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ON THE REPRESENTATION OF DIRICHLET FORMS

by Lars-Erik ANDERSSON

Dédié à Monsieur M. Brelot à l'occasion de son 70e anniversaire.

This article is an abbreviated version of [2] in the reference list.

1. Introduction.

The purpose of this article is to complete and generalize certain results on the representation of Dirichlet forms obtained by A. Beurling and J. Deny (see [3] and [5]) and, recently, by G. Allain (see [1]).

First let us introduce some notations and definitions.

X denotes a locally compact Hausdorff space.

V is a vector space of *realvalued* functions defined on X.

Q and N will denote bilinear forms defined on V.

$$Q(f) = Q(f, f).$$

Q is said to be positive if $Q(f) \geqslant 0$ for all f in the domain V.

$C_{00}(X)$ is the set of all continuous realvalued functions on X, with compact support.

$\Omega \subset R^n$ will denote an open set.

$C^1_{00}(\Omega)$ is the set of all realvalued, once continuously differentiable functions on Ω, with compact support.

$\Lambda^1_{00}(\Omega)$ is the set of all realvalued Lipschitz functions of order one with compact support.

K always denotes a compact subset of $\Omega \subset R^n$.

$\|\text{grad} f\|_\infty = \sup_x |\text{grad} f(x)|$, where $f \in C^1_{00}(\Omega)$ and $|.|$ is the ordinary Euclidean norm.

DEFINITION 1.1. — *A bilinear form* N *is said to be local if* $N(f, g) = 0$ *whenever* f *is constant on a neighbourhood of* supp g *and vice versa.*

DEFINITION 1.2. — *A normal contraction* T *is mapping* $T : \mathbf{C} \to \mathbf{C}$ (*or* $T : \mathbf{R} \to \mathbf{R}$) *such that* $T0 = 0$ *and*

$$|Tz_1 - Tz_2| \leqslant |z_1 - z_2|$$

for all z_1, z_2 *in* $\mathbf{C}$ (*or* $\mathbf{R}$).

If u *and* v *are real or complex valued functions then* u *is said to be a normal contraction of* v *if*

$$|u(x)| \leqslant |v(x)| \quad \textit{and} \quad |u(x) - u(y)| \leqslant |v(x) - v(y)|$$

for all x, y *in the domain.*

It can be shown that u is a normal contraction of v if and only if there exists a normal contraction operator T such that $u = Tv$ (sufficiency is trivial).

DEFINITION 1.3. — T_a *will denote the normal contraction operator, which projects* $\mathbf{C}$ (*or* $\mathbf{R}$) *onto the line segment* $[0, a]$, $a > 0$.

T_1 *is called the fundamental contraction operator. Thus* $T_1 x = \min(x^+, 1)$ *if* x *is real.*

DEFINITION 1.4. — *A normal contraction* T *is said to operate on the positive bilinear form* Q (*with domain* V) *if* $f \in V \Longrightarrow Tf \in V$ *and* $Q(Tf) \leqslant Q(f)$.

A central problem in potential theory is the following: Find all positive, symmetric, bilinear forms defined on a subspace V of $C_{00}(X)$, on which all normal contractions operate (see [1], [3] and [5]).

In [1], essentially, the following theorem is proved.

THEOREM 1.1. — *If* Q *is positive, symmetric and defined on* V, *which is dense in* $C_{00}(X)$ (*in the sup-norm topology*) *and if* T_1 *operates on* Q *then*

$$Q(f, g) = \int f(x)g(x)\, d\mu(x) + \frac{1}{2}\iint (f(x) - f(y))(g(x) - g(y))\, d\sigma(x, y) + N(f, g).$$

Here, μ is a uniquely defined positive Radon measure on X, $\sigma(x, y)$ is a positive, symmetric Radon measure on $X \times X$ which is uniquely defined (except of course on the diagonal) and N is a uniquely defined *positive symmetric* form of *local* type. Moreover T_1 operates on N.

The following problem arises naturally: Characterize the local part when $X = \Omega \subset R^n$, Ω open.

2. Statement of results.

The following theorems are valid.

THEOREM 2.1. — *If, in theorem 1.1, we make the additional assumption that all normal contractions operate on* Q, *then also all normal contractions operate on the local part* N.

This theorem has earlier been discovered by P. Roth. The proof can be found in [2].

