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MANUEL VALDIVIA

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SOME CHARACTERIZATIONS OF ULTRABORNOLOGICAL SPACES *

by Manuel VALDIVIA

In this paper we show that an ultrabornological space E is the inductive limit of a family of nuclear Fréchet spaces and we prove also that E is the inductive limit of a family of nuclear (DF)-spaces.

The vector spaces that we use here are defined over the field K of the real or complex numbers. With the word "space" we shall mean "separated locally convex spaces". Given the space E, then E′ is its topological dual. We denote by $\sigma(\mathrm{E}, \mathrm{E}')$ and $\mu(\mathrm{E}, \mathrm{E}')$ the weak and Mackey topologies, respectively, on E. If A is a bounded closed absolutely convex set in the space E, then E_A is the normed space on the linear hull of A, with A as closed unit ball. If C is a compact set, with non-empty interior, in the n-dimensional euclidean space R^n, $\mathcal{D}_\mathrm{C}$ is the space of all the real or complex valued functions, infinitely differentiable, with compact support contained in C, provided with the topology of the uniform convergence on all the derivatives of order q, $q = 0, 1, 2, \ldots$ $\mathcal{D}'_\mathrm{C}$ is the topological dual of $\mathcal{D}_\mathrm{C}$, with the strong topology. $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are the well-known spaces of L. Schwartz, with the strong topologies, being Ω an open set of R^n. If $x = (x_1, x_2, \ldots, x_n)$ is a point of R^n and $p = (p_1, p_2, \ldots, p_n)$, being p_j a non-negative integer, $j = 1, 2, \ldots, n$, then x^p denotes $x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$, $|p| = p_1 + p_2 + \cdots + p_n$ and $dx = dx_1 dx_2 \ldots dx_n$.

In [4] we have shown the following result : a) *Let* E *be a Banach space. If* $\{x_n\}$ *is a sequence in* E, *such that, for every positive integer p, the sequence* $\{2^{pn} x_n\}$ *converges to the origin then there is in* E *a compact absolutely convex set* B, *so that* E_B *is a Hilbert space and in* E_B $\{x_n\}$ *is a sequence such that* $\{2^{pn} x_n\}$ *converges to the origin.*

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We follow the same way, as we did in [4] for result a), to prove Lemma 1.

LEMMA 1. – *Let* E *be a Banach space. Let* $\{\lambda_n\}$ *be a strictly increasing sequence of positive integers. If* $\{x_n\}$ *is a sequence in* E, *such that, for every positive integer p, the sequence* $\{\lambda_{pn}^p x_n\}$ *converges to the origin, then there is in* E *a weakly compact absolutely convex set* B, *so that* E_B *is a Hilbert space and in* $E_B \{x_n\}$ *is a sequence such that* $\{\lambda_{pn}^p x_n\}$ *converges to the origin.*

Proof. – Clearly, it is sufficient to carry out the proof when $x_n \neq 0$, $n = 1, 2, \ldots$, which we are going to suppose. Let f be the linear mapping of l^2 into E such that

$$f(\{a_n\}) = \sum_{n=1}^{\infty} a_n (x_n / \| x_n \|^{1/2}) .$$

Since $\{\lambda_{2n}^2 x_n\}$ converges to the origin and $\sum_{n=1}^{\infty} \lambda_{2n}^{-2} < \infty$, then $\sum_{n=1}^{\infty} \| x_n \|$ is convergent. On the other hand

$$\left\| \sum_{n=q}^{\infty} a_n (x_n / \| x_n \|^{1/2}) \right\| \leq \left(\sum_{n=q}^{\infty} | a_n |^2 \right)^{1/2} \left(\sum_{n=q}^{\infty} \| x_n \| \right)^{1/2} .$$

