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H-CONES AND POTENTIAL THEORY

by N. BOBOC, Gh. BUCUR and A. CORNEA

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Introduction.

In many recently developed research works in potential theory there were pointed out ordered convex cones, as for instance the cone of positive superharmonic functions on a harmonic space, or the cone of excessive functions with respect to a resolvent family of positive kernels, for which various order theoretical and algebraic properties were proved: lattice-completeness, consistency of algebraic and lattice operations, Riesz splitting property.

It was shown that properties of this type are sufficient in order to build up a good deal of potential theory, especially balayage theory and duality; they were taken as axioms in the definition of the concept of H-cone which is an ordered convex cone satisfying the above mentioned completeness and consistency properties and also Riesz splitting property. In the general theory of H-cones, two concepts are mostly important namely, the balayage and the H-integral. A balayage on an H-cone C is a map from C into C which is additive, increasing, idempotent and continuous in order from below. This generalizes the concept of balayage on sets from classical potential theory. An H-integral on C is a positive, numerical function which is additive, increasing, continuous in order from below and finite on a set which is a subset dense in order from below.

The set of all H-integrals on C organized in a natural way as an ordered cone forms also an H-cone, called the dual of C. It may be shown that if C is the cone of positive supersolutions with respect to a convenient elliptic operator L, then it is an H-cone whose dual is isomorphic with the cone of positive supersolutions with respect to the adjoint operator of L (if such an adjoint does exist). One of the principal aims of this paper is to give an integral representation theorem and a representation theorem of an H-cone as a cone of lower semicontinuous functions on a metrisable space which is also the cone of excessive functions with respect to a resolvent family of continuous kernels. This representation may be performed for a particular type of H-cone called standard H-cone which has the property that its dual is also a standard H-cone, hence representable. Related with duality, very important are those properties, which are simultaneously true on a given H-cone and on its dual. Such a property is axiom D, which in the case of harmonic spaces coincides with the axiom of domination, and which is closely related with a sheaf property when the cone is represented as an H-cone of functions.

The paper is divided in four sections. In the first one we introduce the principal concepts of the theory and in the second we give the representation theorems for standard H-cones.

In sections three and four are studied for standard H-cones of functions, some concepts, inspired from the existing potential theory on harmonic spaces, such as fine topology, thin sets, polar and semipolar sets, balayage on a set. Finally the sheaf property with respect to the fine topology is presented.

We assume that the reader is familiar with the main techniques from potential theory in its axiomatic approach and also with Hunt's potential theory. Therefore many proofs are omitted or only sketched.

Section I.

H-cones and duals of H-cones.

A set C endowed with two composition laws

$$(x, y) \to x + y, \qquad x, y \in C$$
$$(\alpha, x) \to \alpha x, \qquad \alpha \in \mathbf{R}_+, x \in C$$

and with an order relation $\leqslant$ is called *ordered convex cone* if the following axioms are satisfied :

C_1 $x + (y + z) = (x + y) + z, \quad x, y, z \in C$
C_2 $x + y = y + x, \quad x, y \in C$
C_3 there exists an element, denoted by 0, in C, such that

$$x + 0 = x, \qquad x \in C$$

C_4 $(\alpha + \beta)x = \alpha x + \beta x, \quad \alpha, \beta \in \mathbf{R}_+, x \in C$
$\alpha(x + y) = \alpha x + \alpha y, \quad \alpha \in \mathbf{R}_+, x, y \in C$
C_5 $(\alpha . \beta)x = \alpha(\beta x) \quad \alpha, \beta \in \mathbf{R}_+, x \in C$
C_6 $1 . x = x \quad x \in C$
$0 . x = 0 = 0$
C_7 $x \leqslant y \Longrightarrow x + z \leqslant y + z, \quad x, y, z \in C$
$x \leqslant y \Longrightarrow \alpha x \leqslant \alpha y, \quad \alpha \in \mathbf{R}_+, x, y \in C$

We denote by $\vee$ (resp. $\wedge$) the eventual l.u.b. (resp. g.l.b.) in the ordered set C.

An ordered convex cone C is called an H-*cone* if the following axioms are fulfilled :

H_1 $(s \in C) \Longrightarrow (s \geqslant 0)$
H_2 $(s, t, u \in C, s + u \leqslant t + u) \Longrightarrow (s \leqslant t)$
H_3 the ordered set C is a lower complete lattice.
H_4 for any increasing and dominated net $(s_i)_{i \in I}$ and any $s \in C$ we have

$$\bigvee_{i \in I} (s_i + s) = s + \bigvee_{i \in I} s_i$$

H_5 for any net $(s_i)_{i \in I}$ and any $s \in C$ we have

$$\bigwedge_{i \in I} (s_i + s) = s + \bigwedge_{i \in I} s_i$$

H_6 Riesz splitting property holds in C (i.e. for any s, $s_1, s_2 \in C$ such that $s \leqslant s_1 + s_2$ there exists $t_1, t_2 \in C$ such that

$$t_1 \leqslant s_1, \qquad t_2 \leqslant s_2, \qquad s = t_1 + t_2).$$

Exemples of H-*cones.*

Exemple 1. — Let X be a harmonic space [7], [8]. Then the convex cone C of all positive superharmonic functions on X is an H-cone with respect to the pointwise order.

Exemple 2. — Let X be an ordered set. Then the convex cone of all real positive and increasing functions on X is an H-cone iff for any two elements $x, y \in X$ such that there exist $u, v \in X$, with

$$u \leqslant x \leqslant v, \qquad u \leqslant y \leqslant v$$

we have either $x \leqslant y$ or $y \leqslant x$.

Exemple 3. — Let $(X, \mathscr{X}, \mu)$ be a probability space and $\mathcal{V} = (V_\alpha)_{\alpha > 0}$ be a submarkovian resolvent on $(X, \mathscr{X})$ absolutely continuous with respect to μ (i.e. $V_\alpha(f) = 0$ for any α and for any measurable function f which vanishes μ-almost everywhere). Then the convex cone $\mathscr{E}_{\mathcal{V}}$ of all excessive functions with respect to $\mathcal{V}$ is an H-cone.

For the axiom H_6 see [13], or [11]; for the axioma H_4, H_5 see [5].

From now on, throughout this paper we shall denote by C a fixed H-cone.

Using only the above axioms H_1 and H_2 we may construct in a natural way, the ordered vector space $C - C$ of differences of elements of C such that C becomes a subcone of positive elements of $C - C$. Further using axioms H_3 and H_5 one can see that $C - C$ is a vector lattice.

Since from axiom H_5 it follows that for any subset A of C $\wedge A$ is the intersection in $C - C$ of A we shall denote also by $\wedge$ the intersection in $C - C$.

We denote by $\preccurlyeq$ the order relation on C defined by

$$s \preccurlyeq t \Longleftrightarrow (\text{there exists } u \in C \text{ with } s + u = t).$$

This order relation will be called the *specific order.* The initial order relation $\leqslant$ on C will be called the *natural order.* We denote by $\curlyvee$ (resp. $\curlywedge$) the l.u.b. (resp. the g.l.b.) with respect to the specific order. For any $f =: s - t$, s, $t \in C$ we denote

$$\mathbf{R}(f) =: \wedge \{s' \in C | s' \geqslant f\}$$

and we call it *the reduite of* f. One can prove [4] that $R(f) \preccurlyeq s$. Further (see [4], [8], [10]) one can prove that C is a conditionally complete lattice with respect to the specific order. The following proposition is a key to the proof of many important results from this paper.

Proposition 1.1. — *Let* $u, v \in C$ *and let* $(s_i)_{i \in I}$ *be a net in* C *increasing to* $u + v$. *Then the nets* $(R(s_i - v))_i$, $(R(s_i - u))$ *increase to* u *and* v *respectively and we have*

$$[R(s_i - v) + R(s_i - u) \leqslant s_i]$$

A map $B : C \to C$ is called *a balayage* (on C) if it is:

B_1 : additive $(s, t \in C \Longrightarrow B(s + t) = B(s) + B(t)$.
B_2 : increasing $(s \leqslant t \Longrightarrow Bs \leqslant Bt)$.
B_3 : a contraction $(s \in C \Longrightarrow Bs \leqslant s)$.
B_4 : idempotent $(s \in C \Longrightarrow B(Bs) = Bs)$.
B_5 : continuous in order from below (for any $s \in C$ and any net $(s_i)_{i \in I}$ increasing to s we have

$$\bigvee_{i \in I} B(s_i) = Bs)$$

If $f \in C - C$, we denote by B_f the map from C into C defined by

$$B_f(s) = \bigvee_{n \in N} R(nf \wedge s)$$

This map is a balayage (see [4]).

On the set of all balayages on C we may introduce the following order relation

$$[B_1 \leqslant B_2] \Longleftrightarrow [(s \in C) \Longrightarrow (B_1 s \leqslant B_2 s)]$$

We denote, by $\vee$ (resp. $\wedge$) the l.u.b. (resp. g.l.b.), with respect to this order relation.

PROPOSITION 1.2. — *a) If* B *a balayage such that* $B \leqslant B_f$ *where* $f \in C - C$ *then there exists a family* $(f_i)_{i \in I}$ *in* $C - C$ *such that*

$$B = \bigwedge_{i \in I} B_{f_i}.$$

b) The ordered set of all balayages on C *is a distributive and a complete lattice.*

c) For any family $(B_i)_{i \in I}$ *of balayages and any* $s \in C$ *we have*

$$\left(\bigvee_{i \in I} B_i\right)(s) = \bigvee_{i \in I} (B_i(s))$$

d) For any two balayages B_1, B_2 *such that* $B_1 \leqslant B_2$ *we have*

$$B_1 B_2 = B_2 B_1 = B_1$$

THEOREM 1.3. — *Let* B *be a balayage on* C *and denote by* C_B *the subcone of* $C - C$ *of all elements of the form* $s - Bs$ *with* $s \in C$. *Then* C_B *endowed with the natural order from* $C - C$ *is an* H*-cone. Moreover if* (s_i) *is a net in* C *and* $(s_i - Bs_i)$ *increases to* $s - Bs$, *and* $s_i \curlywedge Bs_i = 0$ *then* $(s_i)_i$ *increases,* $s_i \leqslant s$ *and* $s - Bs = \bigvee s_i - \bigvee Bs_i$.