THEOREM 2.2. — *Assume that* N *is a local, positive, symmetric bilinear form defined on* $V = C^1_{00}(\Omega)$ (*or on* V,

$$C^1_{00}(\Omega) \subset V \subset \Lambda^1_{00}(\Omega)$$

where V *is closed in the sense that*

$$f_n \in V,\ \operatorname{supp} f_n \subset K,\ \|\operatorname{grad} f_n - \operatorname{grad} f\|_\infty \to 0 \Longrightarrow f \in V).$$

Then there exists a locally finite point set $E \subset \Omega$ with the following property: If we restrict $N(f, g)$ to functions f, $g \in V$ which are affine on some (arbitrarily small) neighbourhood of each point of E then N is bounded in the sense that for each compact set $K \subset \Omega$ there exists a constant C_K such that

$$|N(f, g)| \leqslant C_K \|\operatorname{grad} f\|_\infty \|\operatorname{grad} g\|_\infty$$

whenever $\operatorname{supp} f$ and $\operatorname{supp} g \subset K$.

Moreover, for functions $f,\ g \in \mathrm{V}$ which have continuous derivatives at every point of E, we have a partition

$$\mathrm{N}(f, g) = \mathrm{N}_0(f, g) + \mathrm{N}_1(f, g).$$

N_1 is a *local positive* bilinear form which is *bounded* in the sense described above.

N_0 has the following properties:

(i) $\mathrm{N}_0(f, g) = 0$ if for every point $x \in \mathrm{E}$, either f or g is constant in some (arbitrarily small) neighbourhood of x.

(ii) $\mathrm{N}_0(f, g) = 0$ if both f and g are affine in a neighbourhood of each point of E.

(iii) N_0 is unbounded (unless $\mathrm{N}_0 \equiv 0$), meaning that inequalities like those for N_1 do not hold.

Furthermore, N_0 and N_1 are symmetric.

With the aid of theorem 2.2 it is possible to prove.

THEOREM 2.3. — *If* N *is a local, positive, symmetric bilinear form on* $\mathrm{V} \supset \mathrm{C}^1_{00}(\Omega)$, *on which all normal contractions operate, then* N *is bounded in the sense described in theorem* 2.2. *when restricted to functions* $f,\ g \in \mathrm{C}^1_{00}(\Omega)$.

THEOREM 2.4. — *If* N *is a local, symmetric, bilinear form defined on* $\mathrm{V} \supset \mathrm{C}^1_{00}(\Omega)$, *which is bounded in the sense of theorem* 2.2, *then there exists a symmetric family* $\{\sigma_{ij}\}_{i,j=1}^n$ *of Radon measures on* Ω *such that for* $f,\ g \in \mathrm{C}^1_{00}(\Omega)$

$$\mathrm{N}(f, g) = \sum_{ij} \int \frac{\delta f}{\delta x_i} \frac{\delta g}{\delta x_j}\, d\sigma_{ij}.$$

The measures σ_{ij} are uniquely defined, provided we demand symmetry, $\sigma_{ij} = \sigma_{ji}$. Moreover, if N is positive then

$$\sum_{ij} h_i h_j \sigma_{ij}$$

is a positive measure for all $h_i \in \mathrm{C}_{00}(\Omega)$, $i = 1, 2, \ldots, n$ (this is equivalent to saying that the matrix $\{\sigma_{ij}(\mathrm{B})\}_{ij}$ is positive semidefinite for all compact Borel sets $\mathrm{B} \subset \Omega$). We also have

$$\int_{\mathring{\mathrm{K}}} |d\sigma_{ii}| \leqslant \mathrm{C}_{\mathrm{K}}$$

and

$$\int_{\dot{K}} |d\sigma_{ij}| \leqslant 2C_K \quad \text{for} \quad i \neq j.$$

If N is positive then

$$\int_{\dot{K}} |d\sigma_{ij}| \leqslant C_K \quad \text{for all} \quad i, j.$$

($\dot{K}$ is the interior of K).

Corollary 2.5. — *If* Q *is a positive, symmetric, bilinear form defined on* $V \supset C^1_{00}(\Omega)$ *such that all normal contractions operate on* Q *then for* $f, g \in C^1_{00}(\Omega)$ *we have*

$$Q(f, g) = \int fg \, d\mu(x) + \frac{1}{2} \iint (f(x) - f(y))(g(x) - g(y)) \, d\sigma(x, y) + \sum_{ij} \int \frac{\delta f}{\delta x_i} \frac{\delta g}{\delta x_j} \, d\sigma_{ij}.$$

(Notations as above.)

3. Proofs of the stated results.

Proof of theorem 2.2. — Using an idea of J. Peetre (see [7]) we introduce the point set E defined by $E = \{x \in \Omega;$ for every neighbourhood $\omega_x \ni x$, $\exists \varphi \in V$, $\text{supp}\, \varphi \subset \omega_x$,

$$\|\text{grad}\, \varphi\|_\infty \leqslant 1 \quad \text{and} \quad N(\varphi) \geqslant 1\}.$$

Lemma 3.1. — E *is locally finite*, i.e. *every compact set contains at most finitely many points of* E.

