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Theory of Bessel potentials. IV. Potentials on subcartesian spaces with singularities of polyhedral type

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THEORY OF BESSEL POTENTIALS. PART IV. POTENTIALS ON SUBCARTESIAN SPACES WITH SINGULARITIES OF POLYHEDRAL TYPE

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Introduction.

This is the fourth (and last) part of the series of papers on Bessel potentials [1], [2] and [3]. At first this part was intended to treat Bessel potentials on manifolds with singularities. It was then noticed that the notion of manifolds with singularities is best introduced by a more general notion of subcartesian spaces. The notion was actually developed with this motivation and led to results in different directions quite apart from applications to Bessel potentials. We should mention [11] and [12], [13] where the notion was introduced independently.

In section 1 we give a brief account of the theory of subcartesian spaces, their singularities and singularities of polyhedral type. The proofs are omitted to shorten the presentation, they will be given in a forthcoming paper [6] and they do not have an essential bearing on the main part of the paper. This section as well as the second put the main result of the paper in a proper perspective.

In section 2 we define local Bessel potentials on subcartesian spaces and prove some of their elementary properties. The general definition is not intrinsic since it is based on the possibility of local adequate extensions of the functions in the image spaces of charts (which define the subcartesian structure of the space). Our main aim in this paper is to give an intrinsic characterization of the local Bessel potentials; we were able to achieve it only in the case of spaces with singularities of polyhedral type. The main results were obtained already in the late 1960s but the publication was delayed by an attempt to obtain the results in the more general case of spaces with singularities of conical type (1); this attempt was not successful and we have an impression that in order to treat the conical singularities one would have to consider more general classes of functions, which would require an additional study analogous to one given for Bessel potentials in the preceeding parts.

Sections 3 and 4 are preliminary to section 5 where the main result is given.

In section 3 we reduce the main problem in the case of polyhedral singularities to the problem of extending a function given on a polyhedral set (2) in some $\mathbf{R}^n$ to a potential of given order on $\mathbf{R}^n$. We give the first incomplete definition of compatibility conditions which form necessary and sufficient conditions for possibility of such extension.

Section 4 gives the main tools for the proof of the main

(1) To describe it vaguely, singularities are of polyhedral type if the space is locally diffeomorphic to *a* (variable) polyhedron, they are of conical type if it is locally diffeomorphic to *a* (variable) rectilinear cone.

(2) Union of a finite number of geometrical polyhedra.

theorem in section 5, and the precise definition of the notion of abstract restriction. This definition allows us in section 5 to complete the description of compatibility conditions outlined in section 3 and to prove the main theorem asserting that the complete set of compatibility conditions is necessary and sufficient for existence of a desired extension.

In section 6 we illustrate the preceding results by a few examples. One of these refers to an application of our results which was used without proof in [7].

The Appendix at the end of the paper contains some remarks pertaining to the notion of abstract restriction and to the possibility of a generalization of the main theorem.

In the first draft of this paper the main result was proved by using an extension operator of the kind considered in [10]; the present version, however, seems to do more justice to the properties of different notions used in this work.

1. Subcartesian spaces.

We begin with a summary of facts and definitions concerning C^∞-subcartesian spaces which are relevant for the remainder of the paper. A complete account in a much more general setting will be given in [6]. The results and definitions as given here are adapted for the sake of expedience to the special C^∞ situation and may not be valid, as stated, in the more general setting.

It is convenient to consider the spaces $\mathbf{R}^n$, $n = 1, 2, \ldots$ as forming an increasing sequence $\mathbf{R}^1 \subset \mathbf{R}^2 \subset \cdots$ with canonical inclusions

$$(x_1, \ldots, x_n) \in \mathbf{R}^n \to (x_1, \ldots, x_n, 0, \ldots, 0) \in \mathbf{R}^m, m > n.$$

We denote by $C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ the class of C^∞ functions with open domains in $\mathbf{R}^n$ and values in $\mathbf{R}^m$ and by $C^\infty(\{\mathbf{R}^n\})$ the union $\bigcup_{n,k=1}^{\infty} C^\infty(\mathbf{R}^n, \mathbf{R}^k)$. Homeomorphisme in $C^\infty(\{\mathbf{R}^n\})$ with inverses in this class are referred to as diffeomorphisms in $\{\mathbf{R}^n\}$. For $f \in C^\infty(\mathbf{R}^n, \mathbf{R}^m)$ we sometimes write $n = n_f$, $m = m_f$.

Let X be a metrizable topological space; a *sub-*$\mathbf{R}$*-atlas* on X (of type C^∞) is a collection Φ of homeomorphisms $\varphi : U_\varphi \to \mathbf{R}^{n_\varphi}$ referred to as *charts* subject to the following conditions:

(1.1) $\{U_\varphi;\ \varphi \in \Phi\}$ *is an open cover of* X.

(1.2) For every $p \in X$ and any two charts $\varphi, \psi \in \Phi$ with $p \in U_\varphi \cap U_\psi$ there is an open neighborhood G of $\varphi(p)$ in $\mathbf{R}^n$, $n \geqslant \max(n_\varphi, n_\psi)$ and a diffeomorphism h in $\{\mathbf{R}^n\}$, such that

$$\mathscr{D}_h = G \quad \text{and} \quad h|_{G\cap\varphi(U_\varphi\cap U_\psi)} = \psi \circ \varphi^{-1}|_{G\cap\varphi(U_\varphi\cap U_\psi)}.$$

h in (1.2) is referred to as a *local extension* about $\varphi(p)$ of the *connecting homeomorphism* $\psi \circ \varphi^{-1}$.

A *subcartesian space* or a *sub-*$\mathbf{R}$ *space* is a metrizable space X with a sub-$\mathbf{R}$ atlas Φ. Φ is also said to *define* a sub-$\mathbf{R}$ structure on X.

Remark 1.1. — If X is a sub-$\mathbf{R}$ space with structure given by an atlas Φ and $X_1 \subset X$ is any subset, then the atlas $\Phi|_{X_1} = \{\varphi|_{X_1\cap U_\varphi};\ \varphi \in \Phi,\ X_1 \cap U_\varphi \neq \emptyset\}$ is a sub-$\mathbf{R}$ atlas on X_1 giving on X_1 the induced structure. Thus any subset of a sub-$\mathbf{R}$ space is a sub-$\mathbf{R}$ space with the induced structure. In the case $X = \mathbf{R}^n$ with the obvious atlas $\{\mathbf{R}^n$, identity$\}$ we conclude that any subset $X_1 \subset \mathbf{R}^n$ is a sub-$\mathbf{R}$ space with the canonical structure given by the inclusion $X_1 \subset \mathbf{R}^n$. We shall refer to this structure as the (canonical) *inclusion structure*.

Remark 1.2. — It is possible to define the notions of *equivalent atlasses, charts compatible with a given atlas* Φ and a *maximal atlas containing* Φ. This is done in the same way as in the theory of manifolds.

Remark 1.3. — If M is a C^∞-manifold with the manifold structure given by an atlas Φ, then Φ defines on M a sub-$\mathbf{R}$ structure. We shall say that a space X with a sub-$\mathbf{R}$ structure given by Φ is a manifold if there is an atlas on X compatible with Φ defining on X a manifold structure.

A function $f\colon \mathscr{D}_f \subset X \to \mathbf{R}^k$ is of class C^∞ (X being a sub-**R** space) if $\mathcal{D}_f$ is open in X and for every $p \in \mathcal{D}_f$ there is a chart φ in the maximal atlas giving the sub-**R** structure on X such that $p \in U_\varphi$ and $f \circ \varphi^{-1}$ can be extended to a function in $C^\infty\{\mathbf{R}^n\}$.

PROPOSITION 1.1. — *For every open cover $\mathcal{U}$ of a sub-**R** space X there exists a C^∞-partition of unity on X subordinate to $\mathcal{U}$; i.e. a family $\mathcal{G}$ of C^∞-functions on X such that a) $g(X) \subset [0, 1]$ for every $g \in \mathcal{G}$, b) $\{g^{-1}((0, 1])\}$ is a locally finite cover of X and for every $g \in \mathcal{G}$,* supp $g \subset U$ *for some $U \in \mathcal{U}$, c) $\sum_{g \in \mathcal{G}} g(p) = 1$ for every $p \in X$.*

If X is a sub-**R** space then for every $p \in X$ we define the *local dimension* of X at p as $\dim_p X = \min\{n_\varphi;\ \varphi \in \Phi,\ p \in U_\varphi\}$, where Φ is the maximal atlas defining the sub-**R** structure on X; a chart φ at $p \in X$ is tangential if $n_\varphi = \dim_p X$. The function $p \in X \to \dim_p X$ is upper semicontinuous and the set $\{p;\ \dim_q X = \dim_p X$ for all q in some neighborhood of $p\}$ is referred to as the *homogeneous part* of X; this is clearly an open dense subset of X. The complement of the homogeneous part is the *nonhomogeneous part of* X, the points in the homogeneous part of X are *points of homogeneity.*

A point $p \in X$ is a *regular point* of X if there is a neighborhood G of p such that G with the structure defined by $\{\varphi|_G,\ \varphi \in \Phi\}$ is a C^∞ manifold (see remark 1.3). Equivalent conditions are : there is a chart φ at p in the maximal atlas such that $\varphi(U_\varphi)$ is an open subset of $\mathbf{R}^{n_\varphi}$ or that there is a tangential chart φ at p defining on U_φ a manifold structure. The collection of all regular points is the *regular part* of X; it is an open, possibly empty, subset of the homogeneous part of X, and if nonempty, it is a union of disjoint open connected manifolds. The complement of the regular part is the *singular* part of X.

The following considerations allow one to define tangent space at each point of a sub-**R** space (and also the « tangent bundle » over such space).

Let $A \subset \mathbf{R}^n$ be an arbitrary set and $x \in A$. We define the (C^∞) *tangent space* to A at x by setting

$$\mathcal{T}_x A = \cap\ \{N_{Df(x)};\ f \in C^\infty(\mathbf{R}^n, \mathbf{R}^1),\ f|_{A \cap \mathcal{D}_f} = 0\},$$

where $Df(x)$ denotes the differential of f at x considered as linear function $Df(x): \mathbf{R}^n \to \mathbf{R}^1$, $N_{Df(x)}$ denotes its null space. $x + \mathscr{C}_x A$ is then the *tangent plane* to A at x. It can be shown that $\dim \mathscr{C}_x A = \dim_x A$; also if $P: \mathbf{R}^n \to \mathscr{C}_x$ is a projection, then for some neighborhood G of x in $\mathbf{R}^n$, $P|_{G \cap A}$ followed by a suitable linear isomorphism is a tangential chart about x. Also $\mathscr{C}_x A$ is independent of the choice of the space $\mathbf{R}^n$ containing A.

If X is a sub-**R** space with a maximal atlas Φ, $p \in X$, then we let for every $\varphi \in \Phi$ with $p \in U_\varphi$, $\mathscr{C}_{p,\varphi} = \mathscr{C}_{\varphi(p)}\varphi(U_\varphi)$- the tangent space in $\mathbf{R}^{n_\varphi}$ to $\varphi(U_\varphi)$ at $\varphi(p)$.

If $\varphi, \psi \in \Phi$, $p \in U_\varphi \cap U_\psi$ and h is a local extension of the connecting homeomorphism $\psi \circ \varphi^{-1}$ then $\xi \to Dh(\varphi(p))\xi$ is a linear isomorphism of $\mathscr{C}_{p,\varphi(p)}$ onto $\mathscr{C}_{p,\psi(p)}$ which is independent of the choice of the local extension h. We denote this isomorphism by $D(\psi \circ \varphi^{-1})(\varphi(p))$.

It is easy to verify that the relation $\xi \sim \eta$ provided that $D(\psi \circ \varphi^{-1})(\varphi(p))\xi = \eta$ is an equivalence relation in the disjoint union of the spaces $\mathscr{C}_{p,\varphi}$, $\varphi \in \Phi$, $p \in U_\varphi$. The space of cosets of this relation, provided with the natural vector space structure is then the *tangent space* $\mathscr{C}_p X$ to X at p. We refer to the space $\mathscr{C}_{p,\varphi}$ as the representative of $\mathscr{C}_p X$ in the chart φ.

Remark 1.4. — It can be shown, using the tangential chart indicated above, that in the condition (1.2) the local extension h of $\psi \circ \varphi^{-1}$ can be taken as a diffeomorphism in $\mathbf{R}^n$ with $n = \max(n_\varphi, n_\psi)$.

In this paper we are interested in sub-**R** spaces *with singularities of polyhedral type* or, to abbreviate, *spaces of polyhedral type*. The definition follows.

A polyhedral set in $\mathbf{R}^n$ is a set of the form

$$K = \cup\{S, S \in \mathscr{K}\},$$

where $\mathscr{K}$ is a simplicial complex in $\mathbf{R}^n$; i.e. a finite collection of simplices with the properties: *a*) if $S \in \mathscr{K}$ then all the faces of S belong to $\mathscr{K}$, *b*) for S and S_1 in $\mathscr{K}$ $S \cap S_1$ is either empty or is a common face of S and S_1.

A k-dimensional closed polyhedron P (briefly-closed polyhedron) in $\mathbf{R}^n$ is a polyhedral set of dimension k at every point, lying in a k-dimensional plane $L \subset R^n$ (the plane of P). A polyhedron is the difference of a closed polyhedron and a polyhedral set of smaller dimension. The interior and boundary of a polyhedron rel. to its plane are called the *inside* and the *border* of the polyhedron.

A (C^∞) sub-**R** space X is of polyhedral type if for every $p \in X$ there is a chart φ at p, in the maximal atlas Φ defining on X the sub-**R** structure such that

$$\varphi(U_\varphi) = Q_\varphi \cap K_\varphi$$

where K_φ is a polyhedral set in $\mathbf{R}^{n_\varphi}$ and Q_φ is an open cube in $\mathbf{R}^{n_\varphi}$ with center at $\varphi(p)$. Equivalently we could stipulate existence for each $p \in X$, of a chart $\varphi \in \Phi$ and an open set U, $p \in U \subset \overline{U} \subset U_\varphi$ such that $\varphi(\overline{U})$ is a polyhedral set.

Any polyhedral set in $\mathbf{R}^n$ is a sub-**R** space of polyhedral type (with the canonical inclusion structure). As other simple examples we could mention a lens obtained by intersection of two closed balls or union of two or more intersecting, nontangential spheres.