Therefore, f is well defined and maps every bounded set of l^2 in a bounded set of E and, thus, f is continuous. Let g be the canonical mapping of l^2 onto $l^2/f^{-1}(0)$. If $f = h \circ g$, then h is a continuous injective linear mapping of the Hilbert space $l^2/f^{-1}(0)$ in E. If U is the closed unit ball of l^2, then $g(\mathrm{U})$ is the closed unity ball of $l^2/f^{-1}(0)$ and, therefore,

$$\mathrm{B} = h(g(\mathrm{U})) = f(\mathrm{U}) = \left\{ \sum_{n=1}^{\infty} a_n (x_n / \| x_n \|^{1/2}) : \sum_{n=1}^{\infty} | a_n |^2 \leq 1, \right. \\ \left. a_n \in \mathrm{K},\ n = 1, 2, \ldots \right\} ,$$

hence E_B can be identified with $l^2/f^{-1}(0)$. Given two positive integers p and r, there exists a positive integer n_1 such that $r\lambda_{pn}^p \| x_n \|^{1/2} < 1$, $n \geq n_1$. Since $x_n / \| x_n \|^{1/2} \in \mathrm{B}$, $n = 1, 2, \ldots$, we have that for $n \geq n_1$

$$\lambda_{pn}^{p} x_n = \lambda_{pn}^{p}(x_n/\|x_n\|^{1/2})\,\|x_n\|^{1/2} \in \lambda_{pn}^{p}\,\|x_n\|^{1/2}\,\mathrm{B} =$$

$$= r\lambda_{pn}^{p}\,\|x_n\|^{1/2}(1/r)\,\mathrm{B} \subset (1/r)\,\mathrm{B}\ ,$$

and, therefore, $\{\lambda_{pn}^{p} x_n\}$ converges to the origin in $\mathrm{E_B}$. Finally, B is weakly compact in E, since U is weakly compact in l^2 and $\mathrm{B} = f(\mathrm{U})$.
Q.E.D.

In [4] we have proved the following result : b) *Let* E *be a Hilbert space of infinite dimension. If* $\{x_n\}$ *is a sequence in* E *such that, for every positive integer p, the sequence* $\{2^{pn} x_n\}$ *converges to the origin, then there is in* E *an orthogonal sequence* $\{y_n\}$, *so that its closed absolutely convex hull contains* $\{x_n\}$ *and* $\{2^{pn} y_n\}$ *converges to the origin.*

We follow the same way, as we did in [4] for result b), to prove Lemma 2.

LEMMA 2. – *Let* $\{\lambda_n\}$ *be a strictly increasing sequence of positive integers. Let* E *be a Hilbert space of infinite dimension. If* $\{x_n\}$ *is a sequence in* E, *such that, for every positive integer p, the sequence* $\{\lambda_{pn}^{p} x_n\}$ *converges to the origin, then there is in* E *an orthogonal sequence* $\{u_n\}$ *so that its closed absolutely convex hull contains* $\{x_n\}$ *and* $\{\lambda_{pn}^{p} u_n\}$ *converges to the origin.*

Proof. – Since E has infinite dimension, we can choose a sequence $\{y_n\}$ in E, $y_n \neq 0$, $n = 1, 2, \ldots$, with infinite dimensional linear hull, such that the closed absolutely convex hull of the sequence $\{y_n\}$ contains $\{x_n\}$, and so that, for every positive integer p, the sequence $\{\lambda_{pn}^{p} y_n\}$ converges to the origin. By induction we select an increasing sequence of positive numbers $\{n_q\}$ setting

$$\|y_{n_1}\| = \sup\{\|y_n\| : n = 1, 2, \ldots\}$$

$$\|y_{n_q}\| = \sup\{\|y_n\| : n = n_{q-1} + 1, n_{q-1} + 2, \ldots\}, q > 1\ .$$

Let $z_n = (y_n/\|y_n\|)\,\|y_{n_q}\|$, $n = n_{q-1} + 1, n_{q-1} + 2, \ldots, n_q$, $q = 1, 2, \ldots$, $n_0 = 0$. The closed absolutely convex hull of the sequence $\{z_n\}$ contains $\{x_n\}$, $\|z_n\| \geqslant \|z_{n+1}\|$, $n = 1, 2, \ldots$, and, for every positive integer p, the sequence $\{\lambda_{pn}^{p} z_n\}$ converges to the origin. We construct, by induction, a family of sequences in E, $\{z_{qn}\}$, $q = 1, 2, \ldots$

We set $z_{1n} = z_n$, $n = 1, 2, \ldots$ We suppose the sequence $\{z_{qn}\}$ already constructed. Let $z_{qn(q)}$ be the first non-zero element of this sequence. If H_q is a hyperplane in E, orthogonal to $z_{qn(q)}$, passing through the origin, we represent by $z_{(q+1)n}$ the orthogonal projection of z_{qn} onto H_q. Clearly, we can choose a positive integer p_0 such that $\sum_{n=1}^{\infty} \lambda_{p_0 n}^{-p_0} < 1$.