The key to the proof is the following.

PROPOSITION 1.4. — *For any* $s, t \in C$ *we have*

$$[(s - Bs + Bt) \wedge t] \in C.$$

Moreover if $s - Bs \leqslant t - Bt$ *and* $s \curlywedge Bs = 0$ *then*

$$s \leqslant t, \quad Bs \preccurlyeq Bt.$$

We shall introduce now a dual for an H-cone. A map

$$\mu : C \to \overline{R}_+$$

will be called an H-*integral* if it is:

I_1 additive: $(s, t \in C) \Longrightarrow \mu(s + t) = \mu(s) + \mu(t)$.

I_2 increasing: $(s \leqslant t) \Longrightarrow \mu(s) \leqslant \mu(t)$.

I_3 continuous in order from below (i.e. for any net $(s_i)_{i \in I}$ increasing to s, we have

$$\mu(s) = \sup \mu(s_i).)$$

I_4 finite on a subset dense in order from below (i.e. for any $s \in C$ there exists a net $(s_i)_{i \in I}$ increasing to s such that $\mu(s_i) < +\infty$ for any $i \in I$).

We shall denote by C^* the ordered convex cone of all H-integrals on C where the order relation and the algebraic operations are defined pointwise (with the convention $0.\infty = 0$). This cone will be called *the dual* of C.

THEOREM 1.5. — *The ordered convex cone* C^* *is an* H-*cone. For any two elements* μ_1, μ_2 *if* C^* *we have*

a) $$(\mu_1 \vee \mu_2)(s) = \sup_{s_1+s_2=s} (\mu_1(s_1) + \mu_2(s_2))$$

b) $$R(\mu_1 - \mu_2)(s) = \sup_{t \leqslant s,\, \mu_2(t) < \infty} (\mu_1(t) - \mu_2(t))$$

Proof. — The axioms H_1, H_2 are obvious from the definitions. Further for any $\mu_1, \mu_2 \in C^*$ the map

$$s \to \sup_{s_1+s_2 \leqslant s} (\mu_1(s_1) + \mu_2(s_2))$$

is an H-integral and represents the least upper bound of μ_1, μ_2.

Since for any increasing and dominated net $(\mu_i)_{i \in I}$ from C^* the map

$$s \to \sup_{i \in I} \mu_i(s)$$

is an H-integral, we deduce that the axioms H_3, H_1 also hold.

Now for any $\mu_1, \mu_2 \in C^*$ the map

$$s \to \sup_{\substack{t \in C,\, \mu_2(t) < \infty \\ t \leqslant s}} (\mu_1(t) - \mu_2(t))$$

is an H-integral μ satisfying the properties

$$\mu_1 - \mu_2 \leqslant \mu \leqslant \lambda \quad \text{for any} \quad \lambda \in C^*,\ \lambda \geqslant \mu_1 - \mu_2.$$

From this, one can easily see that axiom H_5 holds.

Axiom H_6 may be obtained by adaptation to this case of a proof of Mokobodzki (see [13]).

For any balayage B on C and any $\mu \in C^*$ we denote by $B^*\mu$ the map from C into $\overline{\mathbf{R}}_+$ defined by

$$B^*(\mu)(s) = \mu(B(s))$$

One can easily see that the map

$$\mu \to B^*\mu$$

is a balayage on C^*. The map B^* will be called the *adjoint of* B.

Obviously if B_1, B_2 are two balayages on C such that $B_1 \leqslant B_2$, then $B_1^* \leqslant B_2^*$.

Proposition 1.6. — *For any* $s \in C$ *the map* $\tilde{s} : C^* \to \overline{\mathbf{R}}_+$ *defined by*

$$\tilde{s}(\mu) = \mu(s)$$

is an H-*integral on* C^*.

Proof. — Only the property I_4 is somewhat difficult to be proved. Let $\mu \in C^*$ and $C_\mu = \{t \in C | \mu(t) < \infty\}$. Further for any $t \in C_\mu$, let $\mu_t = (B_{t-t\wedge s})^*(\mu)$. It is immediate that $(\mu_t)_t$ is a net increasing to μ and $\tilde{s}(\mu_t) \leqslant \mu(t) < \infty$.

Proposition 1.7. — *For any* $s, t \in C$ *and any* $\mu \in C^*$ *such that* $\mu(s+t) < \infty$, *there exist* $\mu_1, \mu_2 \in C^*$ *such that* $\mu = \mu_1 + \mu_2$ *and* $\mu(s \wedge t) = \mu_1(s) + \mu_2(t)$.

Take μ_1 and μ_2 defined by

$$\mu_1(u) = \sup_{n \in \mathbf{N}} \mu(u \wedge n(t - t \wedge s)),$$

$\mu_2(u) = \mu(u) - \mu_1(u)$ for any $u \in C$ for which $\mu(u) < \infty$. One can show that $\mu_1, \mu_2 \in C^*$ and satisfy the required condition.

Theorem 1.7. — *We have*

a) $\quad s, t \in C, \alpha \in R_+ \Longrightarrow \widetilde{s+t} = \tilde{s} + \tilde{t}, \widetilde{\alpha s} = \alpha \tilde{s}$

b) $\quad s, t \in C, s \leqslant t \Longrightarrow \tilde{s} \leqslant \tilde{t}$

c) for any dominated family $(s_i)_{i\in I}$ *from* C *we have*

$$\widetilde{\bigvee_{i\in I} s_i} = \bigvee_{i\in I} \tilde{s}_i, \quad \widetilde{\bigwedge_{i\in I} s_i} = \bigwedge_{i\in I} \tilde{s}_i.$$

d) for any $s, t \in \mathrm{C}$ *we have* $\widetilde{\mathrm{R}(s-t)} = \mathrm{R}(\tilde{s} - \tilde{t})$.

The map $s \to \tilde{s}$ of C into C** is called the *evaluation map*. If C separates C* this map is an injection. In the sequel, we shall identify the cone C with its image through the evaluation map.

Section II.

Standard H-cones.

In order to get an integral representation theorem for an H-cone or a representation of an H-cone as a cone of functions on a topological space we have to impose some supplementary conditions.

An element u of an H-cone C is called *strictly positive* if for any $s \in C$ we have $s = \bigvee_{n \in N} (s \wedge nu)$.

Let u be a strictly positive element of C. An element $c \in C$ is called *u-continuous* if for any increasing net (s_i) in C such that $\bigvee_i s_i = c$ and any $\varepsilon \in R$, $\varepsilon > 0$ there exists i_ε for which we have $c \leqslant s_i + \varepsilon u$ for any $i \geqslant i_\varepsilon$.

An element $c \in C$ is called *universally continuous* if it is u-continuous for any strictly positive element u of C.

Proposition 2.1. — *Let* u *be a strictly positive element of* C. *Then the set* C_u *of all* u*-continuous elements is a specifically solid subcone of* C. *For any* $s \in C_u$ *there exists* $\alpha > 0$ *such that* $s < \alpha u$.

In order to prove that C_u is a convex cone, use proposition 1.1.

Corollary. — *The set* C_0 *of all universally continuous elements is a specifically solid subcone of* C.

Remark 1. — With the above notation, if s', s'' belong to C_u (respective to C_0) then $s' \vee s''$ belongs to C_u (respective to C_0).

Indeed $s' \vee s'' = R(s' + s'' - s' \wedge s'') \preccurlyeq s' + s''$.

Remark 2. — Let u be a strictly positive element. The map $B : C \to C$ defined by

$$Bs = V\{s' \in C_u | s' \leqslant s\}$$

is a balayage.

PROPOSITION 2.2. — *Let* θ *be a map from* C_0 *into* R_+ *such that*

a) $$s, t \in C_0 \Longrightarrow \theta(s+t) = \theta(s) + \theta(t)$$

b) $$s, t \in C_0,\ s \leqslant t \Longrightarrow \theta(s) \leqslant \theta(t)$$

c) there exists u *strictly positive in* C *such that* $\theta(s) \leqslant 1$ *for any* $s \leqslant u$, $s \in C_0$. *Then* θ *is the restriction to* C_0 *of an element of* C^*.

Proof. — We denote for any $s \in C$

$$\lambda(s) = \sup \{\theta(s') | s' \in C_0,\ s' \leqslant s\}$$

From the definition follows that λ is increasing, additive; $\lambda(u) \leqslant 1$; its restriction to C_0 coincides with θ. It is also continuous from below. Indeed let $(s_i)_i$ be an increasing net and $s = \bigvee_i s_i$, and let $s' \in C_0$ be such that $s' \leqslant s$. Then for any $\varepsilon > 0$ there exists i_ε such that

$$i \geqslant i_\varepsilon \Longrightarrow s' \leqslant s_i + \varepsilon u.$$

Let $s'_i, s''_i \in C$ be such that

$$s' = s'_i + s''_i,\quad s'_i \leqslant s_i,\quad s''_i \leqslant \varepsilon u.$$

Then we have

$$\theta(s') = \theta(s'_i) + \theta(s''_i) \leqslant \lambda(s_i) + \varepsilon \leqslant \sup_i \lambda(s_i) + \varepsilon.$$

Hence

$$\lambda(s) \leqslant \sup_i \lambda(s_i).$$

An H-cone C is called a *standard* H-*cone* if

a) there exists a strictly positive element in C;

b) there exists a countable subset D of universally continuous elements which is dense in order from below

$$(\text{i.e. } s \in C \Longrightarrow s = \vee\{s' | s' \in D,\ s' \leqslant s\})$$

The H-cone in the example 1 is standard provided that the underlying space X has a countable base.