Let X be a sub-**R** space of polyhedral type and consider a component X_n of the homogeneous part of X, $\dim_p X = n$ for $p \in X_n$. Then for every $p \in X_n$ there is a neighborhood $V_p \subset X_n$ of p and a chart $\varphi \in \Phi$ with the following properties : *a*) $\overline{V}_p \subset U_\varphi \subset X_n$, *b*) $\varphi(\overline{V}_p) = K_\varphi$ is a polyhedral set in $\mathbf{R}^{n_\varphi}$. If $q \in U_\varphi$ and $\mathcal{C}_{q,\varphi}$ is the representative of $\mathcal{C}_q X$ in the chart φ then necessarily $\dim \mathcal{C}_{q,\varphi} = n$ and K_φ is a union of maximal closed polyhedra of dimension n with disjoint insides. The point p is then a regular point of X if and only if $\varphi(p)$ is an inner point of one of these polyhedra.

The preceding remark implies that the regular part of a subcartesian space X of polyhedral type is dense in the homogeneous part of X and therefore in X. We note that the regular part of X may be strictly included in the homogeneous part; e.g. if $X = \{(x, y) \in \mathbf{R}^2;\ |y| \leqslant |x|\}$ then X coincides with its homogeneous part, the regular part of X is $\{(x, y) \in \mathbf{R}^2;\ |y| < |x|\}$.

If X is of polyhedral type then it is easy to show that the singular part $X^{(1)}$ of X with the induced structure is also of polyhedral type and by the preceding remarks its regular part is an open and dense subset of $X^{(1)}$. We can thus write a sequence

$$X = X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \cdots \tag{1.3}$$

with the property that $X^{(i+1)}$ is the singular part of $X^{(i)}$. The sequence (1.3) is locally finite in the sense that for every point $p \in X$ there is a neighborhood U of p such that $U \supset X^{(1)} \cap U \supset X^{(2)} \cap U \supset \cdots$ is finite, and the last element of the latter sequence consists of finite number of points.

Since $X^{(i+1)}$ is a singular part of $X^{(i)}$, $X^{(i)} \setminus X^{(i+1)}$ is a union of disjoint connected manifolds, which is locally finite, i.e.

$$U \cap (X^{(i)} \setminus X^{(i+1)}) = \bigcup_{j=1}^{m_i} \mathfrak{M}_j^{(i)} \tag{1.4}$$

The following examples illustrate formulas (1.3) and (1.4).

Example 1. — $X = S_1 \cup S_2$ where S_1, S_2 are nontangential intersecting spheres in $\mathbf{R}^3$. In this case

$$X \setminus X^{(1)} = (S_1 \cup S_2) \setminus (S_1 \cap S_2)$$

is the union of 4 open components of $S_i \setminus (S_1 \cap S_2)$, $i = 1, 2$; $X^{(1)} = S_1 \cap S_2$ is a circle and the singular part of $X^{(1)}$ is empty.

Example 2. — $X = B_1 \cup B_2$ where B_1, B_2 are nontangential intersecting closed balls in $\mathbf{R}^3$. Then $X^{(1)}$ is the boundary $\partial(B_1 \cup B_2)$ of $B_1 \cup B_2$, the singular part $X^{(2)}$ of $X^{(1)}$ is the circle $\partial B_1 \cap \partial B_2$ and $X^{(3)}$ is empty.

Example 3. — $X = \Delta_1 \cup \Delta_2 \cup \Delta_3 \subset \mathbf{R}^3$ where Δ_3 is a closed tetrahedron, Δ_2 is a closed triangle with

$$\Delta_2 \cap \Delta_3 = \text{one-dimensional common face of } \Delta_2 \text{ and } \Delta_3,$$

and Δ_1 is a closed segment $[p_1, p_2]$ perpendicular to Δ_2, p_1 being an inner point of Δ_2 and $[p_1, p_2] \cap \Delta_3 = \emptyset$.

Then $X \setminus X^{(1)} = \Delta_1^{\text{ins}} \cup (\Delta_2^{\text{ins}} \setminus \{p_1\}) \cup \Delta_3^{\text{ins}}$, $X^{(1)} \setminus X^{(2)}$ is the union of $\{p_1\}$, $\{p_2\}$, the insides of 2-dimensional faces of Δ_3 and of the insides of edges of Δ_2 disjoint from Δ_3. $X^{(2)} \setminus X^{(3)}$ is the union of insides of edges of Δ_3 and the vertex of Δ_2 not in Δ_3. $X^{(3)}$ consists of vertices of Δ_3.

2. Local Bessel potentials on subcartesian spaces.

We denote by $P^{\beta}(\mathbf{R}^n)$ the space of Bessel potentials of order β on $\mathbf{R}^n$ and by $\mathfrak{A}^{(n)}_{2\beta}$ its exceptional class (see [1]). We introduce the notion of the Bessel potentials of reduced order α and of corresponding exceptional classes as follows

$$P^{\langle\alpha\rangle}(\mathbf{R}^n) = P^{\alpha+n/2}(\mathbf{R}^n), \qquad \alpha + \frac{n}{2} > 0,$$
$$\mathfrak{A}_{\langle\alpha\rangle} = \{A;\ A \in \mathfrak{A}^{(n)}_{2\alpha+n} \ \text{for some} \ n,\ 2\alpha + n > 0\}.$$

The basic theorem about restrictions and extensions of Bessel potentials can be stated as follows (see [1]).

THEOREM 2.1. — *Let* n, k *be integers,* α *be real and suppose* $n > k > -2\alpha$. *Then for every* $A \in \mathfrak{A}_{\langle\alpha\rangle}$, $A \subset \mathbf{R}^n$, *we have* $A \cap \mathbf{R}^k \in \mathfrak{A}_{\langle\alpha\rangle}$. *Also* $P^{\langle\alpha\rangle}(\mathbf{R}^n)|_{\mathbf{R}^k} = P^{\langle\alpha\rangle}(\mathbf{R}^k)$ *and there is a linear bounded operator of extension*

$$E_\alpha : P^{\langle\alpha\rangle}(\mathbf{R}^k) \to P^{\langle\alpha\rangle}(\mathbf{R}^n)$$

such that $E_\alpha u|_{\mathbf{R}^k} = u$ *for every* $u \in P^{\langle\alpha\rangle}(\mathbf{R}^k)$.

Similarly one could restate the theorem about pointwise differentiability of functions in $P^{\langle\alpha\rangle}$ in the form: if $u \in P^{\langle\alpha\rangle}(\mathbf{R}^n)$ and λ is an n-index such that

$$\alpha - |\lambda| + \frac{n}{2} > 0$$

then $D^\lambda u$ exists exc. $\mathfrak{A}_{\langle\alpha-|\lambda|\rangle}$ and belongs to $P^{\langle\alpha-|\lambda|\rangle}(\mathbf{R}^n)$.

Let X be a sub-$\mathbf{R}$ space with the structure given by an atlas Φ and α be real. We define the class $\mathfrak{A}_{\langle\alpha\rangle, X}$ as the collection of all sets $A \subset X$ with the property that for every $\varphi \in \Phi$, $\varphi(A \cap U_\varphi) \subset \mathfrak{A}_{\langle\alpha\rangle}$, the condition being void for $\alpha \leqslant -\frac{n_\varphi}{2}$. It follows from the known properties of the exceptional classes $\mathfrak{A}_{\langle\alpha\rangle}$ that if the condition above holds for some atlas Φ defining on X a C^∞ structure then it also holds for every atlas equivalent to Φ.

Theorem 2.1 allows us to define the space $P^{\langle\alpha\rangle}_{\mathrm{loc}}(X)$ as follows.

A function $u: X \to \mathbf{C}$ *defined* exc. $\mathfrak{A}_{\langle\alpha\rangle, X}$ *is in* $P^{\langle\alpha\rangle}_{\mathrm{loc}}(X)$ *iff for every* $p \in X$ *there is an open neighborhood* $U \subset X$ *of* p *and* $\varphi \in \Phi$, $U_\varphi \supset U$ *and a function* $u_\varphi \in P^{\langle\alpha\rangle}(\mathbf{R}^{n_\varphi})$ *such that* $u_\varphi|_{\varphi(U)} = u \circ \varphi^{-1}|_{\varphi(U)}$.

It is clear that if the condition is satisfied for some chart $\varphi \in \Phi$ at p then it is also satisfied for any chart at p compatible with Φ.

We shall list now some properties of spaces $P^{\langle\alpha\rangle}_{\mathrm{loc}}(X)$ which are direct consequences of the definition.

$P^{\langle\alpha\rangle}_{\mathrm{loc}}(X)$ is a saturated linear class rel. $\mathfrak{A}_{\langle\alpha\rangle, X}$. Theorem 2.1 and Remark 1.1 imply immediately the first statement of the following

Proposition 2.1. — *If* X *is a* C^∞*-sub-*$\mathbf{R}$ *space and* $X_1 \subset X$ *then for every* $u \in P^{\langle\alpha\rangle}_{\mathrm{loc}}(X)$ *the restriction* $u|_{X_1}$ *belongs to* $P^{\langle\alpha\rangle}_{\mathrm{loc}}(X_1)$. *Also if* $v \in P^{\langle\alpha\rangle}_{\mathrm{loc}}(X_1)$ *and* X_1 *is closed in* X *then there exists* $u \in P^{\langle\alpha\rangle}_{\mathrm{loc}}(X)$ *such that* $u|_{X_1} = v$.

Proof. (of the second statement). — If $v \in P^{\langle\alpha\rangle}_{\mathrm{loc}}(X_1)$ then by the definition there is for every $p \in X_1$ a chart φ_1, $p \in U_{\varphi_1}$, φ_1 in the maximal atlas containing $\Phi|_{X_1}$ such that $v \circ \varphi_1^{-1}$ can be extended to a function $\tilde{v}$ in $P^{\langle\alpha\rangle}(\mathbf{R}^{n_{\varphi_1}})$. There also is a chart $\varphi \in \Phi$ (assuming Φ maximal) such that

$$U_\varphi \cap X_1 \subset U_{\varphi_1}.$$

If h is a local extension of $\varphi \circ \varphi_1^{-1}$ about $\varphi_1(p)$ and $\tilde{v}_1$ is extension of $\tilde{v}$ in $P^{\langle\alpha\rangle}(\mathbf{R}^{n_h})$ (Theorem 2.1) then $\tilde{v}_1 \circ h^{-1}|_{\mathbf{R}^{n_\varphi}}$ is an extension of $v \circ \varphi^{-1}$ in $P^{\langle\alpha\rangle}(\mathbf{R}^{n_\varphi})$. Thus for every $p \in X_1$ there is a chart $\varphi \in \Phi$ with $p \in U_\varphi$ and a function $v_{p,\varphi} \in P^{\langle\alpha\rangle}(\mathbf{R}^{n_\varphi})$ with $v_{p,\varphi} \circ \varphi|_{U_\varphi \cap X_1} = v|_{U_\varphi \cap X_1}$ exc. $\mathfrak{A}_{\langle\alpha\rangle, X_1}$. Denote this chart by (φ^p, U_p) and let $v_{p,\varphi} = v_p$. Since X_1 is closed, for any $p \notin X_1$ we can find a chart $\varphi \in \Phi$ such that $p \in U_\varphi$ and $U_\varphi \cap X_1 = \emptyset$. Again denote this chart by φ^p, set $U_\varphi = U_p$ and let $v_p = 0$ on $\mathbf{R}^{n_\varphi}$. Let $\mathcal{U}$ be a locally finite open refinement of the cover $\{U_p\}_{p \in X}$ of X and assign to each

$U \in \mathcal{U}$ a chart $\varphi_U \in \Phi$ and a function $\nu_U \in P^{\langle\alpha\rangle}(\mathbf{R}^{n_U})$ by setting $\varphi_U = \varphi_p|_U$, $\nu_U = \nu_p$ for some p such that $U_p \supset U$. Let $\mathcal{G} = \{g\}$ be a C^∞ partition of unity subordinated to $\mathcal{U}$ (see Proposition 1.1) and assign to each g a chart $\varphi_g \in \Phi$ and a function $\nu_g \in P^{\langle\alpha\rangle}(\mathbf{R}^{n_g})$ by setting $\varphi_g = \varphi_U$, $\nu_g = \nu_U$ for some $U \in \mathcal{U}$ such that $\operatorname{supp} g \subset U$. Define

$$u = \sum_{g \in \mathcal{G}} g.(\nu_g \circ \varphi_g)$$

with understanding that $g.(\nu_g \circ \varphi_g) = 0$ whenever $g = 0$. We omit the straightforward verification that the function u defined above has the desired properties.

Proposition 2.2. — *Suppose that* $\varphi \in \Phi$, $V \subset \overline{V} \subset U_\varphi$ *is open in* X *and* $\overline{V}$ *is compact. Let* $U_{\varphi_1} = V$ *and* $\varphi_1 = \varphi|_V$. *Then for every* $u \in P^{\langle\alpha\rangle}_{\mathrm{loc}}(X)$ *the function* $u_1 = u \circ \varphi_1^{-1}$ *has an extension* $\tilde{u}_1$ *in* $P^{\langle\alpha\rangle}(\mathbf{R}^{n_\varphi})$.

Proof. — For every point $p \in \overline{V}$ there is a neighborhood V_p of p in U_φ such that the function $u^p = u \circ \varphi_p^{-1}$ has an extension $\tilde{u}^p$ in $P^{\langle\alpha\rangle}(\mathbf{R}^{n_\varphi})$, φ_p denoting the chart $\varphi_p = \varphi|_{V_p}$. Denote by G_p an open set in $\mathbf{R}^{n_\varphi}$ such that $V_p = \varphi_p^{-1}(G_p)$. Since $\varphi(\overline{V}) \subset \mathbf{R}^{n_\varphi}$ is compact, a finite number of the sets G_p covers $\varphi(\overline{V})$ and if h_p denotes a subordinate C^∞ partition of unity on $\varphi(\overline{V})$ then $\Sigma h_p \tilde{u}^p$ is the desired extension $\tilde{u}_1$.

It follows from Proposition 2.2 that if X is locally compact then $P^{\langle\alpha\rangle}_{\mathrm{loc}}(X)$ can be endowed with a locally convex topology given by the family of pseudonorms

$$(2.1) \qquad \|u\|_{\varphi,\langle\alpha\rangle} = \inf\{\|\tilde{u}^\varphi\|_{P^{\langle\alpha\rangle}(\mathbf{R}^{n_\varphi})};\ \tilde{u}^\varphi \in P^{\langle\alpha\rangle}(\mathbf{R}^{n_\varphi}),\ \tilde{u}^\varphi|_{\varphi(U_\varphi)} = u \circ \varphi^{-1}\},$$

where $\varphi \in \Phi$ are restricted by the following two requirements

a) $\overline{U}_\varphi$ is compact;

b) there is a chart $\tilde{\varphi} \in \Phi$ such that

$$\overline{U}_\varphi \subset U_{\tilde{\varphi}} \quad \text{and} \quad \tilde{\varphi}|_{U_\varphi} = \varphi.$$

It is understood that in order to make the above statement meaningful we should consider $P^{\langle\alpha\rangle}_{\mathrm{loc}}$ as a space of equivalence classes of functions equal exc. $\mathfrak{A}_{\langle\alpha\rangle, X}$ rather than a space of functions.