Proof. — Suppose E is not locally finite. Then E has an accumulation point $x_\infty \in \Omega$ and there exists a sequence $\{x_i\}_1^\infty$ of distinct points in E, $x_i \neq x_\infty$, converging to x_∞. Now choose neighbourhoods $\omega_i \ni x_i$ such that $\omega_i \cap \omega_j = \varphi$ if $i \neq j$ and all $\omega_i \subset K$ for some compact K. By the definition of E we can find functions $\varphi_i \in V$ with supp $\varphi_i \subset \omega_i$, $\|\text{grad}\, \varphi_i\|_\infty \leqslant 1$ and $N(\varphi_i) \geqslant 1$. If we take

$$\varphi = \sum_1^\infty \frac{1}{\sqrt{i}} \varphi_i, \quad \text{then} \quad \varphi \in V.$$

N is local and positive, which implies

$$\mathrm{N}(\varphi) = \sum_1^k \frac{1}{i}\,\mathrm{N}(\varphi_i) + \mathrm{N}\left(\sum_{k+1}^{\infty} \frac{1}{\sqrt{i}}\,\varphi_i\right) \geqslant \sum_1^k \frac{1}{i} \to \infty \quad \text{as} \quad k \to \infty.$$

This is a contradiction since $\varphi \in \mathrm{V}$.

Thus the lemma is proved.

LEMMA 3.2. — *If* $\mathrm{K} \subset \Omega \setminus \mathrm{E}$ *is compact, then there exists a constant* $\mathrm{C_K}$ *such that*

$$|\mathrm{N}(f, g)| \leqslant \mathrm{C_K} \|\operatorname{grad} f\|_\infty \|\operatorname{grad} g\|_\infty$$

whenever $\operatorname{supp} f$ *and* $\operatorname{supp} g \subset \mathrm{K}$.

Proof. — For every $x \in \mathrm{K}$, $\exists$ neighbourhood $\omega_x \ni x$ such that

$$\operatorname{supp} \varphi \subset \omega_x \Longrightarrow \mathrm{N}(\varphi) \leqslant \|\operatorname{grad} \varphi\|_\infty^2.$$

K, being compact, can be covered by finitely many such neighbourhoods ω_ν, $\nu = 1, 2, \ldots, n$. We can also find a partition of unity,

$$\sum_{\nu=1}^n \varphi_\nu = 1 \quad \text{on} \quad \mathrm{K},\ 0 \leqslant \varphi_\nu \leqslant 1,$$

$$\operatorname{supp} \varphi_\nu \subset \omega_\nu, \quad \varphi_\nu \in \mathrm{C}^1_{00}(\Omega).$$

$$f = \sum_{\nu=1}^n f\varphi_\nu \quad \text{if} \quad \operatorname{supp} f \subset \mathrm{K}.$$

$$0 \leqslant \mathrm{N}(f) = \mathrm{N}\left(\sum_1^n f\varphi_\nu\right) = \sum_1^n \mathrm{N}(f\varphi_\nu) + \sum_{\nu \neq \mu} \mathrm{N}(f\varphi_\nu, f\varphi_\mu).$$

$$|\mathrm{N}(f\varphi_\nu)| \leqslant \|\operatorname{grad}(f\varphi_\nu)\|_\infty^2 \leqslant (\|\operatorname{grad} f\|_\infty + \|f\|_\infty \|\operatorname{grad} \varphi_\nu\|_\infty)^2$$

But $\|f\|_\infty \leqslant \mathrm{C_1}\|\operatorname{grad} f\|_\infty$ for some constant $\mathrm{C_1}$ (depending only on K). Therefore $\mathrm{N}(f\varphi_\nu) \leqslant \mathrm{C_2}\|\operatorname{grad} f\|_\infty^2$, where

$$\mathrm{C_2} = \left(1 + \mathrm{C_1} \max_\nu \|\operatorname{grad} \varphi_\nu\|_\infty\right)^2.$$

Schwarz' inequality gives

$$|\mathrm{N}(f\varphi_\nu, f\varphi_\mu)| \leqslant \{\mathrm{N}(f\varphi_\nu)\}^{1/2}\{\mathrm{N}(f\varphi_\mu)\}^{1/2} \leqslant \mathrm{C_2}\|\operatorname{grad} f\|_\infty^2.$$

$$\therefore \mathrm{N}(f) \leqslant n^2\mathrm{C_2}\|\operatorname{grad} f\|_\infty^2 = \mathrm{C_K}\|\operatorname{grad} f\|_\infty^2.$$

Next, by Schwarz' inequality

$$|\mathrm{N}(f, g)| \leqslant (\mathrm{N}(f))^{1/2}(\mathrm{N}(g))^{1/2} \leqslant \mathrm{C_K}\|\operatorname{grad} f\|_\infty \|\operatorname{grad} g\|_\infty.$$

LEMMA 3.3. — *If* $K \subset \Omega$ *is compact then there exists a constant* C_K *such that*

$$|N(f, g)| \leqslant C_K \|\text{grad } f\|_\infty \|\text{grad } g\|_\infty$$

whenever f *and* g *are affine in some neighbourhood of each point of* E *and the supports are contained in* K.