PROPOSITION 2.3. — *With the notations from example* 3 (*section* 1) *the* H-*cone* $\mathscr{E}_{\mathfrak{V}}$ *is standard, provided that there exists a proper kernel* V *such that* $V = \sup_\alpha V_\alpha$.

A key to the proof is a result of Mokobodzki [11] after which we may suppose that the set

$$\{V(f) | f \text{ measurable, } 0 \leqslant f \leqslant 1\}$$

is compact with respect to the topology of uniform convergence on X.

First we show that the function $u = V(1)$ is a strictly positive element in $\mathscr{E}_{\mathfrak{V}}$. Indeed if $s \in \mathscr{E}_{\mathfrak{V}}$ then there exists a sequence f_n of positive measurable functions such that $(V(f_n))_n$ increases to s. We have

$$s \geqslant \bigvee_n (nu \wedge s) \geqslant \bigvee_{n, m} (V(\inf(n, f_m)) = s.$$

Further, let α be a positive number and denote

$$A = \{x \in X | u(x) \geqslant \alpha\}.$$

We show that for any strictly positive element $v \in \mathscr{E}_{\mathfrak{V}}$ we have $\inf \{v(x) | x \in A\} > 0$. Indeed if we denote

$$X_n = \{x \in X | u(x) \leqslant nv(x)\}$$

then $\bigcup_{n=1}^{\infty} X_n = X$, hence $V(\chi_{X_n})$ converges uniformely to $V(1) = u$ and therefore for a sufficiently large n, we have $V(\chi_{X_n}) > \frac{\alpha}{2}$ on A. Since V satisfies the complete maximum principle we have $nv \geqslant V(\chi_{X_n})$, $v \geqslant \frac{\alpha}{2n}$ on A.

Let f be a positive, bounded, measurable function which vanishes outside A. We show that $V(f)$ is universally continuous. Let $(s_i)_{i \in I}$ be a net in $\mathscr{E}_{\mathfrak{V}}$ increasing to $V(f)$.

Since the inequality $s \leqslant t$, μ-almost everywhere for two elements of $\mathscr{E}_{\mathfrak{V}}$ implies the inequality $s \leqslant t$ everywhere on X we may, by standard arguments, extract a sequence $(s_n)_n$ such that $(s_n)_n$ increases to $V(f)$. Let ε be a positive number and denote

$$A_n = \{x \in A | s_n(x) + \frac{\varepsilon}{2} v(x) < V(f)(x)\}.$$

Obviously $\cap A_n = \varnothing$. Hence the sequence $V(f.\chi_{A_n})$ converges uniformly to 0 and therefore for a sufficiently arge n, we have $V(f.\chi_{A_n}) < \frac{\varepsilon}{2} \varphi$ on A. Using again the complete maximum principle we have

$$V(f) = V(f.\chi_{A_n}) + V(f\chi_{A \setminus A_n}) \leqslant \frac{\varepsilon}{2} \varphi + s_n + \frac{\varepsilon}{2} \varphi.$$

Let $\mathcal{D}_0$ be a countable uniformly dense subset of the set $\{V(f) | \ f$ measurable, $0 \leqslant f \leqslant 1\}$. From the above considerations we may assume that the elements of $\mathcal{D}_0$ are universally continuous. Let $\mathcal{D}$ be the set

$$\{\gamma' R(s - ru) | s \in \mathcal{D}_0, r', r \text{ positive rational numbers}\}.$$

The set $\mathcal{D}$ is countable and dense in order from below.

PROPOSITION 2.4. — *Let* $\mathcal{V}$ *and* $\mathcal{V}^*$ *be two resolvent families satisfying the conditions of proposition* 2.3 *and assume that for any two positive, measurable functions* f, g *and any* $\alpha > 0$ *we have*

$$\int f.V_\alpha(g)\, d\mu = \int g.V_\alpha(f)\, d\mu$$

Then the H-*cones* $(\mathcal{E}_{\mathcal{V}})^*$ *and* $\mathcal{E}_{\mathcal{V}^*}$ *are isomorphic.*

For the proof see ([5]).

PROPOSITION 2.5. — *Let* $\mathcal{V}$ *be a resolvent family as in proposition* 2.3. *Then there exist two resolvent families* $\mathcal{W}$, $\mathcal{W}^*$ *satisfying the conditions of proposition* 2-4, *such that the* H-*cones* $\mathcal{E}_{\mathcal{V}}$ *and* $\mathcal{E}_{\mathcal{W}}$ *are isomorphic.*

PROPOSITION 2.6. — *Assume that* C *is a standard* H-*cone and let* μ *be an element of* C*. *Then there exists a strictly positive element* $u \in C$ *such that*

a) $\mu(u) < \infty$;

b) u *is the summ of a sequence of universally continuous elements*;

c) *for any* $s \in C$ *we have* $s = V\{s' | s' \leqslant s,\ s' \preccurlyeq \alpha u,$ *for some* $\alpha > 0\}$.

From the definition of a standard H-cone there exists a countable set $(s_n)_{n \in N}$, dense in order from below, of universally continuous elements such that $\mu(s_n) < \infty$ for any n. If u_0 is a strictly positive element of C we may find $\alpha_n > 0$

such that $s_n < \alpha_n u$ (see proposition 2.1). The element

$$u = \sum_{n=1}^{\infty} \frac{1}{2^n(\alpha_n + \mu(s_n))} s_n$$

satisfies the required conditions of the proposition.

Corollary. — *If* C *is standard and* $\mu \in C^*$ *then for any* $s \in C_0$ *we have* $\mu(s) < +\infty$.

Proposition 2.1.7. — *Assume* C *standard. For any decreasing net* $(\mu_i)_{i \in I}$ *from* C^* *and any* $s \in C_0$ *we have*

$$(\wedge \mu_i)(s) = \inf_i \mu_i(s).$$

For the proof, apply proposition 2.2 to the map on C_0

$$s \to \inf_{i \in I} \mu_i(s).$$

The coarsest topology on $C^* - C^*$ which makes continuous the real linear functionals

$$\mu \to \mu(s)$$

for every $s \in C_0$ (where $\mu(s) = \mu_1(s) - \mu_2(s)$ if $\mu = \mu_1 - \mu_2$ $\mu_1, \mu_2 \in C^*$) is called the *natural topology on* $C^* - C^*$.

Theorem 2.8. — *Assume* C *standard. Let* u *be a strictly positive element of* C *and denote*

$$K_u = \{\mu \in C^* | \mu(u) \leqslant 1\}.$$

Then K_u *is a compact metrisable (with respect to the natural topology) cap of the cone* C^*.

Proof. — Since the map $\tilde{u}$ is additive and positively homogeneous on C^* we deduce that the sets K_u and $C^* \setminus K_u$ are convex. We show now that K_u is compact in the natural topology. Let $\mathcal{U}$ be an ultrafilter on K_u and θ be the map on C_0 into $\mathbf{R}_+$ defined by

$$\theta(s) = \lim_{\mathcal{U}} \mu(s).$$

By proposition 2.2 there exists $\mu_0 \in C^*$ such that $\mu_0|_{C_0} = \theta$.

Obviously $\mu_0 \in K_u$ and $\mu_0 = \lim_{\mathcal{U}} \mu$. Hence K_u is compact. Since there exists in C_0 a countable subset which is dense in order from below we deduce that K_u is metrisable.

Remark. — Since C^* is a lattice with respect to the specific order we deduce that K_u is a simplex and C^* is a cone « bien coiffé » (well-capped) in the natural topology.

PROPOSITION 2.9. — *Using the notations from the preceeding theorem, consider* X *the closure of the set of non zero extreme points of* K_u *and denote* $\tilde{C}$ *the cone of functions on* X *of the form* $\tilde{s}|_X$, $s \in C$. *Then there exists a bounded kernel* V *and a finite measure* μ *on the measurable space* $(X, \mathcal{B}(X))$ *such that*

a) V *satisfies the complete maximum principle;*

b) for any positive, bounded, borel measurable function f, $V(f)$ *is continuous and belongs to* $\tilde{C}$;

c) V *is absolutely continuous with respect to* μ;

d) $\tilde{C}$ *is a solid (in the natural order) subcone of* $\mathcal{E}_{\mathcal{V}}$ *where* $\mathcal{V}$ *is the unique submarkovian resolvent family on* $(X, \mathcal{B}(X))$ *such that* $V = \sup V_\alpha$;

e) $\tilde{C}$ *is an* H*-cone and is isomorphic with* C.

For the proof, apply Mokobodzki's procedure [2] and [12] to the cone of potentials on X obtained by uniform closure of the cone $\tilde{C}_0 + R_+$.

THEOREM 2.10. — *If* C *is a standard* H*-cone then we have:*

a) the dual C^* of C *is also a standard of* H*-cone;*

b) C^* *separates* C *(i.e. the evaluation map is an injection);*

c) the image trough the evaluation map is a solid (with respect to the natural order) subcone of C^{**} *and it is dense in order from below.*

For the proof, use the above proposition 2.9, 2.5, 2.4 and 2.3.

PROPOSITION 2.11. — *Assume that in the* H*-cone* C *there exists a strictly positive element* u *and a countable set of* u-

continuous elements which is dense in order from below. Then C *is a standard* H-*cone.*

Let K_u be as in the theorem 2.8. By a slight modification of the proof of theorem 2.8, we may show that K is compact and metrisable in the week topology $\sigma(K_u, C_u)$. Now C may be represented as in the proposition 2.9 and thus using proposition 2.3 we deduce that it is standard.

SECTION III.

Standard H-cones of functions.

A set $\mathcal{F}$ of positive, numerical functions on a set X is called an H-*cone of functions if*:

$\mathcal{F}_1$, $\mathcal{F}$ endowed with the pointwise algebraic operations and order relation is an H-cone (with the convention $0\infty = 0$).

