex space. For any \( \mathcal{R}' \in \Gamma \cup \{\mathcal{R}\} \) let \( \mathcal{M}(\mathcal{R}') \) be the vector space of \( \mathcal{R}' \)-regular \( E \)-valued measures on \( \mathcal{R}' \) endowed with the semi-norm topology, let \( \mathcal{M}(\mathcal{R})' \) be its dual, let \( \varphi_{\mathcal{R}'} \) be the map
\[
\mu \mapsto \mu|_{\mathcal{R}'} : \mathcal{M}(\mathcal{R}) \to \mathcal{M}(\mathcal{R}')
\]
(Proposition 6), and let \( \varphi_{\mathcal{R}'} : \mathcal{M}(\mathcal{R}')' \to \mathcal{M}(\mathcal{R})' \) be its adjoint map. Then
\[
\mathcal{M}(\mathcal{R})' = \bigcup_{\mathcal{R}' \in \Gamma} \varphi_{\mathcal{R}'}(\mathcal{M}(\mathcal{R}')').
\]

Let \( \theta \in \mathcal{M}(\mathcal{R})' \). By Proposition 5 there exists \( \mathcal{R}' \in \Gamma \) and a continuous semi-norm \( p \) on \( E \) such that \( |\theta(\mu)| \leq 1 \) for any \( \mu \in \mathcal{M}(\mathcal{R}) \) with
\[
\sup_{\Lambda \in \mathcal{R}} p(\mu(A)) \leq 1.
\]
By Proposition 6 there exists \( \theta_0 \in \mathcal{M}(\mathcal{R}')' \) such that
\[
\varphi_{\mathcal{R}'}(\theta_0) = \theta.
\]

3. Let \( \mathcal{R} \) be a \( \delta \)-ring of sets, let \( \mathcal{K} \) be a set, let \( \mathcal{M} \) be the vector space of \( \mathcal{K} \)-regular real valued measures on \( \mathcal{R} \) endowed with the semi-norm topology, and let \( \mathcal{M}' \) be its dual. Let further \( E \) be a Hausdorff locally convex space, let \( E' \) be its dual, and let \( \mu \) be a \( \mathcal{K} \)-regular \( E \)-valued
measure on $\mathcal{R}$. Then for any $x' \in E'$, $x' \circ \mu$ belongs to $\mathcal{M}$. If $\theta \in \mathcal{M}'$ then

$$x' \longmapsto \langle x' \circ \mu, \theta \rangle : E' \to \mathbb{R}$$

is a linear form on $E'$. If there exists $x \in E$ such that

$$\langle x' \circ \mu, \theta \rangle = \langle x, x' \rangle$$

for any $x' \in E'$ we say that $\theta$ is $\mu$-integrable. Then $x$ is uniquely defined by the above relation and we shall denote it by $\int \theta \, d\mu$. Any $A \in \mathcal{R}$ may be considered as an element of $\mathcal{M}'$ namely as the linear form $\theta_A$ on $\mathcal{M}$

$$v \longmapsto v(A) : \mathcal{M} \to \mathbb{R}.$$ 

It is easy to see that

$$A \longmapsto \theta_A : \mathcal{R} \to \mathcal{M}'$$

is an injection, that $\theta_A$ is $\mu$-integrable and

$$\int \theta_A \, d\mu = \mu(A).$$

If any $\theta \in \mathcal{M}'$ is $\mu$-integrable we say that the measure $\mu$ is normal. It will be shown in Theorem 10 that if $E$ is quasi-complete then any $E$-valued measure is normal. If $\mathcal{R}$ is a $\sigma$-ring of sets then any bounded $\mathcal{R}$-measurable real function $f$ may be considered as a map $\theta_f$

$$v \longmapsto \int f \, dv : \mathcal{M} \to \mathbb{R}$$

which obviously belongs to $\mathcal{M}'$. For any normal measure $\mu$ we shall write

$$\int f \, d\mu = \int \theta_f \, \mu.$$ 

If $\mu$ is a normal measure then it may be regarded as a map

$$\theta \longmapsto \int \theta \, d\mu : \mathcal{M}' \to E$$

and, identifying $\mathcal{R}$ with a subset of $\mathcal{M}'$ via the above injection, this map is an extension of $\mu$ to $\mathcal{M}'$. If $\mathcal{N}$ is a set of normal $\mathcal{R}$-regular $E$-valued measures on $\mathcal{R}$ then, taking into account the above extensions of the normal measures, it may be regarded as a set of maps of $\mathcal{M}'$ into $E$ and so we may speak of the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{M}'$. 

