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TOPOLOGICAL COUNTABILITY IN BRELOT POTENTIAL THEORY

by Thomas E. ARMSTRONG

0. Notation.

If Y is a topological space C(Y) denotes the space of all countinuous *real valued* functions on Y. In general, by a *function* on Y we will always mean an *extended real valued* function on Y.

If Y is a topological space with subsets A and B we will say that A is *strongly contained* in B, and will write $A \subset\subset B$, iff the closure, $\bar{A}$, of A is a compact subset of the interior, B^0, of B. A sequence $\{A_n : n \in Z^+\}$ of sets in Y is said to be *strongly increasing* iff $A_n \subset\subset A_{n+1}$ for all $n \in Z^+$. A family F of sets in Y is *strongly increasing* iff for any two sets in F there is a third in F strongly containing them.

A topological space Y is *σ-compact* iff it is the union of countably many compact sets. Y possesses an *exhaustion* $\{A_n : n \in Z^+\}$ iff this sequence of subsets of Y is strongly increasing with union Y. If Y is locally compact it is σ-compact iff it possesses an exhaustion.

A topological space Y is *separable* iff it has a countable dense subset. Y is *first countable* iff each point has a countable base of neighborhoods. Y is *second countable* iff it has a countable base of neighborhoods. If Y is uniformizable (i.e. completely regular) it is *metrizable* iff it has a countable base for its uniformity. Any locally compact space is uniformizable, and is metrizable if it is second countable as a topological space. A σ-compact locally compact space is metrizable iff it is second countable.

The concepts of separability, first and second countability, and metrizability form the topological countability properties we shall be concerned with for Brelot harmonic spaces.

1. Brelot Harmonic Spaces.

In this paper X will denote a fixed Brelot harmonic space. We recall that this means that X is supposed to be a locally compact, non-compact, connected, locally connected, Hausdorff space (with no assumptions of topological countability) ; which is fitted with a harmonic sheaf $\mathcal{H} = \{\mathcal{H}_U : U \text{ open in } X\}$. Each $\mathcal{H}_U$ is supposed to be a vector subspace of C(U). The sheaf $\mathcal{H}$ is supposed to satisfy axioms I, II, III of Brelot [3]. Axiom I states that if $V \subset U$ are open and $h \in \mathcal{H}_U$, then $h|_V \in \mathcal{H}_V$ and that if $\{V_\alpha : \alpha \in \Gamma\}$ is a set of open sets with union V and if h is a function on V with $h|_{V_\alpha} \in \mathcal{H}_{V_\alpha}$ for all $\alpha \in \Gamma$ then $h \in \mathcal{H}_V$. Axiom II guarantees that there is a basis for the topology on X consisting of open $\omega \subset\subset X$ which are *regular* for the solution of the Dirichlet problem in that if $f \in C(\partial\omega)$ there is a unique Dirichlet solution for f, $H_f^\omega \in C(\overline{\omega})$, positive if $f \geqslant 0$, such that $H_f|_{\partial\omega} \equiv f$ and $H_f|_\omega \in \mathcal{H}_\omega$. For any regular $\omega \subset\subset X$ and any $x \in \omega$ the positive Radon measure ρ_x^ω on $\partial\omega$, satisfying, for all $f \in C(\partial\omega)$, $\int f \, d\rho_x^\omega = H_f^\omega(x)$, is called the *harmonic measure* for x on $\partial\omega$. Axiom III is Brelot's convergence axiom and states that if U is a domain and $\{h_\alpha : \alpha \in \Gamma\}$ is an increasing family of harmonic functions in U then $\sup \{h_\alpha : \alpha \in \Gamma\}$ is in $\mathcal{H}_U$ iff it is finite at at least one point in U. One may deduce that if U is a domain and $h \geqslant 0$ is in $\mathcal{H}_U$, then either $h > 0$ or $h \equiv 0$.