If

$$u_q = \lambda_{p_0 q}^{p_0} (\| z_{n(q)} \| / \| z_{qn(q)} \|) \, z_{qn(q)} ,$$

then the sequence $\{u_n\}$ is orthogonal. Given any positive number r we obtain a positive integer r_0 such that $n(r_0) \leqslant r < n(r_0 + 1)$. We can set

$$z_r = \sum_{q=1}^{r_0} a_q z_{qn(q)} = \sum_{q=1}^{r_0} b_q u_q .$$

If (x, y) is the inner product of any two elements $x, y \in \mathrm{E}$, then $(z_r, u_q) = b_q (u_q, u_q)$, and so

$$|b_q| \leqslant \| z_r \| / \| u_q \| = \| z_r \| / (\lambda_{p_0 q}^{p_0} \| z_{n(q)} \|) \leqslant$$
$$\leqslant \| z_{n(r_0)} \| / (\lambda_{p_0 q}^{p_0} \| z_{n(q)} \|) \leqslant \lambda_{p_0 q}^{-p_0} ,$$

and, therefore, if $v \in \mathrm{E}'$ and $|\langle v, u_n \rangle \leqslant 1, n = 1, 2, \ldots$, then

$$|\langle v, z_r \rangle| \leqslant \sum_{q=1}^{r_0} |b_q \langle v, u_q \rangle| \leqslant \sum_{q=1}^{\infty} \lambda_{p_0 q}^{-p_0} \leqslant 1 ,$$

hence the closed absolutely convex hull of $\{u_n\}$ contains z_r and, therefore, it contains $\{x_n\}$. Finally, for every pair of positive integers p and q, we get

$$\| \lambda_{pq}^{p} u_q \| = \lambda_{pq}^{p} \lambda_{p_0 q}^{p_0} \| z_{n(q)} \| \leqslant \lambda_{(p+p_0)q}^{p} \lambda_{(p+p_0)q}^{p_0} \| z_q \| =$$
$$= \lambda_{(p+p_0)q}^{p+p_0} \| z_q \|$$

and, therefore, the sequence $\{\lambda_{pn}^{p} u_n\}$ converges to the origin.

Q.E.D.

Markushevich proves in [1] that every separable infinite-dimensional Banach space has a basis in the wide sense, (see also [2] p. 116). In Lemma 3 we shall give a more general result than the one given above. We shall need it later.

LEMMA 3. – *Let* E *be an infinite-dimensional space, with* $E'[\sigma(E', E)]$ *separable. Suppose that there is in* E *a bounded countable total set. If* E *is sequentially complete there is a Markushevich basis* $\{x_n, u_n\}$ *for* E, *such that the sequence* $\{x_n\}$ *is bounded.*

Proof. – Let $\{y_n\}$ be a bounded total sequence in E. Let f be the linear mapping from l^2 into E such that, if $\{a_n\} \in l^2$,

$$f(\{a_n\}) = \sum_{n=1}^{\infty} n^{-1} a_n y_n$$

If q is a continuous seminorm in E, then $q(y_n) < c$, $n = 1, 2, \ldots$, and, therefore,

$$q\left(\sum_{n=m}^{\infty} n^{-1} a_n y_n\right) \leqslant \sum_{n=m}^{\infty} n^{-1} |a_n| q(y_n) \leqslant$$

$$\leqslant c\left(\sum_{n=m}^{\infty} n^{-2}\right)^{1/2} \left(\sum_{n=m}^{\infty} |a_n|^2\right)^{1/2} ;$$

from here, and being E sequentially complete, it follows that f is well-defined and it is bounded and so it is continuous. If B is the closed unit ball of l^2 and $A = f(B)$, then E_A can be identified with the Hilbert space $l^2/f^{-1}(0)$. If $\{v_n\}$ is a total sequence in $E'[\sigma(E', E)]$ whose elements are linearly independent, then $\{v_n\}$ is total in

$$(E_A)' [\sigma((E_A)', E_A)] ,$$

and applying the Gram-Schmidt process we obtain an orthonormal sequence $\{u_n\}$ in $(E_A)'[\mu((E_A)', E_A)]$. If x_n is a continuous linear form on $(E_A)' [\sigma((E_A)', E_A)]$, such that $\langle x_n, u_n \rangle = 1$, $\langle x_n, u_m \rangle = 0$, $n \neq m$, $n, m = 1, 2, \ldots$, then $\{x_n\}$ is total in E_A and, therefore, $\{x_n, u_n\}$ is a Markushevich basis for E, such that $\{x_n\}$ is a bounded sequence in E.