We want to make still another remark. If $F$ is another Hausdorff locally convex space and if $u : E \to F$ is a continuous linear map then for any $\mathcal{K}$-regular $E$-valued measure $\mu$ on $\mathcal{R}$ the map $u \circ \mu$ is a $\mathcal{K}$-regular $F$-valued measure on $\mathcal{R}$. Moreover any $\mu$-integral $\theta \in M'$ is $u \circ \mu$-integral and

$$\int \theta \, d(u \circ \mu) = u \left( \int \theta \, d\mu \right).$$

**Proposition 8.** — Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathcal{K}$ be a set, let $M$ be the vector space of $\mathcal{K}$-regular real valued measures on $\mathcal{R}$ endowed with the semi-norm topology, and let $M'$ be its dual. Let further $E$ be a Hausdorff locally convex space, let $M(E)$ be the vector space of $\mathcal{K}$-regular $E$-valued measures on $\mathcal{R}$ endowed with the topology of pointwise convergence in $\mathcal{R}$, and let $N'$ be a compact set of $M(E)$ such that any measure of $N'$ is normal. Then the topologies on $N'$ of pointwise convergence in $\mathcal{R}$ or in $M'$ coincide.

Since $\mathcal{R}$ may be identified with a subset of $M'$ we have only to show that the topology on $N'$ of pointwise convergence in $\mathcal{R}$ is finer than the topology on $N'$ of pointwise convergence in $M'$. By Proposition 7 we may assume that $\mathcal{R}$ is a $\sigma$-ring of sets. Let $\theta \in M'$ and let $p$ be a continuous semi-norm on $E$. We denote by $E_p$ the normed quotient space $E/p^{-1}(0)$, by $u_p$ the canonical map $E \to E_p$, and by $C(N', E_p)$ the vector space of continuous maps of $N'$ (endowed with the topology of pointwise convergence in $\mathcal{R}$) into $E_p$ endowed with the topology of pointwise convergence. For any $A \in \mathcal{R}$ let $\lambda(A)$ be the map

$$\mu \mapsto u_p \circ \mu(A) : N' \to E_p.$$

Then $\lambda(A) \in C(N', E_p)$ and it is obvious that $\lambda$ is a $\mathcal{K}$-regular measure on $\mathcal{R}$ with values in $C(N', E_p)$. By theorem 3 there exists a $\mathcal{K}$-regular real valued measure $\nu$ on $\mathcal{R}$ such that $\lambda$ is absolutely continuous with respect to $\nu$. By Proposition 4 there exists a bounded $\mathcal{R}$-measurable real function $f$ on $\bigcup_{A \in \mathcal{R}} A$ such that

$$\theta(\rho) = \int f \, d\rho$$
for any $\mathcal{R}$-regular real valued measure $\rho$ on $\mathcal{R}$ which is absolutely continuous with respect to $\nu$. Let $E'_p$ be the dual of $E_p$. Then for any $x' \in E'_p$ and for any $\mu \in \mathcal{N}$ the map $x' \circ u_p \circ \mu$ is a $\mathcal{R}$-regular real valued measure on $\mathcal{R}$ absolutely continuous with respect to $\nu$. Hence

$$\langle x' \circ u_p \circ \mu, \theta \rangle = \int f d(x' \circ u_p \circ \mu)$$

for any $\mu \in \mathcal{N}$ and for any $x' \in E'_p$. We get

$$u_p \left( \int \theta \, d\mu \right) = \int \theta \, d(u_p \circ \mu) = \int f \, d(u_p \circ \mu)$$