It is easy to check that if X is locally compact and σ-compact then $P_{loc}^{\langle\alpha\rangle}(X)$ with the above introduced topology is a Frechet functional space rel. $\mathfrak{A}_{\langle\alpha\rangle, X}$.

If X is compact then $P_{loc}^{\langle\alpha\rangle}(X)$ is a Banach functional space rel $\mathfrak{A}_{\langle\alpha\rangle, X}$ with a norm given by

$$\|u\|_{\langle\alpha\rangle, X, \Phi} = \Sigma\{\|u\|_{\langle\alpha\rangle, \varphi};\ \varphi \in \Phi\} \tag{2.2}$$

where Φ is a finite atlas defining the structure on X and satisfying *a*), *b*). Different such atlasses give rise to equivalent norms. The normed space $P_{loc}^{\langle\alpha\rangle}(X)$ will be referred to in this case as the space of (global) Bessel potentials on X and will be denoted by $P^{\langle\alpha\rangle}(X)$.

The definition of spaces $P_{loc}^{\langle\alpha\rangle}(X)$ given above is by its very nature extrinsic and it is of considerable interest to obtain an intrinsic characterization of functions in $P_{loc}^{\langle\alpha\rangle}(X)$, which would allow one to determine whether a given function is in $P_{loc}^{\langle\alpha\rangle}(X)$ from the properties of the function alone and without necessity of looking at its extensions. This is clearly a local problem which is equivalent to the following one.

Given a set $X \subset \mathbf{R}^n$, $\alpha > 0$, and a function $u : X \to \mathbf{C}$, find necessary and sufficient conditions in order that u be extendable to a function $\tilde{u} \in P^{\langle\alpha\rangle}(\mathbf{R}^n)$. The space of all functions u on $X \subset \mathbf{R}^n$ of the form $u = \tilde{u}|_X$, $\tilde{u} \in P^{\langle\alpha\rangle}(\mathbf{R}^n)$ is denoted by $P^{\langle\alpha\rangle}(X)$. Thus the above problem could be restated as that of finding an intrinsic characterization of functions in $P^{\langle\alpha\rangle}(X)$ for $X \subset \mathbf{R}^n$.

The above problem is difficult unless some additional restrictions are imposed on the set X.

Assume for instance that X is compact and is a union of a finite collection $\mathscr{K}$ of open disjoint manifolds of different dimensions, such that for any $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathscr{K}$, $\overline{\mathfrak{M}_1} \cap \overline{\mathfrak{M}_2}$ is contained in the intersection of the borders of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and is itself a union of manifolds in $\mathscr{K}$.

In this case existence of a function $\tilde{u}$ implies that

$$u_{\mathfrak{M}} = u|_{\mathfrak{M}} = \tilde{u}|_{\mathfrak{M}} \in P^{\langle\alpha\rangle}(\mathfrak{M})$$

(see [3]), also if $\mathfrak{M} \in \mathscr{K}$ is the intersection of the borders of $\mathfrak{M}_i$, then $u_{\mathfrak{M}_i}|_{\mathfrak{M}} = u_{\mathfrak{M}}$ (provided $\dim \mathfrak{M} > -2\alpha$) and all the nontangential derivatives of $u_{\mathfrak{M}_i}$ of any order m such that $2m - 2\alpha < \dim \mathfrak{M}$ satisfy on $\mathfrak{M}$ a system of linear homogeneous relations resulting from the fact that they are all expressible as linear combinations of the restrictions $D^m\tilde{u}|_{\mathfrak{M}}$. These relations are referred to as *compatibility conditions*; they are clearly necessary for existence of $\tilde{u}$.

The objective of this paper is to give a complete list of the compatibility conditions in the case when X is a polyhedral set and to prove that in this case the conditions are not only necessary but also sufficient for existence of an extension $\tilde{u}$. The result solves the problem of characterizing intrinsically $P^{\langle\alpha\rangle}_{\mathrm{loc}}(X)$ in the case when X is of polyhedral type.

The main result of the paper, described above, does not depend on the notion of subcartesian spaces; however, this notion puts the result and its implications in a proper perspective.

3. Compatibility conditions for polyhedral sets.

We proceed now to a description of the compatibility conditions satisfied by functions in $P^{\langle\alpha\rangle}(X)$ where $X \subset \mathbf{R}^n$ is a polyhedral set.

We recall first (see [2]) that if $D \subset \mathbf{R}^k$, is open, $\alpha > 0$, $\alpha = [\alpha] + \beta$ then $P^\alpha(D)$ denotes the space of restrictions to D of functions in $P^\alpha(\mathbf{R}^k)$ (with restriction norm) and $\check{P}^\alpha(D)$ denotes the space consisting of all functions u in $P^\alpha_{\mathrm{loc}}(D)$ with finite norm $|u|_{\alpha,D}$ given by

$$(3.1) \qquad |u|^2_{\alpha,D} = \sum_{|\lambda|\leqslant\alpha} (\|D^\lambda u\|^2_{L^2(D)} + d_{\beta,D}(D^\lambda u))$$

where

$$d_{\beta,D}(v) = \int_D\int_D |x-y|^{-n-2\beta}|v(x) - v(y)|^2\,dx\,dy \qquad \text{for} \qquad \beta > 0$$

and $d_{\beta,D}(v) = 0$ for $\beta = 0$. $d_{\beta,D}(u)$ is referred to as the Dirichlet integral of u of order β. The norm (3.1) is equivalent to the norm introduced in [2].

We have $\check{P}^\alpha(D) \supset P^\alpha(D)$ and if D is sufficiently regular (see [2]) then there exists a bounded linear extension operator from $\check{\mathrm{P}}^{\alpha}(\mathrm{D})$ to $\mathrm{P}^{\alpha}(\mathbf{R}^k)$; in particular in such a case we have

$$(3.2) \qquad \check{\mathrm{P}}^{\alpha}(\mathrm{D}) = \mathrm{P}^{\alpha}(\mathrm{D}).$$

(3.2) is valid for instance if D is the interior of a polyhedron (see 7), section 12, [2]).

Even though a function $u \in \mathrm{P}^{\alpha}(\mathrm{D})$ has infinitely many extensions in $\mathrm{P}^{\alpha}(\mathbf{R}^n)$, all these extensions coincide on $\overline{\mathrm{D}}$; it is thus meaningful to speak of restrictions of u to subsets of $\overline{\mathrm{D}}$ which are not in $\mathfrak{A}^{(k)}_{2\alpha}$.

In the considerations below (e) will denote a set of orthonormal vectors, $(e) = (e_1, \ldots, e_r)$ in $\mathbf{R}^n$, $\mathrm{D}^{\lambda}_{(e)}u$ the partial derivative of u taken in directions of vectors (e), i.e.

$$\mathrm{D}^{\lambda}_{(e)}u = \mathrm{D}^{\lambda_1}_{e_1} \ldots \mathrm{D}^{\lambda_r}_{e_r}u,$$

with the understanding that the length of the multi-index λ is determined by the length of (e).

If $\mathfrak{M}$ is the inside of a polyhedron and $u \in \mathrm{P}^{\langle\alpha\rangle}(\mathfrak{M})$ then u and its derivatives of order m have well defined restrictions to the faces of $\mathfrak{M}$ of dimension larger than $2(m\text{-}\alpha)$. Let $\mathrm{K} \subset \mathbf{R}^n$ be a polyhedral set, u be a function defined on K exc. $\mathfrak{A}_{\langle\alpha\rangle}$ and suppose that $\tilde{u} \in \mathrm{P}^{\langle\alpha\rangle}(\mathbf{R}^n)$, $\tilde{u}|_{\mathrm{K}} = u$. Consider the decomposition (1.3) of $\mathrm{K} = \mathrm{X}^{(0)}$.

$$(3.3) \qquad \mathrm{K} = \mathrm{X}^{(0)} \supset \mathrm{X}^{(1)} \supset \cdots \supset \mathrm{X}^{(\mathrm{N})};\ \mathrm{X}^{(\mathrm{N}+1)} = \varnothing,$$

where for each $k = 0, \ldots, \mathrm{N}$, $\mathrm{X}^{(k)} \setminus \mathrm{X}^{(k+1)} \equiv \mathrm{Y}^{(k)}$ is a disjoint union of insides of polyhedra $\mathfrak{M}$, referred to as *components* of $\mathrm{Y}^{(k)}$, such that the components of $\mathrm{Y}^{(\mathrm{N})}$ are single points.

Observe that if $\mathfrak{M}$ is a component of $\mathrm{Y}^{(k)}$, $k = 0, \ldots, \mathrm{N}$ then

$$(3.4) \qquad u|_{\mathfrak{M}} \in \mathrm{P}^{\langle\alpha\rangle}(\mathfrak{M}) \quad \text{if} \quad \dim \mathfrak{M} > -2\alpha,$$

otherwise $u|_{\mathfrak{M}}$ is not defined.

For fixed k and $\mathfrak{M}$a component of $\mathrm{Y}^{(k)}$ denote by $\mathfrak{M}_1, \ldots, \mathfrak{M}_l$ all the components of $\mathrm{Y}^{(0)}$ whose borders contain $\mathfrak{M}$. Let (e) be an orthonormal basis in the orthogonal complement in $\mathbf{R}^n$ of the plane L of $\mathfrak{M}$ and $(e)^i$ be orthonormal bases in the orthogonal complements of L in the planes L_i of $\mathfrak{M}_i$, $i = 1, \ldots, l$. For every multiindex λ such that $\alpha - |\lambda| > -\frac{1}{2} \dim \mathfrak{M}$ we can write

$$D^{\lambda}_{(e)^i} u_i|_{\mathfrak{M}} = \sum_{|\mu|=|\lambda|} c^{\lambda}_{i,\mu} u^{\mu} \tag{3.5}$$

where $u^{\mu} = D^{\mu}_{(e)} \tilde{u}|_{\mathfrak{M}}$, $c^{\lambda}_{i,\mu}$ are constant coefficients determined by the configuration of $\{\mathfrak{M}_i\}$ and choice of bases (e), $(e)^i$ and $u_i = u|_{\mathfrak{M}_i}$.

Remark. — (3.5) could also be written with bases (e) and $(e)^i$ replaced by orthonormal bases in $\mathbf{R}^n$ and L_i respectively; the resulting set of identities (which would involve derivatives in directions in L) would be a consequence of those appearing in (3.5).

The above consideration could be summarized as follows: if a function u defined on K has an extension $\tilde{u} \in P^{\langle\alpha\rangle}(\mathbf{R}^n)$ then for every component $\mathfrak{M}$ of $Y^{(k)}$, $k = 0, \ldots, N$ we have *a)* $u|_{\mathfrak{M}}$ satisfies (3.4), and *b)* for $\mathfrak{M} \in Y^{(k)}$ and every m such that $\alpha - m > -\frac{1}{2} \dim \mathfrak{M}$ the system of equations (3.5) with $|\lambda| = m$ and the unknowns $u^{\mu} \in P^{\langle\alpha-m\rangle}(\mathfrak{M})$ has a solution. We shall say that a function u on K with property *a)* satisfies on $\mathfrak{M}$ the *compatibility condition of order* m if u has the property *b)*.

If u satisfies these conditions on $\mathfrak{M}$ then a solution of (3.5) can be written in the form

$$u^{\mu} = \sum_{|\lambda|=m} \sum_{i=1}^{l} \gamma^{\mu}_{i,\lambda} D^{\lambda}_{(e)^i} u|_{\mathfrak{M}} \tag{3.6}$$

where $\gamma^{\mu}_{i,\lambda}$ are constant coefficients which in general are not uniquely determined but will be fixed for the remainder of the considerations.

In the case when $\alpha - m = -\frac{1}{2} \dim \mathfrak{M}$ (3.5) (which actually could be written formally for any m) thus far is meaningless since $D^{\lambda}_{(e)^i} u|_{\mathfrak{M}}$ are undefined; this case is referred to as *the exceptional case* and in § 4 we shall define the notion of abstract restriction which also in this case will make (3.5) and (3.6) meaningful. For $\alpha - m < -\frac{1}{2} \dim \mathfrak{M}$ the conditions of order m do not appear on $\mathfrak{M}$.

It is clear from the above that in order that a function u on K be extendable to $\tilde{u} \in P^{\langle\alpha\rangle}(\mathbf{R}^n)$ it is necessary that u satisfy (3.4) and for each component $\mathfrak{M}$ of $Y^{(k)}$ the compatibility conditions of all orders $m < \alpha + \frac{1}{2} \dim \mathfrak{M}$. It will be shown in § 4 that also in the exceptional case

$$\alpha - m = -\frac{1}{2} \dim \mathfrak{M}$$

the conditions are necessary. The objective of this paper is to show that the conditions above are also sufficient for the existence of an extension $\tilde{u} \in P^{\langle\alpha\rangle}(\mathbf{R}^n)$ of u.

We shall make now a few remarks concerning the concept of compatibility conditions.

The compatibility conditions for u on $\mathfrak{M}$ of order m are equivalent to a unique, possibly empty system of linear homogeneous equations with constant coefficients to be satisfied on $\mathfrak{M}$ by $\{D^{\lambda}_{(e)i} u_i | \mathfrak{M}\}_{|\lambda|=m}$, hence they can be expressed in terms of u alone.

The compatibility conditions are in an obvious way invariant with respect to the choice of the bases (e) and $(e)^i$. They could also be stated in terms of arbitrary, not necessarily orthonormal bases.

If u satisfies the compatibility conditions on every component $\mathfrak{M}$ of $Y^{(k)}$ (i.e. compatibility conditions of all orders m, $\alpha - m > -\dim \mathfrak{M}$) then it satisfies these conditions on all the components of Y^{k+1}, $1 \leqslant k \leqslant N$.