Q.E.D.

THEOREM 1. – *Let* F *be a sequentially complete infinite-dimensional space with the following properties :*

1) *There is in* F *a bounded countable total set.*

2) *There is in* $F'[\sigma(F', F)]$ *a countable total set which is equicontinuous in* F.

3) *If u is an injective linear mapping from* F *into* F, *with closed graph, then u is continuous.*

If E *is an infinite-dimensional Banach space then* E *is the inductive limit of a family of spaces equal to* F, *spanning* E.

Proof. – According to Lemma 3 we construct a Markushevich basis $\{x_n, u_n\}$ for E so that the sequence $\{x_n\}$ is bounded. Since in the proof of Lemma 3 we chose the sequence $\{v_n\}$ with the unique conditions that it is total in $F'[\sigma(F', F)]$ and linearly independent, we can suppose, according to property 2), that $\{u_n\}$ is in the linear hull of an equicontinuous set in F. We determine a sequence $\{\lambda_n\}$ of positive integers, strictly increasing, such that the sequence $\{\lambda_n^{-1} u_n\}$ be equicontinuous in F. Let M be the $\sigma(F', F)$-closed absolutely convex hull of $\{\lambda_n^{-1} u_n\}$. Let $\mathfrak{S}$ be the family of all the sequences of E holding the two following properties : α) If $\{y_n\} \in \mathfrak{S}$ then, for every positive integer p, the sequence $\{\lambda_{pn}^{p} y_n\}$ converges to the origin in E. β) If $\{y_n\} \in \mathfrak{S}$ and $\{a_n\}, \{b_n\}$ are two different bounded sequences of K then $\sum_{n=1}^{\infty} a_n y_n$ and $\sum_{n=1}^{\infty} b_n y_n$ are different points of E.

Given an element $\{y_n\} = s \in \mathfrak{S}$ we define a linear mapping f_s from F into E so that, for every $x \in F$,

$$f_s(x) = \sum_{n=1}^{\infty} \langle x, u_n \rangle y_n$$

Let M^0 be the polar set of M in F. We have that $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$ and $\| \lambda_n^3 y_n \| < c,\ n = 1, 2, \ldots,$ being $\| y \|$ the norm of every $y \in E$. Thus, if $x \in M^0$, it results that

$$\left\| \sum_{n=1}^{\infty} \langle x, u_n \rangle y_n \right\| \leqslant \sum_{n=1}^{\infty} \lambda_n^{-2} \, | \langle x, \lambda_n^{-1} u_n \rangle | \cdot \| \lambda_n^3 y_n \| \leqslant c \sum_{n=1}^{\infty} \lambda_n^{-2},$$