$\mathcal{F}_2$ For any net $(s_i)_i$ in $\mathcal{F}$ increasing to an element s of $\mathcal{F}$ we have $s(x) = \sup s_i(x)$ for any $x \in X$.

$\mathcal{F}_3$ For any two elements $s, t \in \mathcal{F}$ and any $x \in X$ we have $(s \wedge t)(x) = \inf(s(x), t(x))$.

$\mathcal{F}_4$ The set $\mathcal{F}$ separates X and contains the positive constant functions.

Remark. — We see that for any $x \in X$ the map $s \to s(x)$ is an H-integral on $\mathcal{F}$ and using especially axiom $\mathcal{F}_3$, one can deduce that it generates an extreme ray in $\mathcal{F}^*$.

The coarsest topology on X for which the elements of $\mathcal{F}$ are continuous maps from X into $\overline{R}_+$ will be called the *fine topology* (with respect to $\mathcal{F}$). Thus X becomes a completely regular topological space.

PROPOSITION 3.1. — *Let* $\mathcal{F}$ *be an* H-*cone of functions on a set* X. *Then*

a) *for any family* $(s_i)_{i \in I}$ *in* $\mathcal{F}$, *the element* $\bigwedge_i s_i$ *coincides with the lower semicontinuous regularisation with respect to the fine topology, of the function*

$$x \to \inf \{s_i(x) | i \in I\}$$

b) *any* $s \in \mathcal{F}$ *is finite on a finely-dense open subset of* X.

Let $\mathcal{F}$ be an H-cone of functions on a set X and assume it is also a standard H-cone. Obviously the universally continuous elements of $\mathcal{F}$ are bounded functions.

The coarsest topology on X for which the universally continuous elements are continuous functions is called the *natural topology* on X (with respect to $\mathcal{F}$).

It is immediate that X endowed with this topology is a metrisable and separable space and the fine topology is finer than the natural one. Also, any element of $\mathcal{F}$ is naturally lower sem-continuous.

An H-integral μ on $\mathcal{F}$ is called *representable* if there exists a positive measure m on the natural-Borel field of X (i.e. the σ-field generated by the naturally open sets) such that

$$s \in \mathcal{F} \Longrightarrow \mu(s) = \int s \, dm.$$

Such a measure m is σ-finite since for any universally continuous element $s \in \mathcal{F}$ we have

$$\int s \, dm = \mu(s) < \infty.$$

From the definition and from the above remark it follows that the representing measure m is unique. Obviously the representable H-integrals on $\mathcal{F}$ form a convex subcone of $\mathcal{F}^*$.

An H-cone of functions $\mathcal{F}$ on X is called a *standard H-cone of functions* if $\mathcal{F}$ is a standard H-cone and the convex cone of all representable H-integrals on $\mathcal{F}$ is a solid subcone of $\mathcal{F}^*$ with respect to the natural order.

One can easily see that the above condition may be replaced with the following simpler one: Any H-integral on $\mathcal{F}$ dominated by a finite measure is representable.

In the sequel we shall identify any representable H-integral with the corresponding unique representing measure.

Let C be a standard H-cone, u be a strictly positive element of C and denote

$$K_u = \{\mu \in C^* | \mu(u) \leqslant 1\}.$$

Further denote X the set of all non zero extreme points of K_u and $\mathcal{F}$ the set of functions on X of the form $\tilde{s}|_X$, $s \in C$. It is immediate that $\mathcal{F}$ is a standard H-cone of functions on X which is isomorphic with C. Moreover X is a G_δ set of a compact metrisable space.

PROPOSITION 3.2. — *Let* $\mathcal{F}$ *be a standard* H-*cone of functions on a set* X. *Then any universally continuous* H-*integral is representable.*

Let $(x_n)_{n\in N}$ be a naturally-dense subset of X and denote $m = \sum_{n=1}^{\infty} \frac{1}{2^n} \varepsilon_n$. It is easy to see that $m \in \mathcal{F}^*$. Since for any $s \in \mathcal{F}$, $s \neq 0$ we have $m(s) \neq 0$ we deduce, using the next lemma, that m is a strictly positive element of $\mathcal{F}^*$ and the assertion follows by application of proposition 2.1.

LEMMA. — *Let* μ *be an* H-*integral on* C *such that for any* $s \in C$, $s \neq 0$, $\mu(s) > 0$. *Then* μ *is a strictly positive element of* C^*.

Let $\nu \in C^*$ and let $s \in C$ be such that $\nu(s) + \mu(s) < \infty$. From proposition 1.7 and theorem 2.10.*c*), for any $n \in N$ there exists $s_n, t_n \in C$ such that $s = s_n + t_n$ and

$$(\nu \wedge n\mu)(s) = \nu(s_n) + n\mu(t_n).$$

We may choose s_n increasing. Since $\mu\left(\bigwedge_n t_n\right) \leqslant \frac{\nu(s)}{m}$ we deduce $\wedge\, t_n = 0$, $\left[\bigvee_n (\nu \wedge n\mu)\right](s) = \nu(s)$.

Throughout this paper, $\mathcal{S}$ will be a standard H-cone of functions on a set X. We shall denote

$$K_1 = \{\mu \in \mathcal{S}^* | \mu(1) \leqslant 1\};$$

by X_1 the set of non zero extreme points of K_1 and by $\mathcal{S}_1$ the standard H-cone of functions on X_1 consisting of all functions of the form $\tilde{s}|_{X_1}$, $s \in \mathcal{S}$. Further we shall identify, in a natural way, X with the subspace of X_1.

It is not difficult to see that X is finely dense in X_1.

THEOREM 3.3. — *a*) $s, t \in \mathcal{S}^{**}$, $s \leqslant t$ *on* $X \Longleftrightarrow s \leqslant t$.

b) *If we identify the elements of* $\mathcal{S}^{**}$ *with their restrictions to* X *then* $\mathcal{S}^{**}$ *is a standard* H-*cone of functions on* X *for which the fine (resp. natural topology coincides with the fine resp. natural) topology given by* $\mathcal{S}$.

c) *Any function* f *on* X *which is finite on a naturally dense subset of* X *and which is the supremum of an increasing net* $(s_i)_i$, $s_i \in \mathcal{S}^{**}$ *belongs also to* $\mathcal{S}^{**}$.

For the proof of c) use a measure on X of the form

$$m = \sum_{n=1}^{\infty} \alpha_n \varepsilon_{x_n}$$

where $(x_n)_n$ is a naturally-dense sequence in X and α_n is a sequence of strictly positive numbers such that

$$\sum_n \alpha_n f(x_n) < \infty, \sum_n \alpha_n < \infty.$$

For any subset A of X and any $s \in \mathcal{S}$ we denote by $R^A s$ (resp. $B^A s$) the numerical function on X defined by:

$$R^A s(x) = \inf \{s'(x) | s' \in \mathcal{S}, \ s' \geqslant s \quad \text{on} \quad A\}$$

(resp. $B^A s = \wedge \{s' \in \mathcal{S} | s' \geqslant s \ \text{ on } \ A\}$).

We remark that $B^A s$ is the lower semicontinuous regularisation of $R^A s$ with respect to both natural or fine topology.

A subset A of X is called *polar* if $B^A 1 = 0$. For any $s \in \mathcal{S}$ the set $\{x \in X | s(x) = +\infty\}$ is polar.

THEOREM 3.4. — *We have*

a) *if* $A_1 \setminus A_2$ *is polar and* $s_1 \leqslant s_2$ *on* $A_1 \cap A_2$ *then* $B^{A_1} s_1 \leqslant B^{A_2} s_2 \leqslant s_2$.

b) $B^A s = \bigwedge_G B^G s$ *where* G *runs through the set of all finely open sets such that* $A \setminus G$ *is polar.*

c) $B^A(s_1 + s_2) = B^A s_1 + B^A s_2$ *and* $B^A(\alpha s) = \alpha B^A s$.

d) $B^{A_1 \cup A_2} s + B^{A_1 \cap A_2} s \leqslant B^{A_1} s + B^{A_2} s$.

e) *If* $A_n \uparrow A$ *and* $s_i \uparrow s$ *on* A, *then* $B^{A_n} s_i \uparrow B^A s$.

f) *If* A *is a finely-open set then* $B^A(B^A s) = B^A s$.

The proof uses standard techniques of balayage theory on harmonic spaces (see [8]).

We remark that the above operator B^A becomes a balayage on the H-cone $\mathcal{S}$ iff it is idempotent (this is true, for instance, when A is finely open).

An element $s \in \mathcal{S}$ is called *a generator* if for any $s' \in \mathcal{S}$ there exists a sequence $(s_n)_n$ increasing naturally to s' such that for any n, s_n is specifically dominated by $\alpha_n s$ where $\alpha_n > 0$. From proposition 2.6 we deduce that there exists a generator which is bounded and continuous in the natural topology.

For any balayage B we call *base* of B and denote it by « $b(B)$ » the set

$$b(B) = \{x \in X | Bs(x) = s(x), (\forall) s \in \mathcal{S}\}.$$

PROPOSITION 3.5. — *Let* B *be a balayage on* $\mathcal{S}$. *Then we have:*

a) for any generator s *of* $\mathcal{S}$

$$b(B) = \{x \in X | Bs(x) = s(x)\}$$

b) the set $b(B)$ *is finely-closed and of type* G_δ *for the natural topology.*

c) For any $x \in X$ *the measure* $B^*_{\varepsilon_x}$ *charges only* $b(B)$.

d) $B = B^{b(B)}$.

For any balayage B on $\mathcal{S}$ denote by $d(B)$ the complement of the base of B. Obviously $d(B)$ is a finely open subset of X of type $F\sigma$. Also for any finite s from $\mathcal{S}$ we denote by s_B the function on $d(B)$ defined by

$$s_B(x) = s(x) - Bs(x).$$

Since the convex cone $\mathcal{S}^f$ of all finite elements of $\mathcal{S}$ is a standard H-cone of functions on X we deduce that the ordered cone of functions on $d(B)$ of the form s_B with $s \in \mathcal{S}^f$ forms an H-cone isomorphic with the H-cone $\mathcal{S}^f_B$. The axioms $\mathcal{F}_2$, $\mathcal{F}_3$ are also verified for this cone of functions (use theorem 1.3 and proposition 1.4).

