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RICHARD SACKSTEDER

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A REMARK ON THURSTON'S STABILITY THEOREM

by Richard SACKSTEDER

Let L be a compact leaf of a smooth transversally oriented foliation of codimension one. Thurston [4] has generalized Reeb's stability theorem by showing that if $H_1(L, R) = 0$, then all nearby leaves are diffeomorphic to L. His theorem answers, for oriented foliations, a question posed by Reeb [2]. If it were true, as has been erroneously asserted in the literature [3, p. 96], that $H_1(L, R) = 0$ implies that $H_1(L', R) = 0$ when L' is a 2-fold cover of L, then Thurston's assymption of transversal orientability would be unnecessary. However the assertion is false (cf. [1, p. 410]), as has been pointed out to the author painfully often.

In fact, the example below shows that Thurston's theorem cannot be generalized to non-oriented foliations, since in the example there is a compact leaf L with $H_1(L, R) = 0$, but which has a neighborhood in which all leaves are non-compact.

The universal covering of L is $S^2 \times R^1$ and $\pi = \pi_1(L)$ is the semi-direct product of $\mathbf{Z}_2 = \{-1, +1\}$ and the integers $\mathbf{Z}$, where $\mathbf{Z}_2$ acts on Z in the obvious way. The product of elements of π is given by

$$(w_1, n_1) \cdot (w_2, n_2) = (w_1 w_2, w_1 n_2 + n_1),$$

where w_i is in $\mathbf{Z}_2$ and n_i is in $\mathbf{Z}$. The action ϕ of π on $S^2 \times R$ is given by

$$\phi((w, n); (s, r)) = (ws, wr + n), \qquad \text{where } s \to -s$$

is the antipodal map of S^2. The quotient L is an oriented manifold that is easily seen to have the properties that $H_1(L, Z) = \mathbf{Z}_2$, hence $H_1(L, R) = 0$, and $S^2 \times S^1$ is a 2-fold cover of L.

To define a foliation of a neighborhood of L it suffices to define a representation ψ of π by C^∞ diffeomorphisms of neighborhoods of $0 \in R$. Let f be any C^∞ diffeomorphism of R satisfying :

$$f(0) = 0, f'(0) = 1, f^{(n)}(0) = 0 \quad \text{if} \quad n > 1, f(x) < x$$

for $x \neq 0$, and

(1) $f(x) = -f^{-1}(-x)$, hence $f^n(x) = -f^{-n}(-x)$ for $n = 0, \pm 1, \ldots$.

An f satisfying these conditions is easily defined for $x \geqq 0$ and can be extended to $x < 0$ by (1). It is easy to check that the derivatives match at 0 so the extended map is C^∞. The second half of (1) shows that $\psi(w, n)(x) = f^n(wx)$ defines a representation of π with the desired properties. The leaves, other than L itself, of the foliation defined by ψ are non-compact, since $f^n(wx) = x$ can only occur if $(w, n) = (1, 0)$, or $x = 0$.

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Richard SACKSTEDER,
The Graduate School and University Center
of the City University of New York
Ph. D. Program in Mathematics
Graduate Center 33 West 42 Street
New York, N.Y. 10036 (U.S.A.).