for any $\mu \in \mathcal{N}$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of step functions with respect to $\mathcal{R}$ converging uniformly to $f$. Since $\mathcal{N}$ is compact the set $\{\mu(A) | \mu \in \mathcal{N} \} \subset E$ is bounded for any $A \in \mathcal{R}$. We deduce that the set $\{\mu(A) | \mu \in \mathcal{N}, A \in \mathcal{R} \}$ is bounded ([5], Corollary 6). Hence the sequence

$$\left( \mu \mapsto \int f_n \, d\mu : \mathcal{N} \to E \right)_{n \in \mathbb{N}}$$

of functions on $\mathcal{N}$ converges uniformly to the function

$$\mu \mapsto \int f \, d\mu : \mathcal{N} \to E.$$ 

The functions of the sequence being continuous with respect to the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{R}$ we deduce that the last function is continuous with respect to this topology. We deduce further that the map

$$\mu \mapsto u_p \left( \int \theta \, d\mu \right) : \mathcal{N} \to E_p$$

is continuous with respect to the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{R}$. Since $\rho$ is arbitrary it follows that the map

$$\mu \mapsto \int \theta \, d\mu : \mathcal{N} \to E$$

is continuous with respect to this topology. Since $\theta$ is arbitrary the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{R}$ is finer than the topology on $\mathcal{N}$ of pointwise convergence in $\mathcal{M}'$. 

\textbf{Corollary.} — Let $\mathcal{R}$ be a $\sigma$-ring of sets, let $\mathcal{K}$ be a set, and let $\mathcal{N}$ be a set of $\mathcal{R}$-regular real valued measures on $\mathcal{R}$
compact with respect to the topology of pointwise convergence in \( \mathbb{R} \). Then any sequence in \( \mathcal{N} \) possesses a convergent subsequence with respect to this topology.

Let \( \mathcal{M} \) be the vector space of \( \mathbb{R} \)-regular real valued measures on \( \mathbb{R} \) endowed with the semi-norm topology. By the proposition, \( \mathcal{N} \) is weakly compact in \( \mathcal{M} \) and the assertion follows from Šumlián theorem. ■

Let \( X \) be an ordered set and let \( Y \) be a topological space. We say that a map \( f: X \to Y \) is order continuous if for any upper directed subset \( A \) of \( X \) possessing a supremum \( x \in X \) the map \( f \) converges along the section filter of \( A \) to \( f(x) \). An ordered set \( X \) is called order \( \sigma \)-complete if any upper bounded increasing sequence in \( X \) possesses a supremum.

**Theorem 9.** — Let \( E \) be an order \( \sigma \)-complete vector lattice, let \( F \) be a locally convex space, and let \( u \) be a linear map of \( E \) into \( F \). If \( u \) is order continuous with respect to the weak topology of \( F \) then it is order continuous with respect to the initial topology of \( F \).

Let \( U \) be a 0-neighbourhood in \( F \), let \( U^0 \) be its polar set in the dual \( F' \) of \( F \) endowed with the induced \( \sigma(F', F) \)-topology, let \( \mathcal{C}(U^0) \) (resp. \( \mathcal{C}_u(U^0) \)) be the vector space of continuous real functions on \( U^0 \) endowed with the topology of pointwise convergence (resp. with the topology of uniform convergence), and let us denote for any \( x \in E \) by \( f(x) \) the map

\[
y' \longmapsto \langle u(x), y' \rangle : U^0 \to \mathbb{R}
\]

which obviously belongs to \( \mathcal{C}(U^0) \).

Let \( (x_n)_{n \in \mathbb{N}} \) be an increasing sequence in \( E \) with supremum \( x \in E \). Then for any \( \mathcal{M} < \mathbb{N} \left( \sum_{\mathcal{M} \subseteq \mathcal{M}} (x_{n+1} - x_n) \right)_{m \in \mathbb{N}} \) is an upper bounded increasing sequence in \( E \) and possesses therefore a supremum. Since \( u \) is order continuous with respect to the weak topology of \( E \) it follows that

\[
(f(x_{n+1} - x_n))_{n \in \mathcal{M}}
\]

is summable in \( \mathcal{C}(U^0) \). The space \( U^0 \) being compact we deduce by [7] Theorem II 4 that \( (f(x_{n+1} - x_n))_{n \in \mathbb{N}} \) is sum-
mable in \( \mathcal{C}_a(U^0) \). Its sum has to be \( f(x - x_0) \). Hence

\[
(f(x_n))_{n \in \mathbb{N}}
\]

converges uniformly to \( f(x) \).