We denote by $\hat{\mathcal{S}}_B$ the convex cone of all functions s on $d(B)$ which are finite on a finely-dense subset and such that there exists a net $(s_i)_{i \in I}$, $s_i \in \mathcal{S}^f$ for which the net $((s_i)_B)_{i \in I}$ increases to s.

THEOREM 3.6. — *The convex cone* $\hat{\mathcal{S}}_B$ *is a standard* H-*cone of functions on the set* $d(B)$ *and it is isomorphic with the* H-*cone* $(\mathcal{S}_B)^{**}$. *Also for any* $s \in \mathcal{S}^{**}$ *the restriction of* s *to* $d(B)$ *belongs to* $\hat{\mathcal{S}}_B$. *The elements of* $\hat{\mathcal{S}}_B$ *are finely-continuous and natural-borel measurable functions on* $d(B)$.

Let p be a bounded generator of $\mathcal{S}$ and $s \in \mathcal{S}^{**}$. Since the function $s_n = (s + nBp) \wedge np \in \mathcal{S}^f$ and

$$(s_n)_B = (s \wedge n(p - B_p))|_{d(B)}$$

we deduce

$$s|_{d(B)} = \sup_n (s_n)_B \in \hat{\mathcal{S}}_B.$$

Since the ordered convex cone $\mathcal{F}$ of functions of the form s_B, $s \in \mathcal{S}^f$ satisfies axioms $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and it is dense in order from below in $\hat{\mathcal{S}}_B$ we deduce that $\hat{\mathcal{S}}_B$ is an H-cone of functions. Since $\mathcal{F}$ is solid in $\hat{\mathcal{S}}_B$ the functions of $\hat{\mathcal{S}}_B$ are finely-continuous and natural-borel measurable.

We show now that $\hat{\mathcal{S}}_B$ is a standard H-cone. Let $\mathcal{V}$ be a resolvant family on X such that $\mathcal{E}_{\mathcal{V}} = \mathcal{S}^{**}$, and V_0 is bounded and absolutely continuous. Denote V the kernel on $d(B)$ defined by

$$Vf = (V_0 f_0 - BV_0 f_0|_{d(B)}$$

where f is a bounded measurable function on $d(B)$ and f_0 is the function on X equal to f on $d(B)$ and equal to 0 elsewhere. Let $s \in \hat{\mathcal{S}}_B$ be such that $Vf \leqslant s$ on the set

$$A = \{x \in X | f(x) > 0\}.$$

If we denote s_0 the function on X equal to s on $d(B)$ and equal to o elsewhere we have

$$V_0(f_0) \leqslant s_0 \wedge (V_0 f_0 - BV_0(f_0)) + BV_0(f_0) \text{ on } A.$$

Since the cone $\mathcal{F}$ from above is solid in $\hat{\mathcal{S}}_B$ we deduce, using proposition 1.4 that the function from the lefthandside belongs to $\mathcal{S}$ and therefore, applying domination principle for V_0, we see that the above inequalities hold on X and on $d(B)$ respectively. From this fact we deduce that V satisfies the complete maximum principle and that there exists a resolvent family $\mathcal{V}' = (\mathcal{V}'_\alpha)_{\alpha \geqslant 0}$ on $d(B)$ such that $V = V'_0$ and $\alpha V'_\alpha(s) \leqslant s$ for any $\alpha > 0$ and any $s \in \hat{\mathcal{S}}_B$ (see [1], [3]). Since V_0 charges any finely open set of X we get that V'_0 also charges any fine open set of $d(B)$ and therefore $\mathcal{E}_{v'} = \hat{\mathcal{S}}_B$. Finaly V'_0 being absolutely continuous we get $\hat{\mathcal{S}}_B$ standard.

In order to show that $\hat{\mathcal{S}}_B$ is a standard H-cone of functions on $d(B)$ it remains only to prove that for any finite measure μ on $d(B)$ and any H-integral μ on $\hat{\mathcal{S}}_B$ dominated by μ, there exists a measure λ' on $d(B)$ which equals λ on $\hat{\mathcal{S}}_B$.

Since $d(\mathrm{B})$ is a F_σ- set in X and the natural Borel-field on $d(\mathrm{B})$ generated by $\hat{\mathscr{S}}_\mathrm{B}$ is the restriction to $d(\mathrm{B})$ of the natural Borel-field from X, we may consider μ as a measure on the whole of X.

If we denote by $\bar{\lambda}$ the map on $\mathscr{S}$ defined by

$$\bar{\lambda}(s) = \lambda(s|_{d(\mathrm{B})})$$

we have $\bar{\lambda}(s) \leqslant \mu(s)$ from which we deduce that $\bar{\lambda}$ is an H-integral on $\mathscr{S}$. Let λ' be the measure on X which represents $\bar{\lambda}$. We have for any, bounded function $p \in \mathscr{S}$,

$$\lambda'(p - \mathrm{B}p) = \lambda(p_\mathrm{B})$$

and therefore

$$\lambda'(1) = \lambda(1|_{d(\mathrm{B})}) = \sup_{\substack{p_\mathrm{B} \leqslant 1|_{d(\mathrm{B})} \\ p \text{ bounded}}} \lambda(p_\mathrm{B}) = \sup_{\substack{p \text{ bounded} \\ p_\mathrm{B} \leqslant 1|_{d(\mathrm{B})}}} \lambda'(p - \mathrm{B}_p) = \lambda'(\chi_{d(\mathrm{B})}).$$

Hence λ' is a measure on $d(\mathrm{B})$ which represents λ.

Corollary. — *For any, $s, t \in \mathscr{S}^{**}$ such that $s \geqslant t$ there exists uniquely $s' \in \hat{\mathscr{S}}_\mathrm{B}$ such that*

$$s|_{d(\mathrm{B})} = s' + \mathrm{B}t|_{d(\mathrm{B})}$$

Particularly there exists $s_\mathrm{B} \in \hat{\mathscr{S}}_\mathrm{B}$ such that

$$s|_{d(\mathrm{B})} = s_\mathrm{B} + \mathrm{B}s|_{d(\mathrm{B})}$$

If $\mathrm{B}t$ is finite we have

$$s_n = : (s - \mathrm{B}t) \wedge n(p - \mathrm{B}p)|_{d(\mathrm{B})} \in \hat{\mathscr{S}}_\mathrm{B}$$

and $s|_{d(\mathrm{B})} = \sup_n s_n + \mathrm{B}t|_{d(\mathrm{B})}$.

Theorem 3.7. — *Let A be a subset of X and x be such that $x \notin \mathrm{A}$. Then for any $s \in \mathscr{S}$ we have*

$$\mathrm{B}^\mathrm{A}s(x) = \mathrm{R}^\mathrm{A}s(x).$$

Using Theorem 3.4. (*e*) and the metrisability of X it is sufficient to prove the theorem for the case where s is bounded and $x \notin \bar{\mathrm{A}}$ (natural closure). Let G be an open set such that $\mathrm{A} \subset \mathrm{G}$, $x \notin \overline{\mathrm{G}}$ and denote $\mathrm{B} = \mathrm{B}^\mathrm{G}$. Then $x \in d(\mathrm{B})$ and we have $\mathrm{B}^\mathrm{A}s = \bigwedge_{s' \geqslant s \text{ on A}} \mathrm{B}(s')$.

From the preceding corollary the set of functions

$$\{Bs'|_{d(B)} : s' \in \mathcal{S}, s' \geqslant s \text{ on } A\}$$

is specifically decreasing in $\hat{\mathcal{S}}_B$ and therefore its infimum in $\hat{\mathcal{S}}_B$ coincides with the pointwise infimum (use proposition 3.1.). Hence $B^A s(x) = \inf_{s' \geqslant s \text{ on } A} Bs'(x) = R^A s(x)$.

A subset A of X is called thin at a point $x \in X$ if there exists $s \in \mathcal{S}$ such that

$$B_s^A(x) < s(x).$$

PROPOSITION 3.8. — *We have*

a) A subset A *is thin at a point* $x \in X$ *iff there exists a natural neighbourhood* V *of* x *such that*

$$B_1^{A \cap V}(x) < 1$$

b) If $x \notin A$ *then* A *is thin at* x *iff* $X \setminus A$ *is a fine neighbourhood of* x *or iff there exists a natural open set* $G, G \supset A$, $x \notin G$ *such that* G *is thin at* x.

c) If A_1, A_2 *are two sets each of which is thin at a point* x, *then* $A_1 \cup A_2$ *is also thin at* x.

For the proof see [8].

Remark. — If X is complete metrisable (for the natural topology) then the Baire property holds for the fine topology.

For any subset A of X we denote

$$b(A) = \{x \in X | A \text{ is not thin at } x\}$$

The set $b(A)$ is called the *base of* A. A subset A of X is called *basic* if $A = b(A)$.

THEOREM 3.9. — *Let* A *be a subset of* X. *Then we have*

a) if s *is a finite generator of* $\mathcal{S}$ *then*

$$b(A) = \{x \in X | B^A s(x) = s(x)\}$$

b) the set $b(A)$ *is finely-closed and naturally of type* G_δ;

c) if $A \subset b(A)$ *then* B^A *is a balayage on* $\mathcal{S}$;

d) for any balayage B *on* $\mathcal{S}$ *the set* $b(B)$ *is the unique basic set* A *for which* $B = B^A$.

Section IV.

Negligible sets and axiom D.