The proof of this lemma is fairly straightforward but lengthy and is therefore omitted. The details can be found in [2].

To define the forms N_0 and N_1 we need only observe that

$$F_1(K) = \{f : f \in V, \text{supp } f \subset K,$$

f affine on a neighbourhood of each point of $E\}$ is dense in

$$F(K) = \{f : f \in V, \text{ supp } f \subset K \text{ and grad } f \text{ is continuous at every point of } E\}$$

under the norm $\|\text{grad } f\|_\infty$.

Now if $f, g \in F(K)$ are given we can take $f_n, g_n \in F_1(K)$ such that

$$\|\text{grad } f_n - \text{grad } f\|_\infty \to 0, \ \|\text{grad } g_n - \text{grad } g\|_\infty \to 0 \text{ as } n \to \infty.$$

DEFINITION. — $N_1(f, g) = \lim_{n \to \infty} N(f_n, g_n)$

$$N_0 = N - N_1.$$

It easily follows that the definition is independent of the particular sequence and that we have the properties:

$$|N_1(f, g)| \leqslant C_K \|\text{grad } f\|_\infty \|\text{grad } g\|_\infty.$$
$$N_1(f) \geqslant 0 \quad \text{for all } f.$$

$N_0(f, g) = 0$ if f and g are affine in a neighbourhood of each point of E.

Property (i) of N_0 is proved in the following way: First, it suffices to consider the case when E consists of a single point, for example 0. Then, if f is constant on a neighbourhood U of 0, we have $N_0(f, g) = N(f, g) - \lim_{n \to \infty} N(f, g_n)$ where $g_n \in F_1(K)$ can be chosen such that

$$\text{supp } (g_n - g) \subset U \ \ (\text{and } \|\text{grad } g_n - \text{grad } g\|_\infty \to 0)$$

consequently $N_0(f, g) = \lim_{n\to\infty} N(f, g - g_n) = 0$ by the locality of N.

Finally, if N_0 were bounded in the sense described, we could redefine $N_1 = N$, $N_0 = 0$.

This completes the proof of theorem 2.2.

Remark. — It should be noted that $N_1(f) \geqslant 0$ but not necessarily $N_0(f) \geqslant 0$. With the aid of a suitably chosen Hamel basis it is possible to construct an example showing this.

Proof of theorem 2.3. — We will prove that in the partition

$$N(f, g) = N_0(f, g) + N_1(f, g)$$

deduced in theorem 2.2 we have

$$N_0(f, g) = 0 \text{ for all } f, g \in C^1_{00}(\Omega).$$

LEMMA 3.4. — *Let* $f \in C^1_{00}(\Omega)$ *have the following properties:*

(i) $f = 0$ *on* E,

(ii) $\operatorname{grad} f = 0$ *on* E,

(iii) $f \geqslant 0$ *on some neighbourhood of* E.

Then $N_0(f) = 0$.

Proof. — By the locality properties of N_0 it is enough to carry out the proof for the case that $f \geqslant 0$ everywhere. Let $h:$ $R^+ \to R^+$ have the following properties:

(i) $h \in C^1([0, \infty[)$,

(ii) $h(x) = x$ for $0 \leqslant x \leqslant \frac{1}{2}$,

(iii) $h(x) = 1$ for $x \geqslant 2$,

(iv) $0 \leqslant h'(x) \leqslant 1$ for all x.

Take $h_\varepsilon(x) = \varepsilon h\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$. Then $f_\varepsilon = h_\varepsilon(f)$ is a normal contraction of f such that

a) $f_\varepsilon \in C^1_{00}(\Omega)$.

b) $f = f_\varepsilon$ on some neighbourhood of E.

c) $f(x) \geqslant 0$ and $f_\varepsilon(x) \geqslant 0$ for a'l x.

d) for all x, y $f(x) - f(y)$ has the same sign as $f_\varepsilon(x) - f_\varepsilon(y)$.

e) $\|\operatorname{grad} f_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0^+$.