from here, f_s is well-defined and transforms M^0, which is a neighbourhood of the origin in F, in a bounded set of E, hence f_s is continuous. According to property β) the mapping f_s is injective. Then, if $E_s = f_s(F)$ we can give to E_s a topology $\mathcal{C}_s$, finer than the induced topology by E, such that $E_s[\mathcal{C}_s]$ can be identified with the space F. Let us see now that E is the locally convex hull of the family of spaces $\{E_s[\mathcal{T}_s] : s \in \mathcal{S}\}$. Let U be an absorbing absolutely convex set in E such that, for every $s \in \mathcal{S}$, $U \cap E_s$ is a neighbourhood of the origin in $E_s[\mathcal{T}_s]$. Let us suppose that U is not a neighbourhood of the origin in E. We take $z_1 \in E$, $\|z_1\| = \lambda_1^{-1}$. Let w_1 be an element of E' such that $\langle w_1, z_1 \rangle = 1$. Supposing constructed $\{w_m, z_m\}_{m=1}^n$ so that $\langle w_m, z_m \rangle = 1$, $\langle w_m, z_p \rangle = 0$, $m \neq p$, $m, p = 1, 2, \ldots, n$, let $H_n = \bigcap_{m=1}^{n} w_m^{-1}(0)$. Since $U \cap H_n$ is not a neighbourhood of the origin in H_n, for the induced topology by E, we choose $z_{n+1} \in H_n$, $z_{n+1} \notin U$, $\|z_{n+1}\| = \lambda_{(n+1)^2}^{-(n+1)}$, and in E' w_{n+1} such $\langle w_{n+1}, z_{n+1} \rangle = 1$ and $\langle w_{n+1}, z_m \rangle = 0$, $m = 1, 2, \ldots, n$. If $y_n = \lambda_n z_n$ the sequence $r = \{y_n\}$ belongs to $\mathcal{S}$ and, therefore, $U \cap E_r$ is a neighbourhood of the origin in $E_r[\mathcal{T}_r]$. The sequence $\{\lambda_n^{-1} x_n\}$ converges to the origin in F, from here the sequence $\{f_r(\lambda_n^{-1} x_n)\} = \{\lambda_n^{-1} y_n\} = \{z_n\}$ converges to the origin in $E_r[\mathcal{T}_r]$ and, therefore, there is a positive integer n_1 such that $z_n \in U \cap E_r$, for $n \geqslant n_1$, which is a contradiction. Given a point $z \in E$, $z \neq 0$, the sequence $\{z_n\}$ can be constructed so that $z_1 = (\lambda_1/\|z\|)z$ and, therefore, $E = \cup\{E_s : s \in \mathcal{S}\}$. Let us see now that the family $\{E_s : s \in \mathcal{S}\}$ is directed by inclusion. Let s_1 and s_2 be two elements of $\mathcal{S}$ so that $s_1 = \{y_n\}$, $s_2 = \{y'_n\}$. We put $t_{2n-1} = \lambda_n^2 y_n$, $t_{2n} = \lambda_n^2 y'_n$, $n = 1, 2, \ldots$ For every positive integer p, the sequence $\{\lambda_{pn}^p t_n\}$ converges to the origin in E. Let A be the closed absolutely convex hull of the sequence $\{t_n\}$. If $y \in E_{s_1}$ there is a element x of F such that

$$y = \sum_{n=1}^{\infty} \langle x, u_n \rangle y_n$$

and, therefore,

$$y = \sum_{n=1}^{\infty} \lambda_n^{-2} \langle x, u_n \rangle t_{2n-1}$$

On the other hand $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$ and $|\langle x, u_n \rangle| < a$, $n = 1, 2, \ldots$, hence if follows that y is in the linear hull of A. Reasoning in an analogous way for the case when y is in E_{s_2}, we have that the linear hull of A contains $E_{s_1} \cup E_{s_2}$. By Lemma 1 there is in E a weakly compact absolutely convex set B such that E_B is a Hilbert space and in $E_B \{t_n\}$ is a sequence so that, for every positive integer p,

$\{\lambda_{pn}^p t_n\}$ converges to the origin. By Lemma 2 we find an orthogonal sequence $\{q_n\}$ in E_B which has the property α) and its closed absolutely convex hull contains $\{t_n\}$. Let $s_3 = \{q_n\}$. Since $\{q_n\}$ is orthogonal in E_B we have that $\{q_n\}$ has the property β), from here $s_3 \in \mathfrak{S}$. Let us see now that $E_{s_3} \supset E_{s_1} \cup E_{s_2}$; it is sufficient to prove that E_{s_3} contains the closed absolutely convex hull P of the sequence $\{q_n\}$. If $z \in P$ it is obvious that

$$z = \sum_{n=1}^{\infty} c_n q_n , \quad \sum_{n=1}^{\infty} |c_n| \leqslant 1$$

Since $\{x_n\}$ is a bounded sequence in the sequentially complete space F, then $\sum_{n=1}^{\infty} c_n x_n$ belongs to F and, therefore,

$$f_{s_3}\Big(\sum_{n=1}^{\infty} c_n x_n\Big) = \sum_{n=1}^{\infty} \langle \sum_{p=1}^{\infty} c_p x_p , u_n \rangle q_n = \sum_{n=1}^{\infty} c_n q_n = z \in E_{s_3}$$

Finally, if $s, r \in \mathfrak{S}$ and $E_s \subset E_r$, let u_s be the canonical injection from $E_s[\mathcal{T}_s]$ into E. The mapping u_s is continuous and so its graph is closed in $E_s[\mathcal{T}_s] \times E_s[\mathcal{T}_r]$. According to property 3) it results that $\mathcal{T}_r$ induces in E_s a topology coarser than $\mathcal{T}_s$.