4. Auxiliary results and abstract restriction.

To prove the sufficiency of compatibility conditions we have to define the notion of abstract restriction and prove several lemmas which will be based essentially on results in [2]. Since we will be interested in very special geometric configurations in $\mathbf{R}^n$ we will prove the lemmas and produce the notion of abstract restriction in very special cases even though many of the results could be extended to much more general configurations. We will introduce the following notations: For a function $u \in P^{\langle\alpha\rangle}(D)$, D being an open set in an l-subplane L of $\mathbf{R}^n$ and F an arbitrary set in L, we will write:

$$(4.1) \qquad I^{(l)}_{\langle\alpha\rangle, D, F}(u) = \int_D \frac{|u(x)|^2}{r_F(x)^{l+2\alpha}}\, dx$$

where $r_F(x) =$ distance from x to F. The integrand is considered $= 0$ when $u(x) = 0$. The integral may be finite or infinite.

For $u \in P^{\langle\alpha\rangle}(D)$ we write further

$$(4.2) \qquad J^{(l)}_{\langle\alpha\rangle, D, F}(u) = \sum_{|\lambda|<\alpha+l/2} I^{(l)}_{\langle\alpha-|\lambda|\rangle, D, F}(D^\lambda u),$$

$D^\lambda u$ are understood as running through derivatives of u in directions of vectors in an orthonormal basis in L ($D^\lambda u = D^\lambda_{(e)} u$ — in notations of § 3, where (e) is an orthonormal basis in L). In [2] the quadratic form J was introduced for $l = n$, D and F being open sets in $\mathbf{R}^n$, and also the non-reduced order of potentials was used, so that the form $J_{\alpha, D, F}$ meant, in the present notation, $J^{(n)}_{\langle\alpha-n/2\rangle, D, F}$. In the integrals (4.1) we will call l the *dimension* and α the *order* of the integral. In the present development we will consider mostly D to be the whole l-plane or the inside of an l-dimensional simplex in L, whereas F will be a closed set.

For two closed non-empty subsets F and F_1 of L and for any $\varepsilon > 0$ we introduce, as in [2], the open sets $U^L_\varepsilon(F, F_1)$ in L called the ε-angular neighborhoods of $F \setminus (F \cap F_1)$ rel. F_1 (in abbreviation ε-neighborhoods of F rel. F_1), by

$$(4.3) \qquad U^L_\varepsilon(F, F_1) = [x \in L : r_F(x) < \varepsilon r_{F_1}(x)].$$

For F and F_1 bounded and $\varepsilon < 1$, U^L_ε is bounded. For any D open in L, $U^L_\varepsilon(\overline{D}, \partial D) = D$ for $\varepsilon < 1$.

As very special cases of Theorems in § 9 and 10 of [2], we get the following two propositions which we will use often:

PROPOSITION 4.1. — *If* $u \in P^{\langle\alpha\rangle}(L)$ *and* u *vanishes on an* l*-dimensional simplex* D, *then* $J^{(l)}_{\langle\alpha\rangle, L, \partial D}(u) < \infty$.

PROPOSITION 4.2. — *Let* D *be an open* l*-dimensional simplex in* L *and* F_i, $i = 1, \ldots, s$ *be* k*-dimensional faces of* D, $0 \leqslant k < l$. *Suppose that* $u \in P^{\langle\alpha\rangle}(D)$ *is such that* $J^{(l)}_{\langle\alpha\rangle, D, F_i}(u) < \infty$, $i = 1, \ldots, s$. *Then for every* ε, $0 < \varepsilon < 1$, *there is an extension* $\tilde{u} \in P^{\langle\alpha\rangle}(L)$ *of* u *vanishing outside* $U^L_\varepsilon\left(\overline{D}, \bigcup_{i=1}^{s} F_i\right)$. *In particular, for any* l*-dimensional simplex* D_1 *in* L, *such that* $\overline{D}_1 \cap \overline{D} = F_i$ *for some* i *there exists such an extension vanishing on* D_1.

As corollaries we obtain:

COROLLARY 4.1. — *If* D *is an open simplex in* L *and* $u \in P^{\langle\alpha\rangle}(D)$ *then* $J^{(l)}_{\langle\alpha\rangle, D, \partial D}(u) < \infty$ *implies that there exists an extension of* u, $\tilde{u} \in P^{\langle\alpha\rangle}(L)$ *which vanishes outside of* D.

This is immediate since for $\varepsilon < 1$, $U^L_\varepsilon(\overline{D}, \partial D) = D$.

COROLLARY 4.2. — *If* $u \in P^{\langle\alpha\rangle}(L)$, D *is an open simplex in* L *and* F *is a* k*-dimensional face of* D, $k < l$, *then* $J^{(l)}_{\langle\alpha\rangle, D, F}(u) < \infty$ *implies* $J^{(l)}_{\langle\alpha\rangle, L, F}(u) < \infty$.

Proof. — By our assumptions and Proposition 4.2 for any open simplex D_1 in L satisfying $\overline{D}_1 \cap \overline{D} = F$ there exists an extension of $u|_D$, $\tilde{u} \in P^{\langle\alpha\rangle}(L)$ vanishing on D_1. Also, $\tilde{u} - u$ vanishes on D. By Proposition 4.1

$$J^{(l)}_{\langle\alpha\rangle, L, F}(\tilde{u}) \leqslant J^{(l)}_{\langle\alpha\rangle, L, \partial D_1}(\tilde{u}) < \infty$$

and similarly, $J^{(l)}_{\langle\alpha\rangle, L, F}(\tilde{u} - u) < \infty$, thus $J^{(l)}_{\langle\alpha\rangle, L, F}(u) < \infty$.

As in [2] we will use the following two propositions.

PROPOSITION 4.3 ([5]) (3). — *Let* $K(x, y) \geqslant 0$, $x, y \in (0, \infty)$, *be a symmetric kernel homogeneous of degree* $-s$, $s \neq 2$ *such that* $\int_0^\infty K(1, t)\, dt < \infty$. *Suppose that* u *and* v *are*

(3) This proposition, in a more general setting, can be found in [4].

measurable functions on $(0, \infty)$ *such that*

$$I \equiv \int_0^\infty \int_0^\infty K(x, y)|u(x) - v(y)|^2 \, dx \, dy < \infty$$

then:

a) *if* $s < 2$ *then* $\int_0^\infty (|u(x)|^2 + |v(x)|^2) x^{-s+1} \, dx \leqslant \text{const } I,$

b) *if* $s > 2,$ *there exists a unique* $w_0 \in \mathbf{C}$ *such that*

$$\int_0^\infty (|u(x) - w_0|^2 + |v(x) - w_0|^2) x^{-s+1} \, dx \leqslant \text{const. } I.$$

If u *and* v *are continuous at* 0 *then* $w_0 = u(0) = v(0)$. *The constants depend only on* K.

Proposition 4.4 (*Hardy's Inequality* [8], p. 246). — *If* F *is absolutely continuous for* $a \leqslant x < b < \infty$ *then for every* $\gamma > \dfrac{1}{2}$

$$\int_a^b |f(x) - f(a)|^2 (x - a)^{-2\gamma} \, dx \leqslant \left(\gamma - \frac{1}{2}\right)^{-2} \int_a^b |f'(x)|^2 (x - a)^{-2\gamma+2} \, dx.$$

Our main purpose in this section will be to establish conditions under which $J^{(l)}_{\langle\alpha\rangle, D, F}(u)$ is finite. To simplify our proofs we will use affine mappings to transfer the configuration of l-dimensional simplex D and its k-dimensional face F into a special position. We should notice that the affine transformation does not change the class of potentials (the norm being transformed into an equivalent one). The derivatives of order $|\lambda|$ are transformed into linear combinations of derivatives of the same order with constant coefficients (determined by the affine mapping). The quadratic forms J are transformed into equivalent quadratic forms, all the constants depending only on the affine mappings and not on the function u. In particular, when we have an l-dimensional simplex D with k-dimensional face F, $k < l$, we can always transform it by an affine mapping of the whole space $\mathbf{R}^n$ into the following situation: $F = \overline{D}_k \subset \mathbf{R}^k$ and there is a sequence of simplices $D_k, \ldots, D_l$ such that D_j is an open simplex in

$\mathbf{R}^j$ which is a face of D_{j+1} such that the orthogonal projection of D_{j+1} on $\mathbf{R}^j$ is D_j and for $x^{(j+1)} \in D_{j+1}$, we have

$$x^{(j+1)} = (x^{(j)}, x_{j+1})$$

with $x^{(j)} \in D_j$ and $0 < x_{j+1} < r_F(x^{(j)})$. In these conditions, assuming that $l - k \geqslant 2$ and $\alpha + \frac{l}{2} \geqslant 1$ [4] we have

$$(4.4) \quad I^{(l)}_{\langle\alpha\rangle, D_l, F}(u) \leqslant 2 I^{(l-1)}_{\langle\alpha\rangle, D_{l-1}, F}(u) + 2^{2+\alpha+l/2} I^{(l)}_{\langle\alpha-1\rangle, D_l, F}\left(\frac{\partial u}{\partial x_l}\right).$$

To prove this inequality we write for $x^{(l)} \in D_l$, $x^{(l)} = (x^{(l-1)}, x_l)$, $u(x^{(l)}) = u(x^{(l-1)}, 0) + (u(x^{(l)}) - u(x^{(l-1)}, 0))$. Then

$$\frac{|u(x^{(l)})|^2}{r_F(x^{(l)})^{l+2\alpha}} \leqslant \frac{2|u(x^{(l-1)}), 0)|^2}{r_F(x^{(l-1)})^{l+2\alpha}} + \frac{2|u(x^{(l-1)}, x_l) - u(x^{(l-1)}, 0)|^2}{r_F(x^{(l-1)})^{l+2\alpha-2} x_l^2}$$

Since $\alpha + \frac{l}{2} \geqslant 1$ the function u on D_l has first derivatives in $L^2(D_l)$; hence for almost all $x^{(l-1)}$ as a function of x_l it is absolutely continuous. We can therefore apply on each of these lines Hardy's inequality which gives, by using the fact $0 < x_l < r_F(x^{(l-1)})$ and that $r_F(x^{(l-1)}) \geqslant \frac{1}{\sqrt{2}} r_F(x^{(l)})$,

$$\int_{D_l} \frac{|u(x)^{(l)})|^2}{r_F(x^{(l)})^{l+2\alpha}} dx^{(l)} \leqslant 2 \int_{D_l} \frac{|u(x^{(l-1)})|^2 r_F(x^{(l-1)})}{r_F(x^{(l-1)})^{l+2\alpha}} dx^{(l-1)}$$
$$+ \frac{2(1/2)^{-2}}{2^{-(l+2\alpha-2)/2}} \int_{D_l} \frac{\left|\frac{\partial}{\partial x_l} u(x^{(l)})\right|^2}{r_F(x^{(l)})^{l+2\alpha-2}} dx^{(l)}$$

which gives the required inequality.

We will consider the integrals I in three cases: when the order $\alpha < -\frac{k}{2}$, $= -\frac{k}{2}$ and $> -\frac{k}{2}$. The distinction between these cases is that in the first case the function and its derivatives have no restrictions whatsoever to F, in the second case the function has no pointwise restriction but has

[4] $\alpha + l/2$ is the non-reduced order of u on D_l and $\alpha + l/2 \geqslant 1$ means that u has first order derivatives belonging to $P^{\alpha+l/2-1}(D_l)$.

an abstract restriction to F (a notion to be introduced later, in the third case, the function and its derivatives $D^\lambda u$, $|\lambda| < \alpha + \frac{k}{2}$, have pointwise restrictions to F.

First Case, $\alpha < -\frac{k}{2}$: By applying (4.4) successively we will evaluate $I^{(l)}_{\langle\alpha\rangle, D_l, F}(u)$ by a linear combination of integrals $I^{(j)}_{\langle\alpha-|\lambda|\rangle, D_j, F}(D^\lambda u)$ for $j = k+1$ or $k+2 < j \leqslant l$ and $0 \leqslant \alpha - |\lambda| + \frac{j}{2} < 1$. If $j = k+1$ then $|\lambda| = 0$ since otherwise the potential $D^\lambda u$ on D_{k+1} would be of order $\alpha + \frac{k+1}{2} - |\lambda| < -\frac{k}{2} + \frac{k+1}{2} - |\lambda| < 0$. If

$$\alpha - |\lambda| + \frac{j}{2} = 0$$

the integral $I^{(j)}_{\langle\alpha-|\lambda|\rangle, D_j, F}(D^\lambda u)$ is the square of the L^2-norm of $D^\lambda u$ over D_j which is finite, hence we can restrict our considerations to the cases where

$$0 < \alpha - |\lambda| + \frac{j}{2} < 1. \qquad \text{Put } \beta = \alpha - |\lambda| + \frac{j}{2},$$

hence $0 < \beta < 1$. We can extend $D^\lambda u$ from D_j to the whole $\mathbf{R}^j$ as a potential φ of order β. The Dirichlet integral $d_{\beta, \mathbf{R}^j}(\varphi)$ is then finite. We can write then

$$(4.5) \quad \infty > d_{\beta, \mathbf{R}^j}(\varphi) = \int_{\mathbf{R}^j}\int_{\mathbf{R}^j} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{j+2\beta}}\, dx\, dy$$
$$\geqslant \int_{\mathbf{R}^k} d_{\beta, \mathbf{R}^{j-k}}(\varphi(x^{(k)}, z^{(j-k)}))\, dx^{(k)}$$

where we have written for $x^{(j)} \in \mathbf{R}^{(j)}$, $x^{(j)} = (x^{(k)}, z^{(j-k)})$, $z^{(j-k)} = (x_{k+1}, \ldots, x_j)$ and in the last integral the Dirichlet integral is taken on the $(j-k)$-dimensional space of $z^{(j-k)}$ [5]. Considering the part S of $\mathbf{R}^{(j-k)}$ composed of points with

[5] By inadvertent omission the last inequality in (4.5) was not stated in Bessel Potentials, Part I, Ch. II, § 8 where it should belong after Propositions 1) and 2). The proof can be easily obtained by Fourier transforms. A more elementary proof in the more general case of potentials of L^p-functions can be found in [9], p. 344.