Let now \( A \) be an upper directed subset of \( E \) with supremum \( x \in E \) and let \( \mathcal{F} \) be its section filter. If \( f \) does not map \( \mathcal{F} \) into a Cauchy filter on \( \mathcal{C}_a(U^0) \) then it is easy to construct an increasing sequence \( (x_n)_{n \in \mathbb{N}} \) in \( A \) such that \( (f(x_n))_{n \in \mathbb{N}} \) is not a Cauchy sequence in \( \mathcal{C}_a(U^0) \). Since \( E \) is order \( \sigma \)-complete and \( (x_n)_{n \in \mathbb{N}} \) is upper bounded by \( x \) it possesses a supremum and this contradicts the above considerations. Hence \( f \) maps \( \mathcal{F} \) into a Cauchy filter on \( \mathcal{C}_a(U^0) \) and therefore, by the completeness of \( \mathcal{C}_a(U^0) \) into a convergent filter on \( \mathcal{C}_a(U^0) \). Using again the hypothesis that \( u \) is order continuous with respect to the weak topology of \( F \) we deduce that \( f(\mathcal{F}) \) converges to \( f(x) \) in \( \mathcal{C}(U^0) \) and therefore in \( \mathcal{C}_a(U^0) \). Since \( U \) is arbitrary it follows that \( u \) converges along \( \mathcal{F} \) to \( u(x) \) in the initial topology of \( F \) which shows that \( u \) is order continuous with respect to this topology. ■

Let \( E \) be a locally convex space, let \( E' \) be its dual endowed with the \( \sigma(E', E) \)-topology, and let \( \hat{E} \) be the set of linear forms \( y \) on \( E' \) such that for any \( \sigma \)-compact set \( A \) of \( E' \) there exists \( x \in E \) such that \( x \) and \( y \) coincide on \( \overline{A} \). We say that \( E \) is \( \delta \)-complete if \( \hat{E} = E \).

**Lemma.** — *Any quasicomplete locally convex space is \( \delta \)-complete.*

Let \( E \) be a quasicomplete locally convex space and let \( y \in \hat{E} \) (with the above notations). Let \( U \) be the neighbourhood filter of \( 0 \) in \( E \) and for any \( U \in \mathcal{U} \) let \( U^0 \) be its polar set in the dual of \( E \) and let \( A_U \) be the set of \( x \in E \) such that \( x \) and \( y \) coincide on \( \bigcup_{n \in \mathbb{N}} U^0 \). It is obvious that there exists \( \alpha_U \in \mathbb{R} \) such that \( A_U \subseteq \alpha_U U \). Let \( \mathcal{F} \) be the filter on \( E \) generated by the filter base \( \{A_U|U \in \mathcal{U}\} \). Then \( \mathcal{F} \) is a Cauchy filter on \( E \) containing the bounded set \( \bigcap_{U \in \mathcal{U}} \alpha_U U \) and converging to \( y \) uniformly on the sets \( U^0(U \in \mathcal{U}) \).
Since $E$ is quasicomplete $y \in E$ and therefore $E$ is $\delta$-complete. \[\square\]

**Remark.** — $\mathcal{F}$ endowed with its weak topology is sequentially complete and $\delta$-complete but it is not quasicomplete.