Now we can choose a sequence ε_i tending to zero fast enough to ensure that

$$g = \sum_{i=1}^{\infty} f_{\varepsilon_i} \in C^1_{00}(\Omega).$$

By *c*) and *d*) it follows that $\sum_{i=1}^{n} f_{\varepsilon_i}$ is a normal contraction of g.

b) Implies that

$$N_0(f_{\varepsilon_i}, f_{\varepsilon_j}) = N_0(f)$$

and

$$N_0\left(\sum_1^n f_{\varepsilon_i}\right) = n^2 N_0(f).$$

Hence

$$N(g) \geqslant N\left(\sum_1^n f_{\varepsilon_i}\right) = N_1\left(\sum_1^n f_{\varepsilon_i}\right) + N_0\left(\sum_1^n f_\varepsilon\right)$$
$$\geqslant N_0\left(\sum_1^n f_{\varepsilon_i}\right) = n^2 N_0(f).$$

Thus $N_0(f) \leqslant 0$. Suppose $N_0(f) < 0$. Then

$$N_0(f_{\varepsilon_i}) = N_0(f) < 0.$$
$$0 \leqslant N(f_{\varepsilon_i}) = N_0(f_{\varepsilon_i}) + N_1(f_{\varepsilon_i}) = N_0(f) + N_1(f_{\varepsilon_i}) \to N_0(f) < 0$$

as $\varepsilon_i \to 0$ which is a contradiction. Therefore $N_0(f) = 0$.

Next we want to get rid of the condition (iii) in the previous lemma.

LEMMA 3.5. — *Let* $f \in C^1_{00}(\Omega)$ *satisfy the conditions* $f = 0$, $\operatorname{grad} f = 0$ *on* E. *Then there exists a function* $\Phi \in C^1_{00}(\Omega)$ *such that*

a) $\Phi = 0$ *on* E;

b) $\operatorname{grad} \Phi = 0$ *on* E;

c) $\Phi \geqslant 0$ *on some neighbourhood of* E;

d) $\Phi - f \geqslant 0$ *on some neighbourhood of* E.

Proof. — It is enough to carry out the proof for the case that $E \cap \operatorname{supp} f$ consists of one single point, say 0.

$$\text{Let } \psi(r) = \sup_{|x| \leqslant r} |\operatorname{grad} f(x)|.$$

ψ is nondecreasing, continuous and $\lim_{r\to 0}\psi(r) = 0$. Take

$$\Phi(x) = \Theta(x)\int_0^{|x|}\psi(r)\,dr$$

where $\Theta \in C^1_{00}(\Omega)$ and $\Theta \equiv 1$ in a neighbourhood of 0. Hence, for small enough x, we have

$$|f(x)| = \left|\int_0^{|x|}\frac{\delta f}{\delta r}\,dr\right| \leqslant \int_0^{|x|}\psi(r)\,dr = \Phi(x)$$

which proves the lemma.

Next let us study $N_0(\lambda\Phi - f)$ when f and Φ are as in the previous lemma. For $\lambda \geqslant 1$ we have $\lambda\Phi - f \geqslant 0$ in a neighbourhood of E. Consequently lemma 3.5 gives

$$0 = N_0(\lambda\Phi - f) = \lambda^2 N_0(\Phi) - 2\lambda N_0(\Phi, f) + N_0(f).$$

But $\lambda \geqslant 1$ was arbitrary, so the polynomial must vanish identically. Therefore $N_0(f) = 0$ whenever $f = 0$, grad $f = 0$ on E.

Next if g is any function in $C^1_{00}(\Omega)$ we can write

$$g = \varphi + f$$

where φ is affine on a neighbourhood of E and $f = 0$, grad $f = 0$ on E. We are going to show that $N_0(g) = 0$. It is no restriction to assume that $\operatorname{supp} g \cap E = \{0\}$. Let

$$\psi(x) = \begin{cases} 1 & \text{for } |x| \leqslant 1, \\ 0 & \text{for } |x| \geqslant 2, \end{cases}$$

$$0 \leqslant \psi \leqslant 1, \quad \psi \in C^1.$$

$$\psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right).$$

Then if $f_\varepsilon = \psi_\varepsilon f$ we have $\|\operatorname{grad} f_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$.

Hence $N_1(f_\varepsilon) \to 0$ and $N_1(f_\varepsilon, h) \to 0$ as $\varepsilon \to 0$ for every $h \in C^1_{00}(\Omega)$. Now

$$N_0(g) = N_0(f) + N_0(\varphi) + 2N_0(\varphi, f) = 2N_0(\varphi, f) = 2N_0(\varphi, f_\varepsilon).$$

But according to Schwarz' inequality we have

$$|N_0(\varphi, f_\varepsilon) + N_1(\varphi, f_\varepsilon)| = |N(\varphi, f_\varepsilon)| \leqslant (N(\varphi))^{1/2}(N(f_\varepsilon))^{1/2}$$
$$= (N(\varphi))^{1/2}(N_0(f_\varepsilon) + N_1(f_\varepsilon))^{1/2} = (N(\varphi))^{1/2}(N_1(f_\varepsilon))^{1/2} \to 0$$

as $\varepsilon \to 0$.