If U is open $\mathcal{H}_U^+$ is the set $\{h \in \mathcal{H}_U : h \geqslant 0\}$, U is said to be of type $\mathcal{H}$ iff there is at least one $h \in \mathcal{H}_U^+$ such that $h > 0$. If U is a domain of type $\mathcal{H}$, $\mathcal{H}_U^+$ is a cone which has base

$$\Phi_U^x = \{h \in \mathcal{H}_U^+ : h(x_0) = 1\}.$$

Axiom III′ states that Φ_U^x is compact for the topology of uniform convergence on compact sets in U if U is a domain of type $\mathcal{H}$. This is equivalent to Harnack's principle and to Axiom III. This was established by Mokobodski, [13], for the case of second countable X and by Loeb and Walsh, [12], in the general case. (Brelot has informed me that Mokobodski has extended his original proof to handle the general case). Note that because of Axiom II any regular $\omega \subset\subset X$ is of type $\mathcal{H}$. Consequently any point has a basis of neighborhoods of type $\mathcal{H}$.

A lower semi-continuous function $s > -\infty$, defined on an open set U in X, is hyperharmonic on U iff $s(y) \geqslant \int s\, d\rho_y^\omega$ for any regular $\omega \subset\subset U$ and any $y \in \omega$. A hyperharmonic s is superharmonic iff it is finite at one point in each component of Y. If s is superharmonic on U it is integrable for ρ_x^ω for any regular $\omega \subset\subset U$ and any $x \in \omega$. Brelot [3] establishes this last fact and all of the other relevant properties of superharmonic functions. $\mathcal{S}_U$ will denote the class of superharmonic functions on U. $\mathcal{S}_U$ is convex cone which contains the infimum of any finite subset. If $F \subset \mathcal{S}_U$ is an increasing family then sup F is hyperharmonic. If $F \subset \mathcal{S}_U$ is a family which is locally lower bounded then $\widehat{\inf F} \in \mathcal{S}_U$. ("$\hat{\ }$" denotes lower semi-continuous regularization). $\mathcal{S}_U^+$ denotes the set $\{s \in \mathcal{S}_U : s \geqslant 0\}$. Any element s of $\mathcal{S}_U$ which possesses one harmonic minorant possesses a largest harmonic minorant h_s dominating any other harmonic minorant of s. The correspondence $s \to h_s$ is additive and positively homogeneous on the cone of a superharmonic functions possessing harmonic minorants. In particular any element of $\mathcal{S}_U^+$ has a greatest harmonic minorant h_s. If $p \in \mathcal{S}_U^+$ and $h_p = 0$, then p is a potential on U, p is a potential on U iff $h \leqslant p$ and $h \in \mathcal{H}_U$ implies $h \leqslant 0$. $\mathcal{P}_U$ will denote the class of potentials on U. If $p_1 + h_1 = p_2 + h_2$ with p_1 and p_2 potentials and h_1 and h_2 harmonic, then $p_1 = p_2$ and $h_1 = h_2$. If $s \in \mathcal{S}_U$ and h_s exists then $p_s = s - h_s \in \mathcal{P}_U$. The decomposition $s = h_s + p_s$ is the Riesz decomposition of s. Countable sums of elements of $\mathcal{S}_U^+$ are in $\mathcal{S}_U^+$ iff they converge to a finite limit at at least one point in each component of U. Countable sums of elements in $\mathcal{P}_U$ lie in $\mathcal{P}_U$ as long as they converge at at least one point in each component of U. U is said to be of type $\mathcal{P}$ iff there is a $p \in \mathcal{P}_U$ with $p > 0$. If U is of type $\mathcal{P}$, then U is of type $\mathcal{H}$. U is of type $\mathcal{H}$ iff there is at least one $s \in \mathcal{S}_U^+$ with $s > 0$. U is said to be of type $\mathcal{H}$-$\mathcal{P}$ iff it is of type $\mathcal{H}$ but not of type $\mathcal{P}$. A domain U is of type $\mathcal{H}$-$\mathcal{P}$ iff $\mathcal{S}_U^+ = \mathcal{H}_U^+ = \{\lambda h : \lambda \in [0, \infty)\} \neq \{0\}$ $(h \in \mathcal{H}_U^+)$. We remark that the notations type $\mathcal{H}$, type $\mathcal{P}$ are due to Constantinescu and Cornea [7] and seem to provide a most convenient shorthand.