A set $A \subset X$ is called totally thin if it is thin at any point of X (i.e. $b(A) = \emptyset$). A countable union of totally thin sets is called *semipolar.*

Theorem 4.1. — *a) (Doob-Bauer). For any set* $(s_i)_{i \in I}$ *in* $\mathcal{S}$, *the set*

$$\left\{ x \in X \,|\, \inf_{i \in I} s_i(x) \neq \left(\bigwedge_{i \in I} s_i \right)(x) \right\}$$

is semipolar.

b) For any $A \subset X$ *the set* $A \setminus b(A)$ *is semipolar.*

c) For any finely closed set $A \subset X$ *there exists a greatest basic set* A_0, $A_0 \subset A$; *moreover* $A \setminus A_0$ *is semipolar.*

Theorem 4.2. — *Any semipolar subset* A *of* X *is negligible with respect to any universally continuous* H-*integral on* $\mathcal{S}$.

We may restrict ourselves to the case where A is totally thin and Borel-measurable. If we denote by μ' the restriction of μ to A and by p a generator element of $\mathcal{S}$ we have, using proposition 2.7, and theorem 2.10,

$$\mu'(p) \leqslant \inf \{\mu'(s) | s \geqslant p \text{ on } A\}$$
$$= \mu'(\wedge \{s | s \geqslant p \text{ on } A\}) = \mu'(B^A(p)).$$

Since $B^A p < p$ we deduce that $\mu' = 0$.

Proposition 4.3. — *Let* B *be a balayage on* $\mathcal{S}$ *and let* A *be a subset of* $d(B)$. *If we denote by* $\hat{B}^A$ *the balayage on* A *with respect to the cone* $\hat{\mathcal{S}}_B$ *then we have:*

$$B^{b(B) \cup A}(s) = B(s) + \hat{B}^A(s_B) \quad \text{on} \quad d(B)$$

for any $s \in \mathcal{S}$.

We may assume s finite. For any $t \in \mathcal{S}$, $t \geqslant s$ on $b(\mathrm{B}) \cup \mathrm{A}$ we have $t \geqslant \mathrm{B}s$ and therefore

$$(t - \mathrm{B}s)|_{d(\mathrm{B})} \in \hat{\mathcal{S}}_{\mathrm{B}}$$

(corollary of theorem 3.6). Since

$$(t - \mathrm{B}s)|_{d(\mathrm{B})} \geqslant s_{\mathrm{B}},$$
$$t \geqslant \mathrm{B}s + \hat{\mathrm{B}}^{\mathrm{A}}(s_{\mathrm{B}}) \quad \text{on} \quad d(\mathrm{B}).$$

Let now $u \in \mathcal{S}$ be such that $u_{\mathrm{B}} \geqslant s_{\mathrm{B}}$ on A and such that $u_{\mathrm{B}} \leqslant a_{\mathrm{B}}$. It follows that

$$u - \mathrm{B}u + \mathrm{B}s \leqslant s \quad \text{on} \quad \mathrm{X}$$
$$u - \mathrm{B}u + \mathrm{B}s \geqslant s \quad \text{on} \quad b(\mathrm{B}) \cup \mathrm{A}.$$

Since $u - \mathrm{B}u + \mathrm{B}s \in \mathcal{S}$ (proposition 1.4) we have

$$u - \mathrm{B}u + \mathrm{B}s \geqslant \mathrm{B}^{b(\mathrm{B}) \cup \mathrm{A}}(s)$$

and therefore

$$\hat{\mathrm{B}}^{\mathrm{A}}(s_{\mathrm{B}}) + \mathrm{B}s = \mathrm{B}^{b(\mathrm{B}) \cup \mathrm{A}}(s) \quad \text{on} \quad d(\mathrm{B}).$$

COROLLARY. — *With the notations from the proposition we have:*

a) for any $x \in d(\mathrm{B})$, A is thin at x with respect to $\mathcal{S}$ iff it is thin at x with respect to $\hat{\mathcal{S}}_{\mathrm{B}}$.

b) A *is semipolar with respect to $\mathcal{S}$ iff it is semipolar with respect to $\hat{\mathcal{S}}_{\mathrm{B}}$.*

For any balayage B on a standard H-cone we denote by B' the smallest balayage B_1 for which

$$\mathrm{B} \vee \mathrm{B}_1 \equiv \mathrm{I}$$

Since any standard H-cone is isomorphic with a standard H-cone of functions it is sufficient to show the existence of B' for the cone $\mathcal{S}$. We have

$$\mathrm{B}_1 \leqslant \mathrm{B}_2 \Longleftrightarrow b(\mathrm{B}_1) \subset b(\mathrm{B}_2)$$

and

$$b(\mathrm{B}_1 \vee \mathrm{B}_2) = b(\mathrm{B}_1) \cup b(\mathrm{B}_2)$$

for any two balayages B_1, B_2. Now it is easy to see that the balayage $\mathrm{B}^{d(\mathrm{B})}$ satisfies the required conditions of B'. Using the relations

$$\mathrm{B}_1 \leqslant \mathrm{B}_2 \Longleftrightarrow \mathrm{B}_1^* \leqslant \mathrm{B}_2^*$$

and

$$B_1^{**} = B_1 \quad \text{on} \quad \mathcal{S}$$

for any two balayages B_1, B_2 it follows that

$$(B^*)' = (B')^*$$

for any balayage B.

We shall say that a standard H-cone C satisfies *axiom* D if for any balayage B on C we have

$$BB' = B'B.$$

From the above considerations we obtain the following.

THEOREM 4.4. — *A standard* H-*cone* C *satisfies axiom* D *if and only if its dual* C^* *satisfies it.*

The following theorem shows that the above axiom D coincides, in the case of harmonic spaces, with the axiom of domination introduced by Brelot ([7], [8]).

THEOREM 4.5. — *The following assertions are equivalent:*

a) $\mathcal{S}$ *satisfies axiom* D.

b) For any balayage B *and any* $x \in d(B)$ *the measure* $B^*(\varepsilon_x)$ *charges only the fine boundary of* $d(B)$.

c) For any balayage B, *any* $s \in \hat{\mathcal{S}}_B$ *and any* $t \in \mathcal{S}$ *such that*

$$\text{fine} \liminf_{d(B) \ni y \to x} s(y) \geq t(x)$$

for any point x *from the fine boundary of* $d(B)$, *the function* s' *on* X *equal to* t *on* $X \setminus d(B)$ *and equal to* $\inf(s, t)$ *on* $d(B)$ *belongs to* $\mathcal{S}$.

d) For any balayage B *and* $s \in \hat{\mathcal{S}}_B$ *and any* $t \in \mathcal{S}$ *such that*

$$\text{fine} \liminf_{d(B) \ni y \to x} s(y) \geq t(x)$$

for any point x *from the fine boundary of* $d(S)$, *we have*

$$Bt \leq s \quad \text{on} \quad d(B).$$

Proof. — $a \Rightarrow b)$. From $d(B) \subset b(B')$ we have

$$x \in d(B) \Rightarrow Bp(x) = B'(Bp)(x)$$

for any finite continuous generator p of $\mathcal{S}$. Since

$$BB' = B'B$$

it follows

$$x \in d(B) \Longrightarrow B(p - B'p)(x) = 0,$$
$$x \in d(B) \Longrightarrow B^*(\varepsilon_x)(p - B'p) = 0$$

and therefore $B^*(\varepsilon_x)$ does not charges $d(B')$ which is equal to the fine interior of $b(B)$.

$b) \Longrightarrow c)$. We may assume t finite. For any real number $\alpha > 0$ denote

$$A_\alpha = : \{x \in d(B) | s(x) + \alpha > t(x)\}$$

and by B_α the balayage $B^{A_\alpha \cup b(B)}$. Obviously A_α and $A_\alpha \cup b(B)$ are finely open sets. From proposition 4.3 it follows that

$$x \in d(B) \Longrightarrow B_\alpha^*(\varepsilon_x)|_{d(B)} = (\hat{B}^{A_\alpha})^*(\varepsilon_x).$$

Hence from the assertion 2) we have

$$x \in d(B) \Longrightarrow B_\alpha^*(\varepsilon_x) = (\hat{B}^{A_\alpha})^*(\varepsilon_x).$$

Further we denote by s_α the function on X equal to t on $X \setminus d(B)$ and equal to $\inf(s + \alpha, t)$ on $d(B)$ and by f_α the function on $d(B_\alpha)$ equal to $s_\alpha - B_\alpha t$. Since, from the above considerations,

$$B_\alpha t = \hat{B}^{A_\alpha}(t|_{d(B)}) \quad \text{on} \quad d(B)$$

we deduce, using the corollary to the theorem [3.6], that $f_\alpha \in \hat{\mathcal{S}}_{B_\alpha}$. Hence there exists a family $(p_i)_{i \in I}$ in $\mathcal{S}$ such that p_i is finite and $(p_i - B_\alpha p_i)_{i \in I}$ is an increasing net to f_α. Since

$$i \in I \Longrightarrow p_i - B_\alpha p_i \leqslant t - B_\alpha t$$

we have

$$i \in I \Longrightarrow p_i - B_\alpha p_i + B_\alpha t \in \mathcal{S} \quad \text{(proposition 1.4)}$$

Hence

$$s_\alpha = \sup_i (p_i - B_\alpha p_i + B_\alpha t) \in \mathcal{S}$$

The assertion $c)$ follows from the immediate relation

$$s' = \inf_\alpha s_\alpha.$$

$d) \Longrightarrow a)$ If B is a balayage and s, t are two elements of $\mathcal{S}$

such that $s = t$ on the fine boundary of $b(\mathrm{B})$ then $\mathrm{B}s = \mathrm{B}t$ on $d(\mathrm{B})$. Now the assertion $a)$ is an immediate consequence of this remark.