Thus

$$N_0(g) = 2(N_0(\varphi, f_\varepsilon) + N_1(\varphi, f_\varepsilon)) - 2N_1(\varphi, f_\varepsilon) \to 0$$

as $\varepsilon \to 0$.

Hence $N_0(g) = 0$ for all $g \in C^1_{00}(\Omega)$. To complete the proof of theorem 2.3 we observe that

$$N_0(f, g) = \frac{1}{4}(N_0(f+g) - N_0(f-g)) = 0$$

for all $f, g \in C^1_{00}(\Omega)$.

Proof of theorem 2.4. — If we restrict N to $D(\Omega)$, the space of all infinitely differentiable realvalued test functions on Ω with compact support, then N is a continuous bilinear functional. We use the kernel theorem (see for example [6]) to conclude that $N(f, g) = \langle f(x)g(y), T(x, y)\rangle$, where $f, g \in D(\Omega)$, $f(x)g(y) \in D(\Omega \times \Omega)$ and $T(x, y)$ is some distribution in $D'(\Omega \times \Omega)$. Furthermore, the locality of N implies that $\operatorname{supp} T \subset \operatorname{diag}(\Omega \times \Omega) = \{(x, y) \in \Omega \times \Omega : x = y\}$.

Now let $K \subset \Omega$ be compact. $\mathring{K}$ is the interior of K. If $\operatorname{supp} f$, $\operatorname{supp} g \subset \mathring{K}$, then we have

$$\langle f(x)g(y), T(x, y)\rangle = \sum_{\substack{\alpha,\beta \\ |\alpha|+|\beta|\leqslant m(K)}} \langle D^\alpha_x D^\beta_y (f(x)g(y)), \sigma^{\alpha,\beta}_K(x, y)\rangle$$

where $\sigma^{\alpha,\beta}_K$ are measures on $\mathring{K} \times \mathring{K}$, $m(K)$ denotes the order of T when restricted to $D(\mathring{K} \times \mathring{K})$. Furthermore, it is possible to choose the measures $\sigma^{\alpha,\beta}_K$ such that

$$\operatorname{supp} \sigma^{\alpha,\beta}_K \subset \operatorname{diag}(\mathring{K} \times \mathring{K}).$$

This follows from the general fact that if a distribution T on R^n has support on a linear subspace of R^n, then the measures « representing » T can be choosen to have supports on that same linear subspace (see [8], chapter 3, § 10).

Thus we have

$$N(f, g) = \sum_{\substack{\alpha,\beta \\ |\alpha|+|\beta|\leqslant m(K)}} \langle D^\alpha f(x) D^\beta g(x), \sigma^{\alpha,\beta}_K(x)\rangle$$

where $\sigma_K^{\alpha,\beta}$ are measures on $\mathring{K}$. We assumed that

$$|N(f, g)| \leqslant C_K \|\text{grad } f\|_\infty \|\text{grad } g\|_\infty,$$

when $\operatorname{supp} f$, $\operatorname{supp} g \subset \mathring{K}$.

LEMMA 3.6. — *If* N *is symmetric, local, and if*

(i) $N(f, g) = \sum_{\substack{\alpha,\beta \\ |\alpha|+|\beta| \leqslant m(K)}} \langle D^\alpha f(x) D^\beta g(x), T_K^{\alpha\beta}(x)\rangle$, $T_K^{\alpha\beta}$ *distributions on* $\mathring{K}$,

(ii) $|N(f, g)| \leqslant C_k \|\text{grad } f\|_\infty \|\text{grad } g\|_\infty$ *when* $f, g \in D(\mathring{K})$ *then there exist uniquely defined distributions* $T_{ij,K}$ *(on* $\mathring{K}$*) such that* $T_{ij,K} = T_{ji,K}$ *and*

$$N(f, g) = \sum_{ij} \left\langle \frac{\delta f}{\delta x_i} \frac{\delta g}{\delta x_j}, T_{ij,K} \right\rangle, \; f, g \in D(\mathring{K}).$$

The fact that $T_{ij,K}$ are uniquely defined, implies that we obtain distributions T_{ij} on $D(\Omega)$ such that

$$N(f, g) = \sum_{ij} \left\langle \frac{\delta f}{\delta x_i} \frac{\delta g}{\delta x_j}, T_{ij} \right\rangle$$

for all $f, g \in D(\Omega)$ $(T_{ij,K} = T_{ij|\mathring{K}})$.