**Theorem 10.** — Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathcal{R}$ be a set, let $\mathcal{M}$ be the vector space of $\mathcal{R}$-regular real valued measures on $\mathcal{R}$ endowed with the semi-norm topology, and let $\mathcal{M}'$ be its dual endowed with the Mackey $\tau(\mathcal{M}', \mathcal{M})$-topology. Let further $E$ be a Hausdorff sequentially complete $\delta$-complete locally convex space, let $E'$ be its dual, let $\mathcal{L}$ be the vector space of continuous linear maps of $\mathcal{M}'$ into $E$ endowed with the topology of uniform convergence on the equicontinuous sets of $\mathcal{M}'$, and let $\mathcal{M}(E)$ be the vector space of $\mathcal{R}$-regular $E$-valued measures on $\mathcal{R}$ endowed with the semi-norm topology. Then for any $\theta \in \mathcal{M}'$ and for any $\mu \in \mathcal{M}(E)$ there exists a unique element $\int \theta \, d\mu$ of $E$ such that

$$\langle x' \circ \mu, \theta \rangle = \langle \int \theta \, d\mu, x' \rangle$$

for any $x' \in E'$. For any $\mu \in \mathcal{M}(E)$ the map $\psi(\mu)$

$$\theta \mapsto \int \theta \, d\mu : \mathcal{M}' \to E$$

belongs to $\mathcal{L}$ and it is order continuous. $\psi$ is a linear injection of $\mathcal{M}(E)$ into $\mathcal{L}$ which induces a homeomorphism of $\mathcal{M}(E)$ onto the subspace $\psi(\mathcal{M}(E))$ of $\mathcal{L}$. For any $\sigma$-ring of sets $\mathcal{R}'$ contained in $\mathcal{R}$ and for any $\mu \in \mathcal{M}(E)$ the closed convex circled hull of $\{\mu(A) | A \in \mathcal{R}'\}$ is weakly compact in $E$.

In order to prove the existence of $\int \theta \, d\mu$ we may assume by Proposition 7 that $\mathcal{R}$ is a $\sigma$-ring of sets. Let $\mathcal{F}$ be the Banach space of bounded $\mathcal{R}$-measurable real functions on $\bigcup A$ with the supremum norm. Since $E$ is sequentially complete we may define in the usual way $\int f \, d\mu \in E$ for any $f \in \mathcal{F}$. Let $A$ be a subset of $E'$ $\sigma$-compact with respect to the $\sigma(E', E)$-topology. By Theorem 3 there exists $\nu \in \mathcal{M}$ such that $x' \circ \mu \ll \nu$ for any $x' \in \overline{A}$. By Proposition 4
there exists \( f \in \mathcal{F} \) such that
\[
\langle x' \circ \mu, \theta \rangle = \int f d(x' \circ \mu) = \langle \int f d\mu, x' \rangle
\]
for any \( x' \in \bar{A} \). Since \( E \) is \( \delta \)-complete there exists
\[
\int \theta \ d\mu \in E
\]
such that
\[
\langle x' \circ \mu, \theta \rangle = \langle \int \theta \ d\mu, x' \rangle
\]
for any \( x' \in E' \).

Let \( \mu \in \mathcal{M}(E) \). It is obvious that \( \psi(\mu) \) is linear and from the relation defining it, it follows that it is continuous with respect to the \( \sigma(\mathcal{M}', \mathcal{M}) \) and \( \sigma(E, E') \) topologies. We deduce that \( \psi(\mu) \) belongs to \( \mathcal{L} \). From Proposition 4 or from the theory of Banach lattices we deduce that \( \psi(\mu) \) is order continuous with respect to the weak topology of \( E \). By the preceding theorem it is order continuous with respect to the initial topology of \( E \).

It is obvious that \( \psi \) is linear. Let \( \mu \in \mathcal{M}(E) \) such that \( \psi(\mu) = 0 \). Let \( A \in \mathcal{R} \) and let \( \theta \) be the map
\[
v \mapsto v(A) : \mathcal{M} \to \mathbb{R}.
\]
Then \( \theta \in \mathcal{M}' \) and we get
\[
\mu(A) = \int \theta \ d\mu = (\psi(\mu))(\theta) = 0.
\]
Since \( A \) is arbitrary we get \( \mu = 0 \). Hence \( \psi \) is an injection.