In classical potential theory the role of negligible sets is played by sets of capacity 0 or polar sets. Brelot introduced the notion of polar sets in the axiomatic setting. A polar subset of an open set U is any subset A of $p^{-1}\{\infty\}$ for a $p \in \mathcal{P}_U$. A subset A of U is locally polar iff for any $x \in U$ there is a neighborhood ω of x such that $A \cap \omega$

is polar in ω. Brelot [3] shows that if U is of type $\mathcal{P}$ any locally polar set in U is polar in U. The converse is true for any open set U. A polar subset of U is of ρ_x^ω measure 0 for any regular $\omega \subset\subset U$ and any $x \in \omega$. As a result if U is a domain and $A \subset U$, then A is not locally polar when $A^0 \neq \emptyset$. Any locally polar set $A \subset U$ has the property that if $V \subset U$ is a domain then V-A is a connected topological space (i.e. A is nowhere disconnecting). As a result if $V \subset U$ is open, U a domain and $\overline{V} \neq U$ then ∂V is not locally polar. In particular, if the open $\omega \subset\subset U$, then $\partial\omega$ is not locally polar in U.

Constantinescu and Cornea in [7] show that if U is a domain of type $\mathcal{H}$-$\mathcal{P}$ and V is an open subset then V is of type $\mathcal{P}$ iff U-V is not locally polar in U. Any open $\omega \subset\subset U$ is of type $\mathcal{P}$. An open set U is of type $\mathcal{H}$ iff every open $\omega \subset\subset U$ is of type $\mathcal{P}$.

An open set U is said to possess a pseudo-exhaustion iff there is a σ-compact open subset V of U such that the relatively closed subset U-V of U is locally polar in U. This is the case iff there is a strongly increasing sequence of subsets of U, $\{A_n : n \in Z^+\}$ such that $U - \bigcup_{n=1}^{\infty} A_n$ is locally polar in U. Any such sequence is called a pseudo-exhaustion of U.

Constantinescu and Cornea [7] show that any domain of type $\mathcal{H}$ possesses a pseudo-exhaustion hence such a domain differs from a σ-compact by a relatively closed locally polar set. Cornea in [9] shows that any relatively closed locally polar subset of a domain of type $\mathcal{H}$ has a metrizable one point compactification hence is σ-compact. Cornea deduces from this that any domain of type $\mathcal{H}$ is σ-compact. We will make considerable use of this fact and the metrizability of relatively closed locally polar subsets of domains of type $\mathcal{H}$.

Cornea, in [9], shows that if every point in a domain U of type $\mathcal{H}$ has a neighborhood which is second countable then U itself is second countable. Kohn, in [11], establishes that if ω is a regular domain such that $\mathcal{H}_\omega$ separates points in ω then ω is second countable. Constantinescu, in [6], deduces that if U is a domain of type $\mathcal{H}$ such that any point has a neighborhood ω separated by $\mathcal{H}_\omega$ then U is second countable. This is one of the few results which have deduced a topological countability result in the axiomatic setting.

2. Topological Countability of Brelot Harmonic Spaces.

The following proposition is our fundamental result.

PROPOSITION 1. – *Let* U *be a domain of type* $\mathcal{H}$. *Let* $F \subset U$ *be relatively closed and locally polar. If* K *is a compact subset of* F *then it is a* $\mathcal{G}_\delta$ *in* U.

Proof. – Since F is metrizable K is a $\mathcal{G}_\delta$ in F. U-F is a domain of type $\mathcal{H}$ hence is σ-compact. Let $\{V_n : n \in Z^+\}$ be a decreasing sequence of open sets in U with $\bigcap_{n=1}^{\infty} (F \cap V_n) = K$. Let $\{A_n : n \in Z^+\}$ be an exhaustion of U-F. $\{V_n - \overline{A}_n : n \in Z^+\}$ is a decreasing sequence of open sets in U whose intersection in K.

The following theorem characterizes when a compact subset of a domain of type $\mathcal{H}$ is a $\mathcal{G}_\delta$, hence characterizes Baire compacts, purely in terms of connectivity properties.