Proof. — To simplify notations let us write $T^{\alpha\beta}$, T_{ij}, m instead of $T_K^{\alpha\beta}$, $T_{ij,K}$, $m(K)$ and let all functions f, g be supported in $\mathring{K}$.

Let $\varphi \in C^\infty$ be any fixed function with $\operatorname{supp} \varphi \subset \mathring{K}$.

Take $f(x) = \varphi(x) e^{-i\langle\lambda, x\rangle}$, $g(x) = \varphi(x) e^{i\langle\mu, x\rangle}$.

(So far we have assumed that the bilinear forms act on real functions, but by linearity we can of course extend them to complex valued functions.) Then

$$\begin{aligned} D^\alpha f(x) &= (-i)^{|\alpha|} \lambda^\alpha \varphi(x) e^{-i\langle\lambda, x\rangle} + 0(|\lambda|^{|\alpha|-1}), \\ D^\beta g(x) &= i^{|\beta|} \mu^\beta \varphi(x) e^{i\langle\mu, x\rangle} + 0(|\mu|^{|\beta|-1}), \end{aligned}$$

for large values of $|\lambda|$ and $|\mu|$.

$$\begin{aligned} \|\text{grad } f\|_\infty &\leqslant \|\text{grad } \varphi\|_\infty + \|\varphi\|_\infty |\lambda| \\ \|\text{grad } g\|_\infty &\leqslant \|\text{grad } \varphi\|_\infty + \|\varphi\|_\infty |\mu| \end{aligned}$$

$$N(f, g) = \sum_{|\alpha|+|\beta| \leqslant m} (-i)^{|\alpha|+|\beta|} (-1)^{|\beta|} \lambda^\alpha \mu^\beta \langle e^{-i\langle\lambda-\mu, x\rangle} \varphi^2, T^{\alpha\beta}\rangle + \cdots,$$

where the dots stand for terms containing derivatives of φ.

Let $\lambda - \mu = \tau$, $\mu = \lambda - \tau$ and let τ be fixed. Then

$$N(f, g) = \sum_{|\alpha|+|\beta|=m} (-i)^m(-1)^{|\beta|}\lambda^\alpha(\lambda-\tau)^\beta\langle e^{-i\langle\tau, x\rangle}\varphi^2, T^{\alpha\beta}\rangle + 0(|\lambda|^{m-1}).$$

$$N(f, g) = \sum_{|\alpha|+|\beta|=m} (-i)^m(-1)^{|\beta|}\lambda^\alpha(\lambda-\tau)^\beta(T^{\alpha\beta}\varphi^2)\hat{}(\tau) + 0(|\lambda|^{m-1}).$$

$$|N(f, g)| \leqslant C_K\|\text{grad } f\|_\infty\|\text{grad } g\|_\infty = 0(|\lambda|^2), \text{ as } |\lambda| \to \infty.$$

The first expression shows that $N(f, g)$ is a polynomial in λ (for fixed τ) and the last that this polynomial is of degree at most two.

Thus, if $m > 2$ and if ν, with $|\nu| = m$, is a fixed index then the coefficient of λ^ν must vanish.

The coefficient of λ^ν is

$$(-1)^m \sum_{\alpha+\beta=\nu} (T^{\alpha\beta}\varphi^2)\hat{}(\tau)(-1)^{|\beta|}.$$

($\hat{}$ denotes the Fourier transform.)

$$\therefore \sum_{\alpha+\beta=\nu} (-1)^{|\beta|}(T^{\alpha\beta}\varphi^2)\hat{}(\tau) = 0.$$

$$\therefore \sum_{\alpha+\beta=\nu} (-1)^{|\beta|}T^{\alpha\beta}\varphi^2(x) = 0.$$

$$\therefore \sum_{\alpha+\beta=\nu} (-1)^{|\beta|}T^{\alpha\beta} = 0, \text{ as } \varphi \text{ was arbitrary.}$$

Now we make repeated partial integrations in the representation

$$N(f, g) = \sum_{|\alpha|+|\beta|\leqslant m} \langle D^\alpha f(x)D^\beta g(x), T^{\alpha\beta}(x)\rangle$$

and in each step we use the relations

$$\sum_{\alpha+\beta=\nu} (-1)^{|\beta|}T^{\alpha\beta} = 0$$

for the terms of the highest order. Then we can reduce the expression to the form

$$N(f, g) = \sum_{|\alpha|+|\beta|\leqslant 2} \langle D^\alpha f D^\beta g, T^{\alpha\beta}\rangle$$

(with abuse of notations).