Let \( p \) be a continuous semi-norm on \( E \) and let \( \mathcal{A} \) be an equicontinuous set of \( \mathcal{M}' \). Then there exists a \( \sigma \)-ring of sets \( \mathcal{R}' \) contained in \( \mathcal{R} \) such that
\[
\alpha := \sup_{v \in \mathcal{A}, \|v\| \leq 1} \|v(A)\| < \infty,
\]
with
\[
\mathcal{N} := \left\{ v \in \mathcal{M} \mid \sup_{A \in \mathcal{R}'} \|v(A)\| \leq 1 \right\}.
\]
Let \( \mu \in \mathcal{M}(E) \) such that
\[
\sup_{A \in \mathcal{R}'} p(\mu(A)) \leq \frac{1}{\alpha + 1}.
\]
Let further \( x' \in E' \) such that \( \langle x, x' \rangle \leq 1 \) for any \( x \in E \) with \( p(x) \leq 1 \). We get

\[
\sup_{\Lambda \in \mathcal{A}} |x' \circ \mu(\Lambda)| = \sup_{\Lambda \in \mathcal{A}} |\langle \mu(\Lambda), x' \rangle| \leq \frac{1}{\alpha + 1}
\]

and therefore \( x' \circ \mu \in \frac{1}{\alpha + 1} \mathcal{N} \) and

\[
|\langle \psi(\mu)(\theta), x' \rangle| = |\int \theta d\mu, x'\rangle| = |\langle x' \circ \mu, \theta \rangle| \leq 1
\]

for any \( \theta \in \mathcal{A} \). Since \( x' \) is arbitrary it follows

\[
p((\psi(\mu))(\theta)) \leq 1
\]

for any \( \theta \in \mathcal{A} \). Hence \( \psi \) is a continuous map of \( \mathcal{M}(E) \) into \( \mathcal{L} \).

Let \( p \) be a continuous semi-norm on \( E \) and let \( \mathcal{R}' \) be a \( \sigma \)-ring of sets contained in \( \mathcal{R} \). Let us denote by \( \mathcal{N} \) the set of \( v \in \mathcal{M} \) such that

\[
\sup_{\Lambda \in \mathcal{R}'} |v(\Lambda)| \leq 1
\]

and by \( \mathcal{N}^0 \) its polar set in \( \mathcal{M}' \). Then \( \mathcal{N}^0 \) is an equicontinuous set of \( \mathcal{M}' \). Let \( \mu \in \mathcal{M}(E) \) such that

\[
\sup_{\theta \in \mathcal{R}'^0} p((\psi(\mu))(\theta)) \leq 1
\]

and let \( A \in \mathcal{R}' \). We denote by \( \theta \) the map

\[
v \mapsto v(A) : \mathcal{M} \to \mathcal{R}.
\]

Then \( \theta \in \mathcal{N}^0 \) and therefore

\[
p(\mu(A)) = p((\psi(\mu))(\theta)) \leq 1.
\]

This shows that \( \psi \) is an open map of \( \mathcal{M}(E) \) onto the subspace \( \psi(\mathcal{M}(E)) \) of \( \mathcal{L} \).

In order to prove the last assertion we may assume by Proposition 5 that any set of \( \mathcal{R} \) contained in a set of \( \mathcal{R}' \) belongs to \( \mathcal{R}' \). The map \( \psi(\mu) \) is continuous if we endow \( \mathcal{M}' \) with the \( \sigma(\mathcal{M}', \mathcal{M}) \)-topology and \( E \) with the weak topology. Let \( \mathcal{N} \) be the set of \( \mu \in \mathcal{M} \) such that

\[
\sup_{\Lambda \in \mathcal{R}'} |\mu(\Lambda)| \leq 1
\]

and therefore...
and let $\mathcal{N}^0$ be its polar set in $\mathcal{M}'$. $\mathcal{N}^0$ is compact with respect to the $\sigma(\mathcal{M}', \mathcal{M})$-topology and therefore $\langle \psi(\mu) \rangle(\mathcal{N}^0)$ is weakly compact in $E$. Since $\mathcal{N}^0$ is circled and convex and since it contains the set $\{\mu(A) | A \in \mathcal{R}'\}$ we infer that the closed convex hull of $\{\mu(A) | A \in \mathcal{R}'\}$ is weakly compact.