$x^i > 0$, $k < i \leqslant j$, we can write

$$d_{\beta, \mathrm{S}}(\varphi(x^{(k)}, z^{(j-k)})) = \int_{\mathrm{S}} \int_{\mathrm{S}} \frac{|\varphi(x^{(k)}, x) - \varphi(x^{(k)}, y)|^2}{|x-y|^{j-k+2\beta}}\, dx\, dy.$$

Introducing polar coordinates $x = \rho\theta$, $y = \rho_1\theta_1$,

$$|\theta| = |\theta_1| = 1$$

we get

$$\begin{aligned} & d_{\beta, \mathrm{S}}(\varphi(x^{(k)}, z^{(j-k)})) \\ & = \iint_{|\theta|=|\theta_1|=1} \int_0^\infty \int_0^\infty \frac{|\varphi(x^{(k)}, \rho\theta) - \varphi(x^{(k)}, \rho_1\theta_1)|^2}{|\rho\theta - \rho_1\theta_1|^{j-k+2\beta}} \\ & \qquad\qquad (\rho\rho_1)^{j-k-1}\, d\rho\, d\rho_1\, d\theta\, d\theta_1 \\ & \geqslant \int_{|\theta|=1} \int_{|\theta_1|=1} \int_0^\infty \int_0^\infty \frac{|\varphi(x^{(k)}, \rho\theta) - \varphi(x^{(k)}, \rho_1\theta_1)|^2}{(\rho+\rho_1)^{j-k+2\beta}} \\ & \qquad\qquad (\rho\rho_1)^{j-k-1} d\rho\, d\rho_1\, d\theta\, d\theta_1. \end{aligned}$$

Since the kernel $\mathrm{K}(\rho, \rho_1) = \dfrac{(\rho\rho_1)^{j-k-1}}{(\rho+\rho_1)^{j-k+2\beta}}$ satisfies the requirements of part *a*) of Proposition 4.3 (because $0 < 2\beta < 2$ for $j \geqslant k+2$ and $2\beta = 2\alpha + j < 1$ for $j = k+1$) we get immediately that

$$\begin{aligned} d_{\beta, \mathrm{S}}(\varphi(x^{(k)}, z^{(j-k)})) & \geqslant \mathrm{C} \int_{\mathrm{S}} \frac{|\varphi(x^{(k)}, z^{(j-k)})|^2}{|z^{(j-k)}|^{2\beta}}\, dz^{(j-k)} \\ & \geqslant \mathrm{C} \int_{\mathrm{S}} \frac{|\varphi(x^{(k)}, z^{(j-k)})|^2}{r_{\mathrm{F}}(x^{(k)}, z^{(j-k)})^{2\beta}}\, dz^{(j-k)}. \end{aligned}$$

where C is a constant dependent only on $j-k$ and β. Restricting $\mathbf{R}^k \times \mathrm{S}$ to D_j and replacing there φ by $\mathrm{D}^\lambda u$ we obtain finally from (4.5) that $\mathrm{I}^{(j)}_{\langle\alpha-|\lambda|\rangle, \mathrm{D}_j, \mathrm{F}}(\mathrm{D}^\lambda u)$ is finite.

By an affine mapping we now translate the obtained result into.

Lemma 4.1. — *For an arbitrary l-dimensional simplex* D *with a k-dimensional face* F, $k < l$, *if* $u \in \mathrm{P}^{\langle\alpha\rangle}(\mathrm{D})$ *for* $\alpha < -\dfrac{k}{2}$ *then* $\mathrm{I}^{(l)}_{\langle\alpha\rangle, \mathrm{D}, \mathrm{F}}(u) < \infty$ *and* $\mathrm{J}^{(l)}_{\langle\alpha\rangle, \mathrm{D}, \mathrm{F}}(u) < \infty$.

Second Case, $\alpha = -\dfrac{k}{2}$: We use again (4.4) to obtain successive evaluation of $\mathrm{I}^{(l)}_{\langle -k/2\rangle, \mathrm{D}_l, \mathrm{F}}(u)$. But now, by virtue of Lemma 4.1 the problem of finiteness will be still uncertain only for integrals of the form $I^{(j)}_{\langle -k/2\rangle, D_j, F}(u)$ and that finally will be reduced to the case $I^{(k+1)}_{\langle -k/2\rangle, D_{k+1}, F}(u)$. On the other hand, if $I^{(l)}_{\langle -k/2\rangle, D_l, F}(u)$ is finite then by virtue of Lemma 4.1, $J^{(l)}_{\langle -k/2\rangle, D_l, F}(u)$ is finite. Hence, by Proposition 4.2 we can extend u to a function $\tilde{u} \in P^{\langle -k/2\rangle}(\mathbf{R}^l)$ which vanishes on any fixed open simplex $D_1 \subset \mathbf{R}^l$ such that $\overline{D}_1 \cap \overline{D}_l = F$. We can choose D_1 so that $D_1 \cap \mathbf{R}^{k+1}$ be a $(k+1)$-dimensional simplex. Thus, restriction of $\tilde{u}$ to $\mathbf{R}^{k+1}$ will vanish on $D_1 \cap \mathbf{R}^{k+1}$ and thus $I^{(k+1)}_{\langle -k/2\rangle, D_{k+1}, F}(u) < \infty$ by virtue of Proposition 4.1.

Again by affine mappings, using also Corollary 4.2 and Lemma 4.1 we transform our result into.

LEMMA 4.2. — *For an arbitrary relatively open l-dimensional simplex D with k-dimensional face F, and an arbitrary $(k+1)$-dimensional simplex $D' \subset D$ with face F, if*

$$u \in P^{\langle -k/2\rangle}(D),$$

then the four quantities $I^{(l)}_{\langle -k/2\rangle, D, F}(u)$, $J^{(l)}_{\langle -k/2\rangle, D, F}(u)$,

$$I^{(k+1)}_{\langle -k/2\rangle, D', F}(u) \quad \textit{and} \quad J^{(k+1)}_{\langle -k/2\rangle, D', F}(u)$$

are all at the same time finite or infinite.

Third Case, $\alpha > -\dfrac{k}{2}$: In this case we will assume that all derivatives $D^\lambda u$ which have pointwise restrictions to F i.e. of order $|\lambda|$ such that $\alpha - |\lambda| > -\dfrac{k}{2}$ have a vanishing restriction to F. In addition, for those derivatives $D^\lambda u$ for which $\alpha - |\lambda| = -\dfrac{k}{2}$ we assume that the integral

$$I^{(l)}_{\langle -k/2\rangle, D_l, F}(D^\lambda u)$$

is finite. By (4.4) we reduce again the consideration of finiteness to the case of integrals $I^{(j)}_{\langle \alpha-|\lambda|\rangle, D_j, F}(D^\lambda u)$ where either

$$j = k+1 \quad \text{or} \quad k+2 \leqslant j \leqslant l$$

and $0 \leqslant \alpha - |\lambda| + \dfrac{j}{2} < 1$. If $\alpha - |\lambda| < -\dfrac{k}{2}$ then the integral if finite by Lemma 4.1. If $\alpha - |\lambda| = -\frac{k}{2}$, by Lemma 4.2 the integral is finite if and only if $I^{(k+1)}_{\langle -k/2\rangle, D_{k+1}, F}(D^\lambda u)$. But this integral is finite by the same lemma and our assumption $I^{(l)}_{\langle -k/2\rangle, D_l, F}(D^\lambda u) < \infty$. Therefore we can restrict ourselves to cases when $\alpha - |\lambda| > -\frac{k}{2}$ and then if $j \geqslant k+2$ we cannot have $\alpha - |\lambda| + \frac{j}{2} < 1$. Therefore, we can restrict ourselves to the integrals $I^{(k+1)}_{\langle \alpha - |\lambda|\rangle, D_{k+1}, F}(D^\lambda u)$. Again by using Lemmas 4.1 and 4.2 we can reduce our consideration to the case when the non-reduced order β of the potential $D^\lambda u$ in D_{k+1} is $> -\frac{k}{2} + \frac{k+1}{2} = \frac{1}{2}$. In this case $D^\lambda u$ has a vanishing pointwise restriction to F. Then if $\beta \geqslant 1$ we can write:

$$\int_{D_{k+1}} \frac{|D^\lambda u(x)|^2}{r_F(x)^{k+1+2(\alpha - |\lambda|)}}\, dx$$
$$\leqslant \int_{D_k} \int_0^{z(x^{(k)})} \frac{|D^\lambda u(x^{(k)}, x_{k+1}) - D^\lambda u(x^{(k)}, 0)|^2}{r_F(x^{(k)})^{k+2(\alpha-|\lambda|)-1} x_{k+1}^2}\, dx_{k+1}\, dx^{(k)}$$

where $x = (x^{(k)}, x_{k+1})$ and $0 < x_{k+1} < z(x^{(k)}) \leqslant r_F(x^{(k)})$. By Hardy's inequality we obtain the majoration by the integral

$$4 \int_{D_k} \int_0^{z(x^{(k)})} \frac{\left|\frac{\partial}{\partial x_{k+1}} D^\lambda u(x^{(k)}, x_{k+1})\right|^2}{r_F(x^{(k)})^{k-1+2(\alpha-|\lambda|)}}\, dx_{k+1}\, dx^{(k)}$$
$$\leqslant 4C \int_{D_{k-1}} \frac{|D^{\lambda'}(x)u|^2}{r_F(x)^{k+1+2(\alpha-|\lambda'|)}}\, dx$$

i.e. by the integral $I^{(k+1)}_{\langle \alpha - |\lambda'|\rangle, D_{k+1}, F}(D^{\lambda'} u(x))$ where

$$|\lambda'| = |\lambda| + 1.$$

By repeating this reduction we get down to an integral of the form $I^{(k+1)}_{\langle \alpha - |\lambda'|\rangle, D_{k+1}, F}(D^\lambda u)$ with $\beta = \alpha - |\lambda| + \frac{k+1}{2}$ satisfying $\frac{1}{2} < \beta < 1$. At this stage we apply again (4.5) by extending $D^\lambda u|_{D_{k+1}}$ to a potential $\varphi \in P^\beta(\mathbf{R}^{k+1})$ and get the finiteness of the integral $\int_{\mathbf{R}^k} d_{\beta,\mathbf{R}^1}(v(x^{(k)}, x_{k+1}))\, dx^{(k)}$. But

$$d_{\beta,\mathbf{R}}(v(x^{(k)}, x_{k+1})) \geqslant \int_0^\infty \int_0^\infty \frac{|v(x^{(k)}, x_{k+1}) - v(x^{(k)}, y_{k+1})|^2}{(x_{k+1}+y_{k+1})^{1+2\beta}}\, dx_{k+1}\, dy_{k+1}.$$

The kernel $K(x_{k+1}, y_{k+1}) = (x_{k+1}+y_{k+1})^{1+2\beta}$ satisfies the conditions of case *b*) of Proposition 4.3. For $x^{(k)} \in D_k$,

$$v(x^{(k)}, 0) = D^\lambda u(x^{(k)}, 0) = 0.$$

Therefore we obtain:

$$\int_{D_k}\int_0^{z(x^{(k)})} \frac{|D^\lambda u(x^{(k)}, x_{k+1})|^2}{x_{k+1}^{2\beta}}\, dx_{k+1}\, dx^{(k)} < \infty$$

which means that $I^{(k+1)}_{\langle \alpha-|\lambda|\rangle, D_{k+1}, F}(D^\lambda u) < \infty$. So, under our assumptions on u and its derivatives we have

$$J^{(l)}_{\langle\alpha\rangle, D_l, F}(u) < \infty.$$

On the other hand, the finiteness of $J^{(l)}_{\langle\alpha, D_l, F}(u)$ implies, by Proposition 4.2, all our assumptions. Transforming this result by an affine mapping we get:

Lemma 4.3. — *Let* D *be a relatively open l-dimensional simplex with k-dimensional face* F, $k < l$. *If* $u \in P^{\langle\alpha\rangle}(D)$ *with* $\alpha > -\frac{k}{2}$ *then the finiteness of* $J^{(l)}_{\langle\alpha\rangle, D, F}(u)$ *is equivalent to the fact that all derivatives* $D^\lambda u$ *with* $\alpha - |\lambda| > -\frac{k}{2}$ *have vanishing pointwise restrictions to* F *and for derivatives satisfying* $\alpha - |\lambda| = -\frac{k}{2}$, $I^{(l)}_{\langle\alpha-|\lambda|\rangle, D, F}(D^\lambda u) < \infty$.

The last lemma suggests the following definition.

Definition 4.1. — *If* D *is a relatively open l-dimensional simplex or subplane of* $\mathbf{R}^n$ *and* F *a finite union of k-dimensional simplices,* $k < l$, *lying in* $\overline{D}$ *we will say that* $u \in P^{\langle -k/2\rangle}(D)$ *has a vanishing* abstract restriction *to* F, $u|^{A}_{F} = 0$, *if* $I^{(l)}_{\langle -k/2\rangle, D, F}(u) < \infty$.

With this definition we could rephrase Lemma 4.3 to say that $J^{(l)}_{\langle\alpha\rangle, D, F}(u) < \infty$ is equivalent to vanishing on F of all derivatives $D^\lambda u$ with restrictions (pointwise or abstract). By Lemma 4.2 we see that if $u \in P^{\langle -k/2\rangle}(D)$ and $u|^A_F = 0$ then for any extension $\tilde{u}$ of u, $\tilde{u} \in P^{\langle -k/2\rangle}(\tilde{D})$, we have $\tilde{u}|^A_F = 0$. Also, for any restriction of u to a simplex with face F the abstract restriction to F will be 0.

DEFINITION 4.2. — *For two functions* $u \in P^{\langle -k/2\rangle}(D)$ *and* $u' \in P^{\langle -k/2\rangle}(D')$ *where* D *and* D' *are two relatively open simplices with k-dimensional proper face* F *we say that* u *and* u' have the same abstract restriction to F, $u|^A_F = u'|^A_F$ *if for any extensions of* u *and* u' *to a common simplex or plane* $\tilde{D}$ *with reduced order* $-k/2$, *we have* $(\tilde{u} - \tilde{u}')|^A_F = 0$. *We write then* $u|^A_F = u'|^A_F$, *which is clearly an equivalence relation.*

By Lemma 4.2 the equality of abstract restrictions does not depend on the choices of the extensions. The abstract restrictions to F, defined as the equivalence classes of the relation above, form a vector space, which can be realized as the algebraic quotient space $P^{\langle -k/2\rangle}(D)/P^{\langle -k/2\rangle}_{0,F}(D)$, where D is any simplex with face F and

$$P^{\langle -k/2\rangle}_{0,F}(D) = \{v \in P^{\langle -k/2\rangle}(D) : \ v|^A_F = 0\}.$$

Note that $P^{\langle -k/2\rangle}_{0,F}(D)$ is dense in $P^{\langle -k/2\rangle}(D)$ but not closed. Our definition of equality of abstract restrictions is extrinsic in nature since it uses extensions of functions even though it is independent of the extensions. To make it more intrinsic we will restrict the functions u to a $(k+1)$-dimensional simplex D with face F. If we have then u and u' defined on two such simplices we can restrict them further to smaller simplices D and D' which are transformed, one into another, by a rotation in a $(k+2)$-dimensional plane L keeping the plane of F invariant. By this rotation T we will transfer the function u' on the simplex D' to a function

$$\check{u}'(x) = u'(Tx)$$

on the simplex D. The intrinsic characterization is then given by the proposition.