A few more fairly simple manipulations (the symmetry of $N(f, g)$ must be used, the details are in [2]) show that we can actually write this

$$N(f, g) = \sum_{i,j} \left\langle \frac{\delta f}{\delta x_i}\frac{\delta g}{\delta x_j}, T_{ij}\right\rangle,$$

where $T_{ij} = T_{ji}$ are distributions on $\dot{K}$. The uniqueness of T_{ij} follows from the following relation, which is easily verified

$$2\langle f, T_{ij}\rangle = N(x_i f, x_j\theta) + N(x_j f, x_i\theta) - N(f, x_i x_j\theta).$$

Here θ is a function such that $\theta \equiv 1$ on a neighbourhood of supp f. Thus lemma 3.6 is proved.

LEMMA 3.7. — *The distributions* T_{ij} *in lemma* 3.6 *are Radon measures on* Ω.

Proof.

$$N(f, g) = \sum_{ij} \left\langle \frac{\delta f}{\delta x_i} \frac{\delta g}{\delta x_j}, T_{ij} \right\rangle, \; T_{ij} = T_{ji}$$

and

$$|N(f, g)| \leqslant C_K \|\text{grad } f\|_\infty \|\text{grad } g\|_\infty$$

when supp f and supp $g \subset K$.

In these formulas we now let $f(x) = f_1(x)e^{i\lambda x_1}$ and

$$g(x) = e^{-i\lambda x_1}\theta(x)$$

where $\theta \equiv 1$ on a neighbourhood of supp f, supp $f \subset \dot{K}$. We can also have $0 \leqslant \theta \leqslant 1$ and supp $\theta \subset \dot{K}$

$$\|\text{grad } f\|_\infty \leqslant \|\text{grad } f_1\|_\infty + |\lambda| \|f_1\|_\infty.$$
$$\|\text{grad } g\|_\infty \leqslant \|\text{grad } \theta\|_\infty + |\lambda| \|\theta\|_\infty = \|\text{grad } \theta\|_\infty + |\lambda|.$$

Then

$$N(f, g) = \lambda^2 \langle f_1\theta, T_{11}\rangle + 0(|\lambda|)$$

for large $\lambda(f_1$ and θ fixed). Also

$$|N(f, g)| \leqslant C_K\{\lambda^2\|f_1\| + 0(|\lambda|)\}.$$
$$\therefore |\lambda^2\langle f_1\theta, T_{11}\rangle + 0(\lambda)| \leqslant C_K\{\lambda^2\|f_1\|_\infty + 0(\lambda)\}.$$

Divide by λ and let $\lambda \to \infty$. Then

$$|\langle f_1, T_{11}\rangle| \leqslant C_K\|f_1\|_\infty$$

which proves that T_{11} is a Radon measure σ_{11} and that

$$\int_{\dot{K}} |d\sigma_{11}| \leqslant C_K.$$

For the same reason T_{ii} are measures σ_{ii}. That also T_{ij} are measures follows after a change of coordinates. The details are omitted.

LEMMA 3.8. — *If* N *is also positive, then* $\sum_{ij} h_i h_j \sigma_{ij}$ *is a positive measure for all continuous functions* $h_i \in C_{00}(\Omega)$, $i = 1, 2, \ldots, n$. *This is equivalent to saying that the matrix* $\{\sigma_{ij}(B)\}_{ij}$ *is positive semidefinite for every compact Borelset* $B \subset \Omega$.

The proof is based on a similar technique as earlier and is omitted. For details see [2].

This completes the proof of theorem 2.4.

Corollary 2.5 is an immediate consequence of theorems 2.1, 2.3 and 2.4.

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BIBLIOGRAPHY

[1] G. ALLAIN, Sur la représentation des formes de Dirichlet, Université Paris XI, Thèse de 3e cycle, Septembre 1973.

[2] L.-E. ANDERSSON, On the representation of Dirichlet forms, Report n° 6, 1974, Institut Mittag-Leffler.

[3] A. BEURLING and J. DENY, Dirichlet spaces, *Proc. Nat. Ac. Sc.*, 45 (1959), 208-215.

[4] J. DENY, Méthodes hilbertiennes en théorie du Potentiel, *cours du C.I.M.E.*, 1969.

[5] J. DENY, Principe complet du maximum et contraction, *Ann. Inst. Fourier*, 15-1 (1965), 259-271.

[6] I. M. GELFAND and N. Ya VILENKIN, Generalized functions, volume 4, Academic press, 1964.

[7] J. PEETRE, Rectification à l'article « une caractérisation abstraite des opérateurs différentiels », *Math. Scand.*, 8 (1960), 116-120.

[8] L. SCHWARTZ, Théorie des distributions, Hermann, Paris, 1973.

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