**Remarks**

1. — J. Hoffmann-Jørgensen proved ([2] Theorem 7) that if $E$ is quasicomplete and if $\mathcal{R}$ is a $\sigma$-algebra then $\{\mu(A) | A \in \mathcal{R}\}$ is weakly relatively compact in $E$, under weaker assumptions about $\mu$.

2. — In the proof we didn’t use completely the hypothesis that $E$ is sequentially complete but only the weaker assumptions that any sequence $(x_n)_{n \in \mathbb{N}}$ in $E$ converges if there exists a bounded set $A$ of $E$ such that for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ with $x_n - x_m \in \varepsilon A$ for any $n, m \in \mathbb{N}$, $n \geq m$.

3. — Let $F$ be another Hausdorff locally convex space, let $\mathcal{M}(F)$ be the vector space of $\mathcal{R}$-regular $F$-valued measures on $\mathcal{R}$ endowed with the seminorm topology, and let $u : E \to F$ be a continuous map. Then for any $\mu \in \mathcal{M}(E)$ we have $u \circ \mu \in \mathcal{M}(F)$, the map

$$\mu \mapsto u \circ \mu : \mathcal{M}(E) \to \mathcal{M}(F)$$

is continuous, and for any $\theta \in \mathcal{M}'$ we have

$$\int \theta \, d(u \circ \mu) = u \left( \int \theta \, d\mu \right).$$

4. — The theorem doesn’t hold any more if we drop the hypothesis that $E$ is $\delta$-complete.

**Theorem 11.** — Let $\mathcal{R}$ be a $\delta$-ring of sets, let $\mathcal{R}$ be a set, let $E$ be a Hausdorff sequentially complete $\delta$-complete locally convex space such that for any convex weakly compact set $K$ of $E$ and for any equicontinuous set $A'$ of the dual $E'$ of $E$ the map

$$(x, x') \mapsto \langle x, x' \rangle : K \times A' \to \mathbb{R}$$

is continuous with respect to the $\sigma(E, E')$-topology on $K$ and $\sigma(E', E)$-topology on $A'$, let $\mathcal{M}(E)$ be the vector space of $\mathcal{R}$-regular $E$-valued measures on $\mathcal{R}$, and let $(\mu_i)_{i \in I}$ be a family in $\mathcal{M}(E)$ such that for any $J \subseteq I$ the family $(\mu_i)_{i \in J}$
is summable in $\mathcal{M}$ with respect to the topology of pointwise convergence in $\mathfrak{R}$. Then for any $J \subseteq I$ the family $(\mu_i)_{i \in J}$ is summable in $\mathcal{M}(E)$ with respect to the semi-norm topology on $\mathcal{M}(E)$.

Let $\mathcal{B}(I)$ be the set of subsets of $I$. The map of $\mathcal{B}(I)$ into $\{0, 1\}^I$ which associates to any subset of $I$ its characteristic functions is a bijection. We endow $\{0, 1\}$ with the discrete topology, $\{0, 1\}^I$ with the product topology, and $\mathcal{B}(I)$ with the topology for which the above bijection is an homeomorphism. Then $\mathcal{B}(I)$ is a compact space. The assertion that any subfamily of a family $(x_i)_{i \in I}$ in a Hausdorff topological additive group is summable is equivalent with the assertion that there exists a continuous map $f$ of $\mathcal{B}(I)$ into $G$ such that $f(J) = \sum_{i \in J} x_i$ for any finite subset $J$ of $I$ ([6]). By the hypothesis there exists therefore a continuous map $f$ of $\mathcal{B}(I)$ into $\mathcal{M}(E)$ endowed with the topology of pointwise convergence in $\mathfrak{R}$ such that $f(J) = \sum_{i \in J} \mu_i$ for any finite subset $J$ of $I$.