PROPOSITION 4.5. — $u|_F^A = u'|_F^A$ *if and only if*

$$(u'(Tx) - u(x))|_F^A = 0 \ (*).$$

Proof. — It is enough to show that $u'|_F^A = \check{u}'|_F^A$. We can obviously assume that $D' \neq D$. Consider in the plane of D, the simplex D'' symmetric to D rel. the k-plane of F, let T_1 be the rotation transforming D'' onto D'. Then the functions $u'(Tx)$ on D and $u'(T_1x)$ on D'' are symmetric and are in $P^{1/2}(D)$ and $P^{1/2}(D'')$ respectively. They form a function u_1 on $D \cup D''$ and a simple evaluation of the Dirichlet integral $d_{1/2, D\cup D''}(u_1)$ shows that

$$u_1 \in P^{1/2}(D \cup D'').$$

Hence, it can be extended to a function $\tilde{u}_1 \in P^1(L)$. There exists a $(k+1)$-plane L_1 separating D from $D' \cup D''$ and containing the k-plane of F. Consider now in L a homeomorphism S which is the identity on the side of L_1 containing D and on the other side is the affine mapping leaving L_1 invariant and transforming D' onto D''. The mapping S is Lipschitzian on the whole of L. Therefore the function $\tilde{u}_1(Sx)$ is still in $P^1(L)$. We have $\tilde{u}_1(Sx)|_D = u'(Tx)$ and $\tilde{u}_1(Sx)|_{D'} = u'(x)$. These two restrictions have the same abstract restrictions to F which finishes the proof.

In the next proposition we will be dealing with the following situation: we will have an l-dimensional relatively open simplex D in $\mathbf{R}^n$. By (e) we will denote an orthonormal system of vectors $(e_1, \ldots, e_{n-k})$ orthogonal to D. For a reduced order $\alpha > -\frac{n}{2}$ we will consider all derivatives $D^\lambda_{(e)}$ for $\alpha - |\lambda| \geqslant -\frac{l}{2}$. For each such $D^\lambda_{(e)}$ with

$$\alpha - |\lambda| > -\frac{l}{2}$$

we will be given a function $f^\lambda \in P^{\langle\alpha-|\lambda|\rangle}(D)$. For each λ satisfying $\alpha - |\lambda| = -\frac{l}{2}$, if they exist, we will be given a function $g^\lambda \in P^{\langle -l/2\rangle}(\mathbf{R}^n)$.

(*) In rather special circumstances conditions of this kind were considered in [14] and [15] under the name of integral compatibility conditions (see also [4]).

PROPOSITION 4.6. — *With the notations explained above there exists always a function* $u \in P^{\langle\alpha\rangle}(\mathbf{R}^n)$ *such that*

$$\text{(4.6)} \qquad D^{\lambda}_{(e)}u|_D = f^{\lambda} \quad \textit{for} \quad \alpha - |\lambda| > -\frac{l}{2},$$

$$\text{(4.6A)} \qquad D^{\lambda}_{(e)}u|^{\Lambda}_{D} = g^{\lambda}|^{\Lambda}_{D} \quad \textit{for} \quad \alpha - |\lambda| = -\frac{l}{2}.$$

Remark 4.1. — It is clear that condition (4.6A) has a meaning only for $0 \leqslant l < n$ and will be required only in this case.

Proof of Proposition 4.6. — If there are no λ with

$$\alpha - |\lambda| = -\frac{l}{2}$$

we extend each f^{λ} to $\tilde{f}^{\lambda} \in P^{\langle\alpha-|\lambda|\rangle}(L)$, where L is the plane of D. Then, by a well known construction [6] we find the desired function $u \in P^{\langle\alpha\rangle}(\mathbf{R}^n)$ satisfying

$$\text{(4.6')} \qquad D^{\lambda}_{(e)}u|_L = \tilde{f}^{\lambda}.$$

If there are λ with $\alpha - |\lambda| = -\frac{l}{2}$ we extend L to an $(l+1)$-dimensional plane $L_1 \subset \mathbf{R}^{n+1}$ containing a vector parallel to $l_{n+1} = (0, \ldots, 0, 1) \in \mathbf{R}^{n+1}$. Then each f^{λ} we extend to $\tilde{f}^{\lambda} \in P^{\langle\alpha-|\lambda|\rangle}(L_1)$ and each g^{λ} to

$$\tilde{g}^{\lambda} \in P^{\langle -l/2\rangle}(\mathbf{R}^{n+1}).$$

The restriction $\tilde{g}^{\lambda}|_{L_1}$ exists and by the same construction as above we obtain a function $\tilde{u} \in P^{\langle\alpha\rangle}(\mathbf{R}^{n+1})$ satisfying

$$\text{(4.7)} \qquad \begin{aligned} D^{\lambda}_{(e)}\tilde{u}|_{L_1} &= \tilde{f}^{\lambda} \qquad \text{for} \quad \alpha - |\lambda| > -\frac{l}{2} \\ D^{\lambda}_{(e)}\tilde{u}|_{L_1} &= \tilde{g}^{\lambda}|_{L_1} \quad \text{for} \quad \alpha - |\lambda| = -\frac{l}{2}. \end{aligned}$$

It follows then immediately that $u = \tilde{u}|_{\mathbf{R}^n}$ is our desired function.

[6] Such constructions were given in [1] and [3]; still another construction was given in [10].

5. The complete formulation of the compatibility conditions and their sufficiency.

In order to use the developments of § 4 we change slightly the setting for the compatibility conditions of § 3. We represent the polyhedral set X as a disjoint union

$$X = \mathrm{U}\{D;\ D \in \mathscr{K}\}$$

where $\mathscr{K}$ is a simplicial complex (see § 1) whose elements are relatively open. We write $\mathscr{K} = \mathscr{K}_0 \cup \mathscr{K}_1 \cup \cdots \cup \mathscr{K}_N$ where $\mathscr{K}_0 = \{D \in \mathscr{K};\ D$ is not a face of any $D' \in \mathscr{K}\}$ and for $k > 0$ $\mathscr{K}_k = \left\{ D \in \mathscr{K} \setminus \left(\bigcup_{s<k} \mathscr{K}_s \right);\ D \right.$ is not the face of any $\left. D' \in \mathscr{K} \setminus \left(\bigcup_{s<k} \mathscr{K}_s \right) \right\}$. We remark that $\mathscr{K}_N$ is a finite set of points, $\mathscr{K}_k = \varnothing$ for $k > N$.

With these notations we have the decomposition analogous to (3.3)

$$(5.1) \quad X = K = K^{(0)} \supset K^{(1)} \supset \cdots \supset K^{(N)},\ K^{(l)} \setminus K^{(l+1)} = \bigcup_{D \in \mathscr{K}_l} D$$

Consider now a fixed $D \in \mathscr{K}_k$, $k \geqslant 1$ and denote by $D_1, \ldots, D_r$ all the simplices in $\mathscr{K}_0$ which have D for a face. As in § 3 denote by (e) and $(e)^i$ orthonormal bases in the orthogonal complements of the plane determined by D in $\mathbf{R}^n$ and in the planes determined by D_i, $i = 1, \ldots, r$, respectively. Let u be a function on K such that

$$u|D \in P^{(\alpha)}(D),\ D \in \mathscr{K}_0$$

and consider, with the same notations as in § 3 with $\mathfrak{M}$, $\mathfrak{M}_i$ replaced by D, D_i, the systems of equations:

$$(5.2) \quad D^{\lambda}_{(e)i} u_i|_D = \sum_{|\mu|=|\lambda|} c^{\lambda}_{i,\mu} \varphi^{\mu},\ |\lambda| = m,$$
$$\alpha - m > -\frac{1}{2} \dim D,\ i = 1, \ldots, r$$

$$(5.2') \quad D^{\lambda}_{(e)i} u_i|^{\Lambda}_D = \sum_{|\mu|=|\lambda|} c^{\lambda}_{i,\mu} \varphi^{\mu}|^{\Lambda}_D,\ |\lambda| = m,$$
$$\alpha - m = -\frac{1}{2} \dim D,\ i = 1, \ldots, r.$$

The compatibility condition of order m for u on D requires by definition that (5.2) or (5.2′) be solvable with solution $\varphi^{\mu} \in \mathrm{P}^{\langle\alpha-m\rangle}(\mathrm{D})$ if $\alpha - m > -\frac{1}{2}\dim \mathrm{D}$ or

$$\varphi^{\mu} \in \mathrm{P}^{\langle -s/2\rangle}(\mathbf{R}^n)$$

for $\dim \mathrm{D} = s$ and $\alpha - m = -\frac{1}{2}\dim \mathrm{D}$. If the condition is satisfied then a solution of (5.2) or (5.2′) can be written in the form

$$(5.3)\quad \varphi^{\mu} = \sum_{|\lambda|=m} \sum_{i=l}^{r} \gamma^{\mu}_{i,\lambda} \mathrm{D}^{\lambda}_{(e)^i} u_i|_{\mathrm{D}},\ |\mu| = m,\qquad \alpha - m > -\frac{1}{2}\dim \mathrm{D},$$

or

$$(5.3')\quad \varphi^{\mu}|^{\mathrm{A}}_{\mathrm{D}} = \sum_{|\lambda|=m} \sum_{i=l}^{r} \gamma^{\mu}_{i,\lambda} \mathrm{D}^{\lambda}_{(e)^i} u_i|^{\mathrm{A}}_{\mathrm{D}},\ |\mu| = m,\qquad \alpha - m = -\frac{1}{2}\dim \mathrm{D}.$$

We remark that if u has an extension $\tilde{u} \in \mathrm{P}^{\langle\alpha\rangle}(\mathbf{R}^n)$ then for every $\mathrm{D} \in \mathscr{K}_k$, $k \geqslant 1$, u satisfies on D the compatibility condition of all orders $m \leqslant \alpha + \frac{1}{2}\dim \mathrm{D}$, the solutions of (5.2), (5.2′) being given respectively by $\varphi^{\mu} = \mathrm{D}^{\mu}_{(e)}\tilde{u}|_{\mathrm{D}}$ for $|\mu| < \alpha + \frac{1}{2}\dim \mathrm{D}$ and $\varphi^{\mu} = \mathrm{D}^{\mu}_{(e)}\tilde{u}$ if $|\mu| = \alpha + \frac{1}{2}\dim \mathrm{D}$.

The compatibility conditions as stated above are equivalent to those in § 3; in fact, if $\mathrm{D} \in \mathscr{K}_k$, $k \geqslant 1$ and D is contained in one of the components of $\mathrm{Y}^{(k)}$ then they amount to the same as conditions in § 3, if D is not contained in any of the components of $\mathrm{Y}^{(k)}$, $k \geqslant 1$ then D is contained in a component of $\mathrm{Y}^{(0)}$ and the conditions are automatically satisfied.

All the remarks in § 3 concerning compatibility conditions remain valid also in the present setting.

THEOREM 5.1. — *In order that a function u defined on* K *be extendable to a function in* $\mathrm{P}^{\langle\alpha\rangle}(\mathbf{R}^n)$ *it is necessary and* *sufficient that* $u \in \mathrm{P}^{\langle\alpha\rangle}(\mathrm{D})$ *for every* $\mathrm{D} \in \mathscr{K}_0$ *and that for every* $\mathrm{D} \in \mathscr{K} \setminus \mathscr{K}_0$ u *satisfy the compatibility conditions of all orders* $m \leqslant \alpha + \frac{1}{2} \dim \mathrm{D}$.

Proof. — It was already remarked that the conditions are necessary.

To prove sufficiency we will construct by induction on decreasing $k \geqslant 0$ functions $w_k \in \mathrm{P}^{\langle\alpha\rangle}(\mathbf{R}^n)$ with the property that for every $k \leqslant \mathrm{N}+1$ and $\mathrm{D} \in \mathscr{K}_k$ the functions $\{\mathrm{D}^{\mu}_{(e)^i} w_k|_{\mathrm{D}}\}_{|\mu|=m}$, $m \leqslant \frac{1}{2} \dim \mathrm{D} + \alpha$, form the solution of (5.2) or (5.2′) given by (5.3) or (5.3′).

For $k = \mathrm{N}+1$, $\mathscr{K}_{\mathrm{N}+1} = \varnothing$ and we let $w_{\mathrm{N}+1} = 0$. Suppose that w_k is constructed for some k, $1 \leqslant k \leqslant \mathrm{N}+1$. To define w_{k-1} let D be any simplex in $\mathscr{K}_{k-1}$.

For brevity's sake we will say restriction, meaning pointwise restriction as well as abstract restriction.

