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CHARLES FEFFERMAN

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CONVERGENCE ON ALMOST EVERY LINE FOR FUNCTIONS WITH GRADIENT IN $L^p(\mathbb{R}^n)$

by Charles **FEFFERMAN**

This note answers a question asked by L.D. Kudrjačev ([1], p. 264, problem 1). The question was suggested by the following result of Uspenskiĭ [2] : if $u(x)$ is a smooth function $\mathbb{R}^n \rightarrow \mathbb{R}$ and $\int_{\mathbb{R}^n} |\text{grad } u|^p dx < \infty$ ($1 < p < n$), there exists a constant c such that $\lim_{r \rightarrow \infty} u(rx') = c$ for almost every $x' \in S^{n-1}$. Professor Kudrjačev kindly informed me that V. Portnov answered his question independently by another method [3].

We use now the notation

$$x = (x_1, x'), \quad \text{where } x_1 \in \mathbb{R} \quad \text{and} \quad x' \in \mathbb{R}^{n-1}.$$

THEOREM. — *Let $u(x_1, x')$ be a smooth function : $\mathbb{R}^n \rightarrow \mathbb{R}$ and suppose $\int_{\mathbb{R}^n} |\text{grad } u|^p dx < \infty$ ($1 < p < n$). Then for a constant c , $\lim_{x_1 \rightarrow \infty} u(x_1, x') = c$ for almost all x' .*

Prof. — Set $u_j = \frac{\partial u}{\partial x_j}$, let R_j denote j^{th} Riesz transform, and let I^1 denote fractional integration of first order in \mathbb{R}^n .

We begin with the standard formula

$$\sum_i R_i R_j v_j = v_i + \sum_j R_j I^1 \left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} \right),$$

valid for C^∞ functions of compact support. (To check this, just take Fourier transforms). Take a function φ_N on \mathbb{R}^n equal to one for $|x| \leq 2^N$, supported in $|x| \leq 2^{N+1}$, and satisfying $|\nabla \varphi_N| \leq C 2^{-N}$; and apply the above formula to the functions $v_j = \varphi_N u_j \in C_0^\infty$. As

$N \rightarrow \infty$, the left-hand side of the formula tends to $\sum_j R_i R_j u_j$ in L^p , since the Riesz transforms are bounded on L^p . On the other hand, $\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} = - \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \varphi_N + u_j \frac{\partial \varphi_N}{\partial x_i} - u \frac{\partial \varphi_N}{\partial x_j}$, and the first term is zero since (u_i) is a gradient. Thus,

$$\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} \leq \| \text{grad } u \|_p \| \text{grad } \varphi_N \|_\infty \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Since $R_j I^1$ is bounded from L^p to L^s with $\frac{1}{s} = \frac{1}{p} - \frac{1}{n}$ ($1 < s < \infty$ since $1 < p < n$), we see that as $N \rightarrow \infty$, the final term in our formula tends to zero in L^s . Thus, taking the limit in measure of both sides of our formula, we obtain $u_i = \sum_j R_i R_j u_j$.

In other words, setting $f = \sum_j R_j \frac{\partial u}{\partial x_j} \in L^p$, we have $u_i = R_i f$.

Now for $f \in L^p$, we know that $\frac{\partial}{\partial x_i} (I^1 f) = R_i f$ in the sense of tempered distributions. (It's trivial for $f \in C_0^\infty$; in general we express f as the limit in L^p of smooth compactly supported f_N as $N \rightarrow \infty$. $R_i f_N \rightarrow R_i f$ in L^p and hence weakly as distributions, and $I^1 f_N \rightarrow I^1 f$ in L^s and hence weakly, so that $\frac{\partial}{\partial x_i} (I^1 f_N)$ also converges weakly to $\frac{\partial}{\partial x_i} (I^1 f)$.)

On the other hand, the Riesz transforms or I^{rst} fractional integral of a smooth function in L^p are smooth functions, so that $\frac{\partial}{\partial x_i} (u - I^1 f) = R_i f - R_i f = 0$, not just as distributions, but as pointwise derivatives of smooth functions. Thus $u = \text{constant} + I^1 f$. So to prove the claim, it will be enough to show that

$$\lim_{x_1 \rightarrow \infty} I^1 f(x_1, x') = 0$$

for almost all x' , whenever f is a smooth function in L^p .