Corollary 1. — *Suppose that* $\mathcal{S}$ *satisfies axiom* D *and let* B *be a balayage. Then for any* $s \in \hat{\mathcal{S}}_{\mathrm{B}}$ *and for any* $t \in \mathcal{S}$ *such that* $s \geqslant \mathrm{B}t$ *on* $d(\mathrm{B})$ *we have*

$$s \succcurlyeq \mathrm{B}t|_{d(\mathrm{B})}$$

We may assume that t is finite. Let p be a finite continuous generator of $\mathcal{S}$. It is sufficient to show that for any $\varepsilon > 0$ the element

$$\inf(\varepsilon p + s - \mathrm{B}t,\ p - \mathrm{B}p)$$

belongs to $\hat{\mathcal{S}}_{\mathrm{B}}$. We have

$$\inf(\varepsilon p + s - \mathrm{B}t,\ p - \mathrm{B}p) = u_\varepsilon - \mathrm{B}v \geqslant 0$$

where

$$u_\varepsilon = (\varepsilon p + s + \mathrm{B}p) \wedge (p + \mathrm{B}t)$$
$$v = p + t.$$

If we denote

$$\mathrm{A}_\varepsilon = \{x \in d(\mathrm{B}) | (\varepsilon p + s + \mathrm{B}p)(x) > (p + \mathrm{B}t)(x)\}$$

we remark that the sets A_ε, $\mathrm{A}_\varepsilon \cup b(\mathrm{B})$ are finely open.

We have

$$u_\varepsilon(x) = (p + \mathrm{B}t)(x) \quad \text{on } \mathrm{A}_\varepsilon$$
$$u_\varepsilon(x) = (\varepsilon p + s + \mathrm{B}p)(x) \leqslant (p + \mathrm{B}t)(x) \quad \text{on } d(\mathrm{B}) \setminus \mathrm{A}_\alpha.$$

Let us denote by $\overline{u}_\varepsilon$ the function on X equal to $p + \mathrm{B}t$ on $b(\mathrm{B}^{\mathrm{A}_\varepsilon \cup b(\mathrm{B})})$ and equal to $\varepsilon p + s + \mathrm{B}p$ on $d(\mathrm{B}^{\mathrm{A}_\varepsilon \cup b(\mathrm{B})})$. We have fine $\liminf\limits_{d(\mathrm{B}^{\mathrm{A}_\varepsilon \cup b(\mathrm{B})}) \ni y \to x} (\varepsilon p + s + \mathrm{B}p)(y) \geqslant (p + \mathrm{B}t)(x)$ at any point x of the fine boundary of $d(\mathrm{B}^{\mathrm{A}_\varepsilon \cup b(\mathrm{B})})$. From this fact we deduce, using assertion $a \Leftrightarrow c$ from the preceeding theorem, that $\overline{u}_\varepsilon \in \mathcal{S}$. We have

$$u_\varepsilon - \mathrm{B}v = \overline{u}_\varepsilon - \mathrm{B}v = \overline{u}_\varepsilon - \mathrm{B}(\overline{u}_\varepsilon)$$

and therefore $u_\varepsilon - \mathrm{B}v \in \hat{\mathcal{S}}_{\mathrm{B}}$.

Corollary 2. — *Assume that* $\mathscr{S}$ *satisfies axiom* D. *Then for any balayage* B, $\hat{\mathscr{S}}_B$ *satisfies also axiom* D.

The assertion follows from theorem 4.5 ($(a) \Longleftrightarrow (b)$) using proposition 4.3.

Proposition 4.6. — *If* $\mathscr{S}$ *satisfies axiom* D, *then for any subset* A, B^A *is a balayage on* $\mathscr{S}$.

Let B be a balayage on $\mathscr{S}$ and $x \in d(B)$. We show first that for any decreasing net $(s_i)_{i\in I}$ of finite elements of $\mathscr{S}$ we have

$$\inf_{i\in I} (Bs_i)(x) = B(s)(x)$$

where

$$s = \bigwedge_{i\in I} s_i.$$

Denote

$$h = \bigwedge_{i\in I} (Bs_i)|_{d(B)}.$$

Since $((Bs_i)|_{d(B)})_{i\in I}$ is a specifically decreasing net in $\hat{\mathscr{S}}_B$ we have

$$h = \inf_{i\in I} Bs_i \quad \text{on} \quad d(B)$$

and there exists a sequence $(i_n)_{n\in I}$ in I such that the sequence $(Bs_{i_n})_n$ is decreasing to h on $d(B)$ (use the metrisability of X and Choquet lemma).

For any $\varepsilon > 0$ take

$$G_\varepsilon = \{y | Bs(y) + \varepsilon p(y) > h(y)\} \cup b(B).$$

Since $Bs \leqslant h \leqslant s$ on $d(B)$ it follows that G_ε is a fine neighbourhood of $b(B)$. We have

$$n \in N \Longrightarrow B^{G_\varepsilon}(Bs_{i_n}) = Bs_{i_n}.$$

Hence

$$x \notin G_\varepsilon \Longrightarrow (B^{G_\varepsilon})^*(\varepsilon_x)(h) = \inf_n (B^{G_\varepsilon})^*(\varepsilon_x)(Bs_{i_n})$$
$$= \inf_n Bs_{i_n}(x) = h(x) \leqslant (Bs + \varepsilon p)(x).$$

Letting ε tend to zero we get

$$Bs(x) = \inf_{i\in I} Bs_i(x).$$

Let now A be a subset of X and G be a fine neighbourhood of A. Using the above result we get

$$x \notin G \Longrightarrow B^G(B^A p) = B^A p$$

for any finite element $p \in \mathcal{S}$. The set G being arbitrary we get

$$B^A(B^A p) = B^A p$$

for p of $\mathcal{S}$. Hence B is a balayage.

We say that a standard H-cone C satisfies the *axiom of polarity* is for any decreasing net $(B_i)_{i \in I}$ of balayages on C and for any universally continuous element p we have

$$k \in I \Longrightarrow B_k\left(\bigwedge_{i \in I}(B_i p)\right) = \bigwedge_{i \in I}(B_i p).$$

THEOREM 4.7. — *The following assertions are equivalent;*

a) $\mathcal{S}$ *satisfies axiom of polarity;*

a′) for any decreasing net of balayages $(B_i)_{i \in I}$ *on* C *such that* $\bigwedge_{i \in I} B_i = 0$ *we have* $B_i\left(\bigwedge_{k \in I} B_k p\right) = \bigwedge_{k \in I} B_k p$ *for any* $i \in I$ *and any* $p \in \mathcal{S}_0$;

b) any semipolar subset of X *is polar;*

c) for any two finite measures μ, ν *on* X *such that* $\mu \leqslant \nu$ *and for any semipolar set* A *such that* $\nu(A) = 0$ *we have* $\mu(A) = 0$;

c′) for any point $x \in X$ *and any measure* μ *on* X *such that* $\mu \leqslant \varepsilon_x$ *we have* $\mu(A) = 0$ *for any semipolar set* A *for which* $x \notin A$;

d) for any subset A *of* X, B^A *is a balayage on* $\mathcal{S}$.

Proof. — *a)* $\Longrightarrow$ *d)* Let A be a subset of X and denote $\mathcal{B}_A$ the set of balayages B on $\mathcal{S}$ such that

$$A \subset b(B).$$

Obviously

$$s \in \mathcal{S} \Longrightarrow \bigwedge_{B \in \mathcal{B}_A}(Bs) = B^A s.$$

Hence

$$B \in \mathcal{B}_A \Longrightarrow B(B^A p) = B^A p,$$
$$B^A(B^A p) = B^A p$$

for any universally continuous element p of $\mathcal{S}$.

d) $\Longrightarrow$ *b)* follows immediately.

b) $\Longrightarrow$ *c)* Let A be a semiplan subset of X such that $\nu(A) = 0$. Since A is polar there exists an element $s \in \mathcal{S}^{**}$

such that $s = +\infty$ on A and

$$\nu(s) < \infty.$$

Obviously

$$\mu(\chi_A) \leqslant \inf_n \mu\left(\frac{1}{n} s\right) \leqslant \inf_n \nu\left(\frac{1}{n} s\right) = 0$$

$c')\Longrightarrow a)$ Let $(B_i)_{i\in I}$ be a decreasing net of balayages on $\mathcal{S}$, p a universally continuous element of $\mathcal{S}$ and denote

$$s = \bigwedge_{i\in I} B_i p, \quad A = \{y \mid \inf_{i\in I} (B_i p)(y) > s(y)\}.$$

Using Choquet's lemma we may assume that I is countable.

Further let $i \in I$ and $x \in d(B_i)$. Since

$$B_i^*(\varepsilon_x) \leqslant \varepsilon_x$$

we have

$$B_i^*(\varepsilon_x)(A) = B_i^*(\varepsilon_x)(A \cap b(B_i)) = 0,$$

Hence

$$B_i^*(\varepsilon_x)(s) = B_i^*(\varepsilon_x)\left(\inf_{k\in I} B_k p\right) = \inf_{k>i} B_i^*(\varepsilon_x)(B_k p)$$
$$= \inf_{k\in I} B_k(p)(x) = \bigwedge_{k\in I} B_k p(x) = s(x).$$

$a) \Longrightarrow a')$ is obvious.

$a') \Longrightarrow b)$ as in the proof $a) \Longrightarrow d)$ we may show that B^A is a balayage

$$B^A(B^A p) = B^A p$$

for any totally thin subset of X and therefore:

$$B^A(p - B^A p) = 0, \quad B^A = 0$$

Corollary. — *If axiom* D *holds for* C, *then axiom of polarity holds on* C.

Theorem 4.8. — *Let* C *be a standard* H-*cone. Then the following assertions are equivalent.*

a) For any decreasing net $(B_i)_{i\in I}$ *of balayages on* C *such that* $\bigwedge_{i\in I} B_i = 0$ *we have*

$$\bigwedge_{i\in I} B_i(p) = 0$$

for any universally continuous element $p \in C$.

b) For any decreasing net $(B_i)_{i\in I}$ *of balayages on* C *we have*

$$\Big(\bigwedge_{i\in I} B_i\Big)(p) = \bigwedge_{i\in I}(B_i(p))$$

for any universally continuous element $p \in C$.

c) C *and* C* *satisfy the axiom of polarity.*

We show first that the assertion *a)* holds simultaneously for C and C*.