Q.E.D.

THEOREM 2. – *If* E *is an ultrabornological space, then* E *is the inductive limit of a family of nuclear Fréchet spaces, spanning* E.

Proof. – If the topology of E is the finest locally convex topology, then E is the inductive limit of the finite-dimensional subspaces of E. In the other case, E is the inductive limit of a family of infinite-dimensional Banach spaces spanning E, and, therefore, it is enough to make the proof for the case that E is an infinite-dimensional Banach space. We take F, in Theorem 1, equal to $\mathcal{D}_C$, which is nuclear and separable, and its topology is defined by a countable family of norms, and so properties 1), 2) and 3) of Theorem 1 hold, hence E is the inductive limit of a family of spaces equal to $\mathcal{D}_C$, spanning E.

Q.E.D.

THEOREM 3. – *If* E *is an ultrabornological space, then* E *is the inductive limit of a family of nuclear* (DF)*-spaces, spanning* E.

Proof. – It is analogous to the proof of Theorem 2, changing $\mathcal{D}_C$ to its strong dual $\mathcal{D}'_C$.

Q.E.D.

THEOREM 4. – *If* E *an infinite-dimensional Banach space, then* E *is the inductive limit of a family of spaces equal to* $\mathcal{D}(\Omega)$, *spanning* E.

Proof. – Let $\{w_m\}$ be a linearly independent sequence in $\mathcal{D}'(\Omega)$, whose elements are the monomial functions x^p with p_j any non-negative integer, $j = 1, 2, \ldots, n$. The sequence $\{w_m\}$ is total in $\mathcal{D}'(\Omega)$. If $w_m = x^{p(m)}$, let $v_m = w_m \cdot m^{-|p(m)|-n-1}$. If $\varphi \in \mathcal{D}(\Omega)$ there exists a positive integer m_0 such that the support A of φ is contained in the ball with center 0 and radius m_0. If $m \geqslant m_0$ we have

$$|\langle v_m, \varphi\rangle| = m^{-|p(m)|-n-1} \left| \int_A \varphi(x)\, x^{p(m)}\, dx \right|$$

$$\leqslant m^{-|p(m)|-n-1} m_0^{|p(m)|+n} \sup_{x\in A} |\varphi(x)| \leqslant m^{-1} \sup_{x\in A} |\varphi(x)|$$

hence $\{v_n\}$ converges weakly to the origin in $\mathcal{D}'(\Omega)$ and, therefore, $\{v_n\}$ is equicontinuous in $\mathcal{D}(\Omega)$.

Let C be a compact set in Ω with non-empty interior. Then $\mathcal{D}_C$ is a subspace of $\mathcal{D}(\Omega)$ and $\{v_n\}$ is total and linearly independent in $\mathcal{D}'_C$, being also equicontinuous in $\mathcal{D}_C$. We apply now Lemma 3 and we obtain from $\{v_m\}$ a Markushevich basis $\{x_m, u_m\}$ for $\mathcal{D}_C$, such that $\{x_m\}$ is bounded in $\mathcal{D}_C$, and also in $\mathcal{D}(\Omega)$. Obviously $\{u_n\}$ is total in $\mathcal{D}'(\Omega)$. On the other hand if u is a linear mapping from $\mathcal{D}(\Omega)$ into $\mathcal{D}(\Omega)$, with closed graph, then u is continuous, (see [3], p. 17). Following now the same method as in the proof of Theorem 1, the conclusion of the theorem is obtained.

Q.E.D.

COROLLARY 1.4. – *If* E *is an ultrabornological space such that* $\sigma(E', E) \neq \mu(E', E)$, *then* E *is the inductive limit of a family of spaces equal to* $\mathcal{D}(\Omega)$, *spanning* E.

COROLLARY 2.4. – *The space* $\mathcal{D}'(\Omega)$ *is the inductive limit of a family of spaces equal to* $\mathcal{D}(\Omega)$, *spanning* $\mathcal{D}'(\Omega)$.

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Manuel VALDIVIA,
Facultad de Ciencias
Paseo Valencia al Mar, 13
Valencia (España).