Now

$$\begin{aligned} I^1 f(x_1, x') &= \int_{\mathbb{R}^n} \frac{f(x) dy}{|x-y|^{n-1}} = \int_{|x-y| < 1} \frac{f(y) dy}{|x-y|^{n-1}} + \\ &\quad + \int_{|x-y| \geq 1} \frac{f(y) dy}{|x-y|^{n-1}} = I + II. \end{aligned}$$

Then II is easy ; it tends to zero as $x \rightarrow \infty$. (In fact, we just breakup f into $f = g + b$ where $\|g\|_p \leq \|f\|_p$ and g lives on a bounded set, while $\|b\|_p < \epsilon$. The contribution of g to II clearly tends to zero, while b produces at most

$$\begin{aligned} \|b\|_p \cdot \left(\int_{|x-y| \geq 1} |x-y|^{-(n-1)q} dy \right) \left[\frac{1}{p} + \frac{1}{q} = 1 \right] &\leq \\ &\leq C \|b\|_p \leq C \epsilon, \quad \text{since } p < n. \end{aligned}$$

So the main problem is term I.

The main step in dealing with term I is to prove the

LEMMA. — Let $f \in L^p(Q)$ where Q is a cube of side 10, and $1 < p < n$. Say $Q = [0, 10] \times Q' \subseteq \mathbb{R}^1 \times \mathbb{R}^{n-1}$. Then

$$M(x') = \sup_{0 \leq x_1 \leq 10} \left| \int_Q \frac{f(y_1, y') dy_1 dy'}{((y_1 - x_1)^2 + (y' - x')^2)^{\frac{n-1}{2}}} \right|$$

belongs to $L^p(Q')$, and $\|M\|_p \leq C \|f\|_p$.

Proof. — Say $f \geq 0$ and $f = 0$ outside Q . Let $f^{\downarrow}(\cdot, y')$ be the decreasing re-arrangement of $f(\cdot, y')$ on $[0, 10]$, for each fixed $y' \in Q'$. Of course $\|f\|_p = \|f^{\downarrow}\|_p$. We claim that $Mf(x') \leq 2Mf^{\downarrow}(x')$ pointwise. In fact, fix x', x_1, y' , and consider

$$\int_{y_1 > x_1} \frac{f(y_1, y') dy_1}{((y_1 - x_1)^2 + (y' - x')^2)^{\frac{n-1}{2}}}.$$

Since $\chi_{\{y_1 | y_1 > x_1\}}(y_1) \cdot ((y_1 - x_1)^2 + (y' - x')^2)^{-\frac{(n-1)}{2}}$ is monotone decreasing in the interval (x_1, ∞) in which it is supported, we know that

$$\int_{y_1 > x_1} \frac{f(y_1, y') dy_1}{((y_1 - x_1)^2 + (y' - x')^2)^{\frac{n-1}{2}}} \leq \int_0^{10} \frac{f^+(y_1, y') dy_1}{((y_1)^2 + (y' - x')^2)^{\frac{n-1}{2}}}.$$

Integrating over y' yields

$$\int_{\substack{y_1 > x_1 \\ y' \in Q'}} \frac{f(y_1, y') dy_1}{((y_1 - x_1)^2 + (y' - x')^2)^{\frac{n-1}{2}}} \leq \int_Q \frac{f^+(y_1, y') dy_1}{(y_1^2 + (y' - x')^2)^{\frac{n-1}{2}}}.$$

Similarly,

$$\int_{\substack{y_1 < x_1 \\ y' \in Q'}} \frac{f(y_1, y') dy_1 dy'}{((y_1 - x_1)^2 + (y' - x')^2)^{\frac{n-1}{2}}} \leq \int_Q \frac{f^+(y_1, y') dy_1 dy'}{(y_1 + (y' - x')^2)^{\frac{n-1}{2}}}.$$

Adding and taking the sup over x_1 gives us

$$Mf(x') \leq 2 \int_Q \frac{f^+(y_1, y') dy_1 dy'}{(y_1^2 + (y' - x')^2)^{\frac{n-1}{2}}}. \quad (*)$$

which is stronger than the claim.