We may assume $C = C^{**}$ and let $(B_i)_{i\in I}$ be a decreasing net of balayages on C* such that $\bigwedge_{i\in I} B_i = 0$ and assume *a)* true for C. It follows that the family $(B_i^*)_{i\in I}$ is a decreasing net of balayages on C such that $\bigwedge_{i\in I} B_i = 0$. Let now p (resp. μ) be a universally continuous element from C (resp. C*). We have

$$\wedge B_i^*(p) = 0,$$

$$0 = \mu\Big(\bigwedge_{i\in I} B_i^*(p)\Big) = \inf_{i\in I} \mu(B_i^*(p))$$
$$= \inf_{i\in I}\,[B_i(\mu)](p) = \Big[\bigwedge_{i\in I} B_i(\mu)\Big](p)$$

and therefore, p being arbitrary,

$$\bigwedge_{i\in I}(B_i(\mu)) = 0$$

Now *a)* $\Longrightarrow$ *c)* follows immediately.

c) $\Longrightarrow$ *b)* Let $(B_i)_{i\in I}$ be a decreasing net of balayages on C and let p be a universally continuous element of C.

Denote

$$B = \bigwedge_{i\in I} B_i$$
$$s = \bigwedge_{i\in I}(B_i p).$$

Since C satisfies the axiom of polarity we have

$$i \in I \Longrightarrow B_i s = s.$$

Hence if we assume C* represented as a standard H-cone of functions on a set Y we may consider s as a measure on Y carried by $b(B_i^*)$ for any $i \in I$, and therefore by $\bigcap_{i\in I} b(B_i^*)$ (passing eventually to a countable subset of I).

Since C^* satisfies also the axiom of polarity, and

$$\bigcap_{i \in I} b(B_i^*) - b(B^*)$$

is semipolar it follows that s is carried by $b(B^*)$, using the fact that $s \leqslant p$ and theorem 4.7.

Hence

$$Bs = s$$

and therefore

$$B(s) \leqslant Bp \leqslant s = B(s),$$
$$Bp = s.$$

$b) \Longrightarrow a)$ is immediate.

Let C be a finely open subset of X and denote by B the greatest balayage on $\mathcal{S}$ dominated by $B^{(X \setminus G)}$. It is known that the set $b(X \setminus G) \subset X \setminus G$ and

$$(X \setminus G) \setminus b(X - G) \quad \text{is a semipolar set.}$$

(see theorem 4.1.(*c*)). Hence $d(B) \setminus G$ is a semipolar set. We shall denote by $\mathcal{S}(G)$ the set of restrictions to G of elements of $\hat{\mathcal{S}}_B$. It is easy to see that $\mathcal{S}(G)$ is an H-cone of functions on G. Although it is a standard H-cone, being isomorphic with $\hat{\mathcal{S}}_B$, we do not know in general whether it is a standard cone of functions.

This is true if axiom D is fulfiled (use theorem 4.7 and corollary 2 of Theorem 4.5).

THEOREM 4.9. — *The map*

$$G \to \mathcal{S}(G)$$

is a sheaf on X endowed with the fine topology if and only if axiom D holds.

For the only « if » part of the theorem let B be a balayage on $\mathcal{S}$ and denote G the fine interior of $b(B)$ and A the complement of G. Obviously A is the fine closure of $d(B)$ thus it is a basic set. Further let p be a finite generator of $\mathcal{S}$ and denote $p' = B^A Bp$. We want to show that $Bp' = p'$.

Indeed for any $\varepsilon > 0$ denote

$$G_\varepsilon = \{x \in X | Bp'(x) + \varepsilon > p(x)\}$$

Since $p' = p$ on $A \setminus d(B)$, G_ε is a neighbourhood of this set. From the sheaf property the function s on X defined by

$$s(x) = \begin{cases} p(x) & x \in G \\ \inf(Bp(x), Bp'(x) + \varepsilon) & x \in G_\varepsilon \cup d(B) \end{cases}$$

belongs to $\mathcal{S}$, and $s \geqslant Bp \geqslant p'$, $Bp' + \varepsilon \geqslant p'$.

Since ε is arbitrary we get $Bp' \geqslant p'$.

Now for any $x \in d(B)$, $B^*_{\varepsilon_x}(p - p') = 0$ and since

$$G = \{x' | p(x') > p'(x')\} \text{ we have } B^*_{\varepsilon_x}(\chi_G) = 0 \text{ i.e.}$$

the measure $B^*_{\varepsilon_x}$ charges only the fine boundary of $b(B)$.

Assume now that axiom D holds. We show only that if $(G_i)_{i \in I}$ is a family of fine open sets and s is a function on $\bigcup_{i \in I} G_i$ such that $s|_{G_i} \in \mathcal{S}(G_i)$ then $s \in \mathcal{S}\left(\bigcup_{i \in I} G_i\right)$. We may assume $X = \bigcup_{i \in I} G_i$ and also that for any G_i there exists a balayage B_i for which $G_i = d(B_i)$. Using Doob's procedure [8] we may assume I is countable.

Let p be a universally continuous element of $\mathcal{S}$. We shall show that the function $\inf(s, p)$ belongs to $\mathcal{S}$. Assume first that $t \in \mathcal{S}$ is such that $\inf(s + t, p) \in \mathcal{S}$. Then using theorem 4.5 ($a) \Longrightarrow c)$) we deduce that for any $i \in I$,

$$\inf(s + B_i t,\ p) \in \mathcal{S}.$$

Hence by a simple induction for any finite sequence $(i_k)_{1 \leqslant k \leqslant n}$ we have $\inf(s + B_{i_1} B_{i_2}, \ldots, B_{i_n} p,\ p) \in \mathcal{S}$. Denote by $\mathcal{F}$ the set of elements of $\mathcal{S}$ of the form $B_{i_1}, B_{i_2}, \ldots, B_{i_n}(p)$ where $(i_k)_{1 \leqslant k \leqslant n}$ is any finite sequence in I. The proof is complete if we show that $u = \bigwedge \mathcal{F} = 0$.

If we represent $\mathcal{S}^*$ as a standard cone of functions on a set Y then the element u of $\mathcal{S}$ may be represented as a measure on Y, being dominated by the universally continuous element p.

As in the proof of proposition 4.6 we see that for any $i \in I$, we have $B_i u = u$, hence the measure u on Y is carried by $b(B_i^*)$ for any $i \in I$. Since $\bigcup_i G_i = X$ we deduce that $\bigcap_{i \in I} b(B_i^*)$ is a finaly closed subset of Y which contains no basic set. Hence it is semipolar and therefore polar (see proposition 4.6 and theorem 4.7). Using again theorem 4.7 we deduce $u = 0$.

Remark. — If in the example 1 the positive constant functions are superharmonic we get a standard H-cone of functions for which the natural topology is locally compact. Moreover if Brelot's convergence axiom and axiom D holds, then the cone $\mathscr{S}(G)$ (where G is a fine open set) introduced above, coincides with the set of positive finely superharmonic functions defined by B. Fuglede in [9].

Also in the example 1 with countable base, Brelot's axiom and axiom of domination, under the supplementary conditions of « proportionality » and existence of a base of completely determining sets (see R. M. Hervé [10]) one can see that the dual H-cone $\mathscr{S}^*$ may be represented as a standard H-cone of functions on the same space X (Hervé [10]). In these conditions one may deduce that the axiom of proportionality holds for the dual. Indeed if $x \in X$ is the carrier of $\mu \in \mathscr{S}^*$, then for any neighbourhood Vx, $(B^V)^*(\mu) = \mu$. Hence μ is representable, x is its support and $\mu = \alpha\varepsilon x$. We see that a relatively compact open set D is completely determining (resp. regular) iff the balayage $B^{X \setminus D}$ is a continuous map from $\mathscr{S}$ into $\mathscr{S}$ endowed with the natural topology $\sigma(\mathscr{S}, \mathscr{S}_0^*)$ (resp. maps the set $\mathscr{S}_0$ of universally continuous elements of $\mathscr{S}$ into itself).

For the further development of the theory it seems interesting to find out formulations and solutions in terms of an abstract H-cone C, of the following questions:

1. C may be represented as a standard H-cone of functions for which the natural topology is locally compact. Moreover the dual has the same property.

2. If C is represented as a standard H-cone of functions $\mathscr{S}$ on X then $G \to \mathscr{S}(G)$ is a sheaf for the natural topology.

3. There exists a set $(B_i)_i$ of balayages such that the set $d(B_i)$ forms a base for the natural topology and such that any B_i is continuous in $\sigma(C, C_0^*)$ (completely determining) resp. $B_i(C_0) \subset C_0$ (regular).

4. If $(\mathscr{S}, X)$ and $(\mathscr{F}, Y)$ are two H-cones of functions,

what does mean tensor product $\mathcal{S} \otimes \mathcal{F}$ on $X \times Y$ (certainly $(\mathcal{S} \otimes \mathcal{F})^* = \mathcal{S}^* \otimes \mathcal{F}^*$ is also understood)?

5. If we are given on C a bilinear form (u, v) such that $(u, u) + (v, v) \geqslant 2(u, v)$ and $u \to (u, v)$ belongs to C for any $v \in C$, try to develop a Dirichlet space theory for C, where (u, v) is the energy.

6. If C and C^* are represented as standard H-cones of functions on X and X^* respectively, then the map

$$A \to b((B^A)^*)$$

is a one-to-one correspondence between the basic set of X and X^* respectively. Find out conditions under which this correspondence is produced by a pointwise bijection φ between X and X^*. Moreover find out conditions under which the map φ is continuous (naturally or finely) or preserves thinness, polarity, semi-polarity.

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