PROPOSITION 2. – 1/ *Let* K *be a compact set with contained in a domain* U *of type* $\mathcal{H}$. K *is a* $\mathcal{G}_\delta$ *iff* U-K *has at most countable many components.*

2/ *A point* x *has a countable basis of neighborhoods iff for some (hence for any) domain* U *of type* $\mathcal{H}$ *containing* x, $U - \{x\}$ *has at most countably many components.*

Proof. – 1/ Let K be compact in U. Each component of U-K is σ-compact since each is of type $\mathcal{H}$. If U-K has only countably many components it is σ-compact. If $\{A_n : n \in Z^+\}$ is an exhaustion of U-K then $K = \bigcap_{n=1}^{\infty} U\text{-}\overline{A}_n$ is a $\mathcal{G}_\delta$.

Let us now assume that K is a $\mathcal{G}_\delta$ in U and that $\{G_n : n \in Z^+\}$ is a strongly decreasing sequence of relatively compact open sets in U with intersection K. Let V be a non-empty component of U-K. $(\partial V \cap U) \subset K$ for V has no limit points in any other component of U-K. $\emptyset \neq \partial V \cap U$ for otherwise V would be a non-empty proper component of the domain U. If $n \in Z^+$ and V does not meet ∂G_n then $V = (V \cap G_n) \cup (V - \overline{G}_n)$. Since V is a domain $V \cap G_n = \emptyset$ or $V - \overline{G}_n = \emptyset$. Since $\emptyset \neq \partial V \cap U \subset K \subset G_n$, $V - \overline{G}_n = \emptyset$. Consequently, if $V \cap \partial G_n = \emptyset$ then $V \subset G_n$. For any $n \in Z^+$, ∂G_n is compact hence meets at most finitely many components of U-K. For each $n \in Z^{+}$ at most finitely many components of U-K are not contained in G_n. $K = \bigcap_{n=1}^{\infty} G_n$ contains all but countably many components of U-K so U-K has only countably many components.

2/ x has a countable basis of neighborhoods iff $\{x\}$ is a $\mathcal{G}_\delta$. x is a $\mathcal{G}_\delta$ in a domain U of type $\mathcal{H}$ iff $U - \{x\}$ has at most countably many components. 2/ follows from the fact that any point in X has a neighborhood of type $\mathcal{H}$.

COROLLARY.

i) *Any polar point in a* $\mathcal{G}_\delta$.

ii) *A domain* U *of type* $\mathcal{H}$ *is first countable iff no point uncountably disconnects* U.

iii) *If* X *is separable it is first countable.*

iv) *Let* U *be a domain of type* $\mathcal{H}$. *Every Borel set in* U *is Baire iff no compact in* U *uncountably disconnects* U.

Proof. –

i) follows from Proposition 1, since any point has a neighborhood of type $\mathcal{H}$.

ii) follows from Proposition 2, since U is first countable iff each point in a $\mathcal{G}_\delta$.

iii) Let X be separable and $x \in X$. Let U be a domain of type $\mathcal{H}$ containing x. Let $\{x_1, \ldots, x_n, \ldots\}$ be a countable dense set in X. Every component of $U - \{x\}$ contains at least one x_i so there are at most countably many components of $U - \{x\}$. Thus x is a $\mathcal{G}_\delta$. Since x was arbitrary iii) is established.

iv) If every Borel set in U is Baire then every compact in U is a $\mathcal{G}_\delta$. As a result no compact in U is uncountably disconnecting.

If no compact uncountably disconnects U then every compact in U is Baire. Since U is σ-compact so is any closed subset. We deduce that all closed subsets of U are Baire hence that all Borel subsets of U are Baire.

PROPOSITION 3. – *Let* U *be a domain such that* $\overline{\mathrm{U}}$ *possesses a neighborhood* V *of type* $\mathcal{H}$.

1/ *Every Borel set in* $\partial\mathrm{U}$ *is Baire.*

2/ *In the topological space* $\overline{\mathrm{U}}$ *every point* $x \in \partial\mathrm{U}$ *has a countable basis of neighborhoods.*

3/ *Any relatively open set* $\theta \subset \partial\mathrm{U}$ *is* σ*-compact.* θ *has at most countably many disjoint relatively clopen subsets.*

4/ *If* $\overline{\mathrm{U}}$ *is compact then every bounded Borel function on* $\partial\mathrm{U}$ *is resolutive for the Perron-Wiener