We shall use what we proved in full strength. For a fixed y_1 , the function $M_{y_1}(x') = \int_{y' \in Q'} \frac{f^+(y_1, y') dy'}{(y_1^2 + (y' - x')^2)^{\frac{n-1}{2}}}$ is just the convolution of $f^+(y_1, \cdot)$ with $(y_1^2 + (y' - x')^2)^{-\frac{n-1}{2}}$ on the cube $Q' \subseteq \mathbb{R}^{n-1}$. Thus

$$\begin{aligned} \|M_{y_1}(\cdot)\|_p &\leq \|\text{convolution kernel}_{y_1}\|_{L^1(Q')} \|f(y_1, \cdot)\|_p \sim \\ &\sim C \left(\log \frac{1}{|y_1|} \right) \|f^+(y_1, \cdot)\|. \end{aligned}$$

Therefore by estimate (*), $Mf(x') \leq \int_0^{10} M_{y_1}(x') dy_1$, and

$$\begin{aligned} \|Mf\|_p &\leq \int_0^{10} C \left(\log \frac{1}{|y_1|} \right) \|f^+(y_1, \cdot)\|_p dy_1 \leq \\ &\leq \left(\int_0^{10} \left(C \log \frac{1}{|y_1|} \right)^q dy_1 \right)^{1/q} \left(\int_0^{10} \|f(y_1, \cdot)\|_p^p dy_1 \right)^{1/p} \end{aligned}$$

(Note : $q < \infty$ since $p > 1$) $\leq C \|f\|_p = C \|f\|_p$.

Q.E.D.

Now we return to term I above. Divide R^{n-1} into a mesh of cubes of side 2, $R^{n-1} = \bigcup_j Q'_j$, and write $R^n = \bigcup_{i,j} Q_{ij}$ where

$$Q_{ij} = \{(x_1, x') \in R^n \mid x' \in Q_j, 2i \leq x_1 < 2(i+1)\} \quad (-\infty < i < \infty).$$

Then let Q^* be the cube concentric with Q_{ij} but with side 10, and set $f_{ij} = f \chi_{Q^*_{ij}}$. For

$$g_{ij}(x') = \begin{cases} \sup_{2i < x_1 < 2(i+1)} \int_{|x-y| < 1} \frac{|f(y_1, y')| dy_1 dy'}{|x-y|^{n-1}}, \\ 0 \text{ otherwise} \end{cases}$$

where $x = (x_1, x')$ if $x' \in Q'_j$ we have $\|g_{ij}\|_{L^p(R^{n-1})} \leq C \|f_{ij}\|_{L^p(R^n)}$ by the lemma. For

$$\begin{aligned} x' \in Q_j, \sup_{x_1 \in R^1} \int_{\substack{|x-y| < 1 \\ \text{where } x=(x_1, x')}} \frac{|f(y)| dy}{|x-y|^{\frac{n-1}{2}}} &= \\ &= \sup_i g_{ij}(x') \leq \left(\sum_i g_{ij}^p(x') \right)^{1/p}. \end{aligned}$$

So certainly

$$\begin{aligned} \int_{R^{n-1}} \left[\sup_{x_1 \in R^1} \int_{|(x_1, x')-y| < 1} \frac{|f(y)| dy}{|x-y|^{n-1}} \right]^p dx' &= \\ = \sum_j \int_{x' \in Q'_j} [\sup (\text{etc. etc.})]^p dx' &\leq \sum_j \int_{R^{n-1}} \left(\sum_i g_{ij}^p(x') \right)^{p/p} dx' = \\ = \sum_{i,j} \int_{R^{n-1}} g_{ij}^p(x') dx' &\leq C \sum_{i,j} \|f_{ij}\|_p^p \leq C \|f\|_{L^p(R^n)}^p. \end{aligned}$$

In other words, if

$$N(x') = Nf(x') = \sup_{x_1 \in R^1} \int_{|(x_1, x')-y| < 1} \frac{|f(y)| dy}{|x-y|^{n-1}},$$

then

$$\|Nf\|_{L^p(R^{n-1})} \leq C \|f\|_{L^p(R^n)}.$$

Now term I becomes easy. Again split up if $f = g + b$ where g is supported in a bounded set and $\|b\|_p < \epsilon$.

In evaluating the claim

$$\lim_{x_1 \rightarrow \infty} \int_{|(x_1, x') - y| < 1} \frac{|f(y)| dy}{|x - y|^{n-1}} = 0, ((x_1, x') = x)$$

we find that g makes no contribution at all to the last integral for x_1 large enough. On the other hand, for b we know that

$$\left| \limsup_{x_1 \rightarrow \infty} \int_{|(x_1, x') - y| < 1} \frac{|f(y)| dy}{|(x_1, x') - y|^{n-1}} \right|_{L^p(dx')} < C \epsilon \quad \text{by} \quad (**)$$

Since $\epsilon > 0$ is arbitrary, the proof is complete.

Q.E.D.

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Charles FEFFERMAN,
 Departement of Mathematics
 University of Chicago
 5734 University Avenue
 Chicago, Ill. 60637.