Let $\mathcal{M}$ be the vector space of $\mathfrak{R}$-regular real valued measures on $\mathfrak{R}$ endowed with the semi-norm topology, and let $\mathcal{M}'$ be its dual. By Theorem 10 any measure of $\mathcal{M}(E)$ is normal and therefore $\mathcal{M}(E)$ may be considered as a set of maps of $\mathcal{M}'$ into $E$. By Proposition 8 the above map $f$ is continuous with respect to the topology on $\mathcal{M}(E)$ of pointwise convergence in $\mathcal{M}'$. It follows that for any $J \subseteq I$ the family $(\mu_i)_{i \in J}$ is summable in $\mathcal{M}(E)$ with respect to this last topology.

Let us endow $\mathcal{M}'$ with the Mackey $\tau(\mathcal{M}', \mathcal{M})$-topology, let $\mathcal{L}$ be the vector space of continuous linear maps of $\mathcal{M}'$ into $E$, and let $\psi$ be the injection $\mathcal{M}(E) \rightarrow \mathcal{L}$ defined in Theorem 10. It is obvious that $\psi$ is continuous with respect to the topology on $\mathcal{M}(E)$ and $\mathcal{L}$ of pointwise convergence in $\mathcal{M}'$. Hence for any $J \subseteq I$ the family $(\psi(\mu_i))_{i \in J}$ is summable in $\mathcal{L}$ with respect to the topology of pointwise convergence in $\mathcal{M}'$.

Let $U$ be a closed convex $0$-neighbourhood in $E$ and let $U^0$ be its polar set in $E'$ endowed with the $\sigma(E', E)$-topology. Let $\mathfrak{R}'$ be a $\sigma$-ring of sets contained in $\mathfrak{R}$, let $N$
be the set \( \{ \gamma \in \mathcal{M} \mid \sup_{A \in \mathcal{R}'} |\gamma(A)| \leq 1 \} \), and let \( \mathcal{N}^0 \) be its polar set in \( \mathcal{M}' \) endowed with the \( \sigma(\mathcal{M}', \mathcal{M}) \)-topology. For any \( \mu \in \mathcal{M}(E) \) the map
\[
\theta \mapsto \int \theta \, d\mu : \mathcal{N}^0 \to E
\]
is continuous with respect to the weak topology of \( E \). It follows that the image of \( \mathcal{N}^0 \) through this map is a convex weakly compact set of \( E \). By the hypothesis about \( E \) the map \( \hat{\mu} \)
\[
(\theta, x') \mapsto \langle \int \theta \, d\mu, x' \rangle : \mathcal{N}^0 \times U^0 \to \mathbb{R}
\]
is continuous. Let \( \mathcal{C}(\mathcal{N}^0 \times U^0) \) be the vector space of continuous real functions on \( \mathcal{N}^0 \times U^0 \). By the above proof for any \( J \subset I \) the family \( (\hat{\mu}_i : i \in J) \) is summable in \( \mathcal{C}(\mathcal{N}^0 \times U^0) \) with respect to the topology of pointwise convergence. By [7] Theorem II 4 the same assertion holds with respect to the topology of uniform convergence. Let \( J \subset I \). Then there exists a finite subset \( K \) of \( J \) such that
\[
|\sum_{i \in L} \hat{\mu}_i(\theta, x') - \sum_{i \in J} \hat{\mu}_i(\theta, x')| \leq 1
\]
for any finite subset \( L \) of \( J \) containing \( K \) and for any \( (\theta, x') \in \mathcal{N}^0 \times U^0 \). We get
\[
\sum_{i \in L} \mu_i(A) - \sum_{i \in J} \mu_i(A) \in U
\]
for any finite subset \( L \) of \( J \) containing \( K \) and for any \( A \in \mathcal{R}' \). Since \( \mathcal{R} \) and \( U \) are arbitrary this shows that the family \( (\mu_i : i \in J) \) is summable in \( \mathcal{M}(E) \) with respect to the seminorm topology.

**BIBLIOGRAPHY**


Manuscrit reçu le 23 décembre 1974.

Corneliu Constantinescu,
ETH, Mathematisches Seminar
8006 Zürich, Switzerland.