By hypothesis, if D' is on border $\partial\mathrm{D}$, $\mathrm{D}' \in \mathscr{K}_k$, all the derivatives $\mathrm{D}^{\lambda}_{(e)^j} u_j$ figuring in the compatibility condition for D' have the same restriction to D' as $\mathrm{D}^{\lambda}_{(e)^j} w_k$. It follows that for the derivatives $\mathrm{D}^{\lambda}_{(e)^i}$ figuring in compatibility conditions for D all the functions $\mathrm{D}^{\lambda}_{(e)^i}(u_i - w_k)$ have restrictions to D which together with their normal derivatives (as far as they exist) vanish on $\partial\mathrm{D}$. By (5.3) we obtain then that $v^{\mu} - \mathrm{D}^{\mu}_{(e)} w_k|_{\mathrm{D}}$ vanish together with all their existing normal derivatives on $\partial\mathrm{D}$. By Proposition 4.6 we construct now $v_{\mathrm{D}} \in \mathrm{P}^{\langle\alpha\rangle}(\mathbf{R}^n)$ so that $v_{\mathrm{D}}|_{\mathrm{D}} = u|_{\mathrm{D}}$, $\mathrm{D}^{\mu}_{(e)} v_{\mathrm{D}}|_{\mathrm{D}} = v^{\mu} - \mathrm{D}^{\mu}(e) w_k|_{\mathrm{D}}$ for $\alpha - |\mu| > -\frac{1}{2} \dim \mathrm{D}$ and,

$$\mathrm{D}^{\mu}_{(e)} v_{\mathrm{D}}|^{\mathrm{A}}_{\mathrm{D}} = v^{\mu}|^{\mathrm{A}}_{\mathrm{D}} - \mathrm{D}^{\mu}_{(e)} w_k|^{\mathrm{A}}_{\mathrm{D}}$$

for $\alpha - |\mu| = -\frac{1}{2} \dim \mathrm{D}$. We define the open sets $\mathrm{U}_{\varepsilon}(\mathrm{D})$ in $\mathbf{R}^n$ by $\mathrm{U}_{\varepsilon}(\mathrm{D}) = \mathrm{U}^{\mathbf{R}^n}_{\varepsilon}(\overline{\mathrm{D}}, \partial\mathrm{D})$ for $\dim \mathrm{D} > 0$ and

$$\mathrm{U}_{\varepsilon}(\mathrm{D}) = \{x \in \mathbf{R}^n : \ |x - \mathrm{D}| < \varepsilon\}$$

for D a single point. We choose $\varepsilon > 0$ small enough so that all the $\mathrm{U}_{\varepsilon}(\mathrm{D})$ for $\mathrm{D} \in \mathscr{K}_{k-1}$ be mutually disjoint (ε must

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be < 1). Then we choose for each D with $\dim D > 0$ an n-dimensional open simplex $\tilde{D} \subset U_{\varepsilon/3}(D)$ with face D. By construction, all existing derivatives $D^{\mu}\varphi_D$ in directions orthogonal to a face D' of D have restriction 0 to D'. Hence all derivatives $D^{\mu}\varphi_D$ with restriction to ∂D have there restriction $= 0$. We take $\varphi_D|_{\bar{D}}$ and apply Lemma 4.3 and Proposition 4.2 to obtain an extension φ'_D of $\varphi_D|_{\tilde{D}}$, $\varphi'_D \in P^{\langle\alpha\rangle}(\mathbf{R}^n)$ and φ'_D vanishing outside of

$$U^{\mathbf{R}^n}_{\varepsilon/3}(\bar{\tilde{D}}, \partial D) \subset U_{\varepsilon}(D) \text{ (7)}.$$

If D is a single point we take a function $\varphi_0 = 1$ on $U_{\varepsilon/3}(D)$ and $= 0$ outside of $U_{\varepsilon/2}(D)$ and put $\varphi'_D = \varphi_D \varphi_D$. It is then clear that the function

$$w_{k-1} = w_k + \sum_{D \in \mathcal{K}_{k-1}} \varphi'_D \tag{5.4}$$

satisfies our requirements.

Obviously, the function w_0 is the function required in our theorem. The last step of induction leading from w_1 to w_0 is completely similar to the other steps except that on $D \in \mathcal{K}_0$ there are no compatibility conditions and we define φ_D by the requirement that $\varphi_D|_D = (u - w_1)|_D$, $D^{\mu}\varphi_D|_D = 0$ for all existing derivatives in directions orthogonal to D.

6. Examples.

We will give here two examples to illustrate the notions and procedures used in Theorem 5.1.

Example 1. — Let X be the union of l distinct 2 dimensional spheres $S_1, \ldots, S_l$ in $\mathbf{R}^3$ intersecting in a circle

$$C_0 = \bigcap_{k=1}^{l} S_k.$$

We remark that the regular part of X is $X \setminus C_0$, the singular

(7) We use here the following general statement pertaining to angular neighborhoods in arbitrary metric space M: If $F_2 = U^M_{\varepsilon}(F_1, F)$ then

$$U^M_{\eta}(F_2, F) \subset U^M_{\varepsilon+\eta+\varepsilon\eta}(F_1, F).$$

part, $X^{(1)} = C_0$, is a manifold. In particular, the decomposition (1.3) is of the form

$$X = X^{(0)} \supset X^{(1)} \supset X^{(2)}, \qquad X^{(2)} \equiv \emptyset.$$

For every point $p \in C_0$ we have to construct a chart whose image is a polyhedral set. To this effect we consider the point $p_0 \in C_0$ diametrically opposed to p and consider the inversion mapping T in $\mathbf{R}^3$ defined by

$$Tx = \frac{|p - p_0|^2}{|x - p_0|^2}(x - p_0) + p_0.$$

Putting $U_\varphi = X \setminus (p_0)$, $\varphi = T|_{U_\varphi}$ we obtain a chart with the property that $\varphi(C_0 \setminus (p_0)) = L$ where L is a straight line in $\mathbf{R}^3$ passing through p and $\varphi(S_k \setminus (p_0)) = Q_k$ where Q_k are 2-dimensional planes containing L. The image $\varphi(U_\varphi)$ is not strictly speaking a polyhedral set or contained in one since it is unbounded but obviously we could restrict the neighborhood U_φ to a smaller neighborhood of p so that the image will be a polyhedral set presenting on the line l a relatively open segment I_0 centered at p and on each plane Q_k two relatively open equilateral triangles, Q_k^1 and Q_k^2, on the two sides of L with common face I_0. The given function u on X is transferred by the chart to a function v on the polyhedral set $K = \bigcup_{k=1}^{l} (\overline{Q_k^1} \cup \overline{Q_k^2})$. It is quite clear that all the compatibility conditions will be satisfied if they are satisfied on I_0 which belongs to $\mathscr{K}_1$ (in the notation of § 5).

To describe the compatibility conditions on I_0 we can choose the coordinate basis (e_1, e_2, e_2) in $\mathbf{R}^3$ so that $e_3 \in L$, the origin being at p. On each plane Q_k we choose a vector $e^k \perp e_3$. We put $e^k = \theta_k e_1 + \tau_k e_2$ and write the compatibility conditions on I_0 of order m:

$$\text{(6.1)} \quad D^m_{e^k} v|_{Q_k^1}|_{I_0} = D^m_{e^k} v|_{Q_k^2}|_{I_0} = \sum_{\mu=0}^{m} \binom{m}{\mu} \theta_k^\mu \tau_k^{m-\mu} v^{\mu, m-\mu}, \; k = 1, 2, \ldots, l$$

$$0 \leqslant m < \alpha + \frac{1}{2}.$$

and

$$\text{(6.1}') \quad D^m_{e^k}\varphi|_{Q^1_k}|^A_{I_0} = D^m_{e^k}\varphi|_{Q^2_k}|^A_{I_0} = \sum_{\mu=0}^{m} \binom{m}{\mu}\theta^\mu_k \tau^{m-\mu}_k \varphi^{\mu,\, m-\mu}|^A_{I_0},$$
$$k = 1, 2, \ldots, l, \qquad m = \alpha + \frac{1}{2}.$$

The equality of the two first terms in each of these equations means simply that the function $\varphi|_{Q^1_k \cup Q^2_k} \in P^{\langle\alpha\rangle}(Q^1_k \cup Q^2_k)$. This is obviously a necessary condition which we can accept from the start as satisfied so that in the above formulas we will need only the first term and the third term.

The nature of the compatibility conditions depends obviously on l and α. Let us consider a few cases.

$l = 2$, $\alpha < -\frac{1}{2}\cdot$ There are no compatibility conditions and there exists always an extension.

$l = 2$, $\alpha = -\frac{1}{2}\cdot$ No compatibility conditions of order > 0. For $m = 0$ the condition is $\varphi|_{Q^1_1}|^A_{I_0} = \varphi_{Q^1_2}|^A_{I_0}$.

$l = 2$, $h - \frac{1}{2} < \alpha < h + \frac{1}{2}$; h being a non-negative integer. The only compatibility conditions will be of orders $0 \leqslant m \leqslant h$. Only equation (6.1) is to be considered. For $m = 0$ they mean that $\varphi|_{Q^1_1}|_{I_0} = \varphi|_{Q^1_2}|_{I_0}$. For $m \geqslant 1$ equations (6.1) are always solvable. Hence only the conditions for $m = 0$ are relevant.

$l = 2$, $\alpha = h + \frac{1}{2}$, h being a non-negative integer. We have compatibility conditions for orders $0 \leqslant m \leqslant h + 1$. For $m = h + 1$ the equation (6.1′) is to be considered. As in the last case, only the condition for $m = 0$ is relevant.

$l = 3$, $\alpha < -\frac{1}{2}\cdot$ No compatibility conditions.

$l = 3$, $\alpha = -\frac{1}{2}\cdot$ Only compatibility conditions of order $m = 0$ exist and they mean that $\varphi|_{Q^1_i}|_{I_0}$ is independent of $i = 1, 2, 3$.

$l = 3$, $h - \frac{1}{2} < \alpha < h + \frac{1}{2}$, h being a non-negative integer. As previously, in the similar case, only conditions of orders $0 \leqslant m \leqslant h$ exist and are given by (6.1). The only conditions for which there may not exist solutions are of orders $m = 0$ and $m = 1$. For $m = 0$ the conditions mean that $\varphi|_{Q_i^1}|_{I_0}$ is independent of i. For $m = 1$ they mean that the over determined system (6.1) with three equations and two unknowns has to be solvable.

$l = 3$, $\alpha = h + \frac{1}{2}$, h being a non-negative integer. Only compatibility conditions for $0 \leqslant m \leqslant h + 1$ exist (for $m = h + 1$ they are of the form (6.1')). Only for $m = 0,1$ we have actual restrictions on φ which are the same as in the preceding case except when $\alpha = \frac{1}{2}$, when the compatibility conditions for $m = 1$ is given by the over-determined system (6.1') dealing with abstract restrictions.

For $l > 3$ it is clear now what the situation will be: for orders $m \leqslant \alpha + \frac{1}{2}$ the system of equations (6.1) (or (6.1')) will give an underdetermined, determined, or overdetermined system of equations for the $m + 1$ unknowns $\varphi^{\mu, m-\mu}$. The first two cases present themselves when $m + 1 \geqslant l$ and the equations are solvable; whereas in the third case where $m + 1 < l$ the compatibility conditions actually give some linear relations between the restrictions to I_0 of the different $D^m_{e_k}\varphi|_{Q_k^1}$, $k = 1, \ldots, l$.

It is easy to transfer the compatibility conditions back to the original subcartesian space X. If S_k, instead of being spheres, were C^∞-compact submanifolds of dimension 2 in $\mathbf{R}^n$ intersecting along a C^∞-submanifold C_0 so that no two $S_{k'}$, $S_{k''}$, $k' \neq k''$ are tangent at any point of C_0, the same procedures will give us the compatibility conditions. The only difference will be that the chart transforming a neighborhood of $p \in C_0$ onto a polyhedral set will be much more complicated to write explicitly.

Example II. — Let X be a closed n-dimensional polyhedron in $\mathbf{R}^n$. If a function u is given on X and we want u to be extendable to $\tilde{u} \in P^{\langle\alpha\rangle}(\mathbf{R}^n)$ it is obviously necessary and sufficient that $u|_{X^{\text{int}}} \in P^{\langle\alpha\rangle}(X^{\text{int}})$ and on all faces $\mathfrak{M}$ of X, $u|_{\mathfrak{M}} = u|_{X^{\text{int}}}|_{\mathfrak{M}}$. We could write also the compatibility conditions among which most will be redundant and the remaining will reduce to the conditions which we just wrote.

A seemingly different problem which arises in the study of boundary value problems for differential equations is the one where we want to find a function $u \in \mathrm{P}^{\langle\alpha\rangle}(\mathrm{X}^{\mathrm{int}})$ giving its values and the values of all its normal derivatives on all $(n-1)$-dimensional faces $\mathfrak{M}$ of X, the derivatives being of orders $m < \alpha + \dfrac{n-1}{2}$ so that the normal derivatives have a pointwise restriction to $\mathfrak{M}$. In these conditions we can write formally all the compatibility conditions as if the function u was defined also on $\mathrm{X}^{\mathrm{int}}$ and we notice then that the compatibility conditions on the faces of dimension $n-1$ do not exist (since u actually is not defined on $\mathrm{X}^{\mathrm{int}}$) and that all compatibility conditions on faces of dimensions $< n-1$ can be expressed in terms of the function and its normal derivatives on the faces $\mathfrak{M}$ of dimension $n-1$. Therefore, we can apply our inductive procedure from the proof of Theorem 5.1, constructing the functions w_k down to and including $k = 1$. The function $w_1 \in \mathrm{P}^{\langle\alpha\rangle}(\mathbf{R}^n)$ gives then the required function $u = w_1|_{\mathrm{X}^{\mathrm{int}}}$.

We will illustrate this procedure by a simple example where in the plane $\mathbf{R}^2$ with variables x_1, x_2, we put

$$\mathrm{X} = [x_1, x_2 : 0 \leqslant x_1 \leqslant a_1, 0 \leqslant x_2 \leqslant a_2].$$

We will consider the case $\alpha = 1$. This is the case which was used in [7] where the compatibility conditions were given without proof. The compatibility conditions here have to be written only for the faces of dimension < 1 which means the four vertices of X. The configuration, by an affine mapping, can be transformed into itself with any given vertex being mapped onto any other given vertex. It is therefore enough to consider the compatibility conditions at vertex (0, 0). Considering the two adjacent 1-dimensional faces $\mathfrak{M}_1$ and $\mathfrak{M}_2$ in the direction of the x_1-axis and the x_2-axis respectively, we know that the normal derivatives are to be taken of orders 0 and 1 and we have the necessary conditions that

$$u\Big|_{\mathfrak{M}_k} \in \mathrm{P}^{\langle 1\rangle}(\mathfrak{M}_k), \ \frac{\partial u}{\partial x_{3-k}}\Big|_{\mathfrak{M}_k} \in \mathrm{P}^{\langle 0\rangle}(\mathfrak{M}_k) \quad \text{for} \quad k = 1,\ 2.$$

The compatibility conditions of order 0 reduce themselves to

$$u|_{\mathfrak{M}_1}(0,\ 0) = u|_{\mathfrak{M}_2}(0,\ 0).$$

The compatibility conditions of order 1 mean that

$$\frac{\partial u}{\partial x_1}\Big|_{\mathfrak{M}_1}\Big|^{\mathrm{A}}_{(0,\,0)} = \frac{\partial u}{\partial x_1}\Big|_{\mathfrak{M}_2}\Big|^{\mathrm{A}}_{(0,\,0)}, \quad \frac{\partial u}{\partial x_2}\Big|_{\mathfrak{M}_2}\Big|^{\mathrm{A}}_{(0,\,0)} = \frac{\partial u}{\partial x_2}\Big|_{\mathfrak{M}_1}\Big|^{\mathrm{A}}_{(0,\,0)}.$$

The formulas written in [7] are just an intrepretation of the last relations by the use of Proposition 4.5.

Appendix.

We shall give here some additional results and remarks related to the material in the preceeding sections. The proofs will be omitted or only briefly outlined. We will use, without further explanations, notations introduced in sections 3, 4 and 5.

We recall that the class $P^{\langle\alpha\rangle}$ was defined only for manifolds of dimensions larger than -2α which excluded from considerations potentials of (non-reduced) order 0, i.e. functions in L^2. This is due to the fact that k-dimensional manifolds in $\mathbf{R}^n$ are exceptional for the class $P^{\langle -k/2\rangle}$ and functions in $P^{\langle -k/2\rangle}$ have in general no pointwise restrictions to such manifolds.

It is conceivable however, that a function u given on a polyhedral set X in $\mathbf{R}^n$ may be defined a.e. as a measurable function on some of the polyhedra $\mathfrak{M}_j^{(l)}$ or simplices in $\mathscr{K}$ of dimension -2α. In such a case it is natural to ask if, and in what sense, this property is preserved by the extension $\tilde{u} \in P^{\langle -\alpha\rangle}$ given by Theorem 5.1.

Let $\varphi \in P^{\langle -k/2\rangle}(\mathbf{R}^n)$, $n > k$ and L be a plane in $\mathbf{R}^n$ of dimension larger than k. We denote by $\varphi_{c\mathrm{L}}$ the Lebesgue correction of the restriction $\varphi|_{\mathrm{L}}$, defined by the condition $\lim_{r\to 0} \int_{\mathrm{B}} |\varphi|_{\mathrm{L}}(x+ry) - \varphi_{c\mathrm{L}}(x)|\, dy = 0$, where B is any open bounded set in the subspace parallel to L such that $0 \in \mathrm{B}$; $\varphi_{c\mathrm{L}}$ does not depend on choice of B. Recall that

$$\varphi_{c\mathrm{L}}(x) = \varphi \text{ exc. } \mathfrak{A}_{\langle -k/2\rangle}(\mathrm{L}).$$

Since any k-dimensional subplane of L is in $\mathfrak{A}_{\langle -k/2\rangle}(L)$ φ_{cL} may not exist at any point of such a subplane and, even if it exists there, it does not have to be even in L^2.

Let F be a k-dimensional simplex in $\mathbf{R}^n$ and

$$\varphi \in P^{\langle -k/2\rangle}(\mathbf{R}^n), \quad n > k.$$

Proposition A.1. — *If* $\varphi|_F^A = 0$ *then for every plane* L *in* $\mathbf{R}^n$ *such that* $F \subset L$ *and* $\dim L > k$, $\varphi_{cL}(x) = 0$ *for a.e.* $x \in F$ *(rel. to k-dimensional Lebesgue measure).*

A proof is given in [10] in the case when $\dim L = k+1$, the same idea works also in the general case.

With the same notations as in Proposition A.1 we also have :

Proposition A.2. — *Suppose that for some plane* $L_1 \supset F$ $\varphi_{cL_1}(x)$ *exists for a.e.* $x \in F$. *Then the same is true for every plane* $L \supset F$, $\dim L > k$, *and* $\varphi_{cL} = \varphi_{cL_1}$ *a.e. on* F.

In the case when $\dim L = \dim L_1 = k+1$ the result is an immediate consequence of Proposition 4.5 and Proposition A.1. Otherwise, it is easy to see that it is sufficient to establish the statement in the case when

$$|\dim L\text{-}\dim L_1| = 1 \quad \text{and} \quad L \subset L_1 \quad \text{or} \quad L_1 \subset L.$$

Assuming as we may that the two planes are $\mathbf{R}^l$ and $\mathbf{R}^{l-1}$ we define for an arbitrary $w \in P^{\langle -k/2\rangle}(\mathbf{R}^l)$, $r > 0$ and $x \in F$:

$$\begin{aligned}
\Phi_r(x) &= r^{-l}\int_{|x-y'|<r}\int_{-r}^{r} |w(y', y_l)|\, dy_l\, dy' ;\\
\Phi_r'(x) &= r^{-l+1}\int_{|x-y'|<r} |w(y', 0)|\, dy' ;\\
\Phi_r''(x) &= \left(r^{-l+2}\int_{|x-y'|<r}\int_{-r}^{r}\left|\frac{\partial w(y', y_l)}{\partial y_l}\right|^2 dy_l\, dy'\right)^{1/2}.
\end{aligned}$$

With these notations we have the inequalities

$$\Phi_r(x) \leqslant \Phi_r'(x) + \Phi_r''(x)$$

and $\Phi_r'(x) \leqslant \Phi_r(x) + \Phi_r''(x)$. Using Lemma 4.1 one can show that $\Phi_r''(x) \to 0$ a.e. on the plane determined by F. The result is now obtained for $L_1 \subset L$ by using the first inequality and for $L \subset L_1$ by using the second, in each case with $w(y)$ replaced by $w(y) = \varphi(y) - \varphi_{cL_1}(x)$.

Propositions A.1 and A.2 justify the following definition.

DEFINITION. — *If* $u \in P^{\langle -k/2\rangle}(\mathbf{R}^n)$ *then the abstract restriction* $u|_{F}^{A}$ *is realized by a measurable function* f *(or* f *is the realization of* $u|_{F}^{A}$*) if for some (and therefore every) plane* $L \supset F$, $\dim L > k$, *we have* $u_{cL}(x) = f(x)$ a.e. *on* F.

PROPOSITION A.3. — *Every finitely valued measurable function* f *on* $\mathbf{R}^k$ *is a realization of* $u|_{\mathbf{R}^k}^{A}$ *for some* $u \in P^{\langle -k/2\rangle}(\mathbf{R}^n)$.

In the proof we assume, as we may, by Proposition A.2, that $n = k + 1$ and take advantage of the following two facts:

a) every function in $L^2(\mathbf{R}^{n-1})$ is a realization of the abstract restriction of a function in $P^{1/2}(\mathbf{R}^n)$ (extension operators of [3] or [10]);

b) $P^{1/2}_{0,\mathbf{R}^{n-1}}(\mathbf{R}^n) = \{v \in P^{1/2}(\mathbf{R}^n);\ v|_{\mathbf{R}^{n-1}}^{A} = 0\}$ is dense in

$$P^{1/2}(\mathbf{R}^n).$$

Let $\{A_p\}_{p=0}^{\infty}$ be a sequence of mutually disjoint compact sets in $\mathbf{R}^{n-1}$ such that $f|_{A_p}$ is bounded and $\mathbf{R}^{n-1}\setminus\left(\bigcup_{p=0}^{\infty} A_p\right)$ is of n-1-dimensional measure 0; denote by f_p the function $f_p(x) = f(x)$ on A_p, $f_p(x) = 0$ on $\mathbf{R}^{n-1}\setminus A_p$. Consider any strictly decreasing sequence, $\rho_p \searrow 0$, $p \to \infty$, such that $\min\left\{r_{A_p}(x);\ x \in \bigcup_{s=0}^{p-1} A_s\right\} \geqslant 3\rho_p$. Let $\varphi_p \in C^{\infty}(\mathbf{R}^n)$ be such that $\varphi_p = 1$ on $B_{\rho_p/2}(A_p)$, $\varphi_p = 0$ outside $B_{\rho_p}(A_p)$ where $B_\varepsilon(A)$ denotes the ε-neighborhood of A in $\mathbf{R}^n$. Let

$$v_p \in P^{1/2}(\mathbf{R}^n)$$

be such that $v_p|_{\mathbf{R}^{n-1}}^{A}$ is realized by f_p (note that $f_p \in L^2(\mathbf{R}^{n-1})$ and use *a*)). By *b*) we can find w_p such that

$$\|\varphi_p(v_p - w_p)\|_{P^{1/2}(\mathbf{R}^n)} \leqslant \omega_n^{-1/2}\rho_p^{n/2}2^{-p}$$

($\omega_n =$ the volume of the unit ball in $\mathbf{R}^n$) and $w_p|_{\mathbf{R}^{n-1}}^{A} = 0$. Then $u_p = \varphi_p(v_p - w_p)$ has the property that $u_p|_{\mathbf{R}^{n-1}}^{A}$ is realized by f_p (Proposition A.1), and we define $u = \sum_{p=0}^{\infty} u_p$. Clearly $u \in P^{1/2}(\mathbf{R}^n)$ and the point is to show that $u|_{\mathbf{R}^{n-1}}^{A}$ is realized by f. To this effect we consider for fixed p a point

$x \in A_p$ such that $(u_p)_{c\mathbf{R}^n}(x) = f_p(x)$. The last property means that for every m there is $r_m > 0$ such that for $r \leqslant r_m$, $r^{-n} \int_{|x-y|<r} |u_p(y) - f_p(x)|\, dy < 2^{-m}$; the following argument shows that the same inequality remains valid with u_p replaced by u, 2^{-m} replaced by 2^{-m+1} provided that $r \leqslant \rho_l$ where l is the least index $\geqslant \max(m, p)$ such that $\rho_l \leqslant r_m$. This, of course, will complete the proof.

For $r \leqslant \rho_l$ choose $i \geqslant 0$ such that $\rho_{l+i+1} < r \leqslant \rho_{l+i}$ and write

$$r^{-n} \int_{|x-y|<r} |u(y) - f_p(x)|\, dy \leqslant r^{-n} \int_{|x-y|<r} \left| \sum_{s=0}^{l+i} u_s(y) - f_p(x) \right| dy + r^{-n} \sum_{s=l+i+1}^{\infty} \int_{|x-y|<r} |u_s(y)|\, dy.$$

The second sum is estimated by 2^{-m} by Cauchy Schwartz inequality applied to each term. On the ball $\{y;\, |x-y| < r\}$ the sum appearing in the first integral is by construction equal to u_p which completes the argument.

We shall next give an example showing that the converse of Proposition A.1 is false, in particular that two abstract restrictions may be different even if they are both realized by the same function.

Example. — We indicate the construction of a function f on $\mathbf{R}^1$ with the following properties:

a) $f \in L^2(\mathbf{R}^1)$, $f(x) = 0$ for $x < 0$,

b) $f(x) + f(-x) \in P^{1/2}(\mathbf{R}^1)$ equivalent by *a*) to

$$f|_{\mathbf{R}^1_+} \in P^{1/2}(\mathbf{R}^1_+),\ \ f(x) + f(-x) \in C(\mathbf{R}^1),\ \ f(0) = 0,$$

c) $f \notin P^{1/2}(\mathbf{R}^1)$, equivalent by *a*), *b*) to $f|_{\mathbf{R}^1_+}|_0^A \neq 0$.

In particular, the correction of f at 0 is 0 but

$$f|_{\mathbf{R}^1_+}|_0^A \neq 0.$$

The function f is given in terms of its Fourier transform:

$$\hat{f}(\xi) = \frac{1}{(3 + i\xi)\,(\log(2 + i\xi))^{1/2}},$$

considered as extended to an analytic function of $\xi \in \mathbf{C}^1$

regular in $\mathbf{C}^1\setminus\{i + i\mathbf{R}^1_+\}$ determined by the requirement that it be positive on the halfline $i - i\mathbf{R}^1_+$. Note that for real ξ

$$-\frac{1}{2}\pi \leqslant \arg(2 + i\xi) = -\arg(2 - i\xi) \leqslant \frac{1}{2}\pi.$$

It is immediate that $\hat{f} \in L^2(\mathbf{R}^1)$ and that its extension converges to 0 for $|\xi| \to \infty$ in the lower halfplane of $\mathbf{C}^1$. Hence f satisfies a).

Property b) follows from the finiteness of the integral $\int_{-\infty}^{\infty} |\hat{f}(\xi) + \hat{f}(-\xi)|^2(1 + \xi)^{1/2}\, d\xi$, which is checked by a simple computation. Furthermore $\hat{f}(\xi) + \hat{f}(-\xi)$ is in $L^1(\mathbf{R}^1)$ and has the integral equal to 0.

To prove c) we check that $\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2(1 + \xi^2)^{1/2}\, d\xi = \infty$. It should be noted that by extending the function f given above in a suitable way we could obtain for an arbitrary $n \geqslant 1$ a function $\varphi \in P^{1/2}(\mathbf{R}^n)$ such that $\varphi|^{A}_{\mathbf{R}^{n-1}} \neq 0$ but $\varphi|^{A}_{\mathbf{R}^{n-1}}$ is realized by the function 0.

We shall next make some comments on the compatibility conditions and Theorem 5.1. First of all, if all the data appearing in (3.5) or respectively in (5.2′) have realizations then the compatibility conditions on $\mathfrak{M}^{(l)}_i$ (or D) are satisfied in pointwise sense provided that they are satisfied in the sense of abstract restrictions (Proposition A.1). The converse of course is false. Furthermore, the solutions of (5.2′) given by (5.3′) are also realized by functions and it is clear from the proof of Theorem 5.1, Propositions A.1 and A.2 that the extension $\tilde{u}$ has the property that for every $D \in \mathscr{K}_l$, $l > 0$ where the abstract restrictions of u, or of its derivatives, are realized by functions, the same is true for $\tilde{u}$ and realizations are preserved.

On the other hand, if the reduced order $\alpha = -k/2$ then in the setting of section 5 the simplices D of dimension $\leqslant k$ are ignored. In the case when for some $D \in \mathscr{K}_0$ with $\dim D = k$, $u|_D$ is a measurable function it is still possible to construct the extension $\tilde{u}$ in Theorem 5.1 in such a way that $\tilde{u}|^{A}_{D}$ is realized by $u|_D$.

To end this section we mention briefly an extension of the concept of compatibility conditions and the content of Theorem 5.1 to the setting when the function u given on a polyhedral set $X \subset \mathbf{R}^n$ has the property that

$$u|_{\mathfrak{M}_j^{(0)}} \in P^{\langle\alpha_j\rangle}(\mathfrak{M}_j^{(0)}),$$

where α_j are not necessarily all equal. It is then natural to ask for an extension of u to $\mathbf{R}^n$, say $\tilde{u}$, such that for each $\mathfrak{M}_j^{(0)}$ $\tilde{u} \in P^{\langle\alpha_j\rangle}$ on an open neighborhood in $\mathbf{R}^n$ of $\mathfrak{M}_j^{(0)}$. The necessary compatibility conditions become somewhat more cumbersome since on each $\mathfrak{M}_i^{(l)}$, $l > 0$ conditions (3.5) of different orders may involve different sets of $\mathfrak{M}_j^{(0)}$-s, containing $\mathfrak{M}_i^{(l)}$ in their borders. An exact analogue of Theorem 5.1 remains valid with a similar proof except that additional care has to be taken in the choice of the extension operator in Proposition 4.6; in particular, operators $E: P^{\langle\alpha\rangle}(\mathbf{R}^l) \to P^{\langle\alpha\rangle}(\mathbf{R}^n)$ with the property that for every f on $\mathbf{R}^l$ $Ef \in C^\infty$ on $\mathbf{R}^n \setminus \operatorname{supp} f$, are suitable. For instance, extension operators considered in [10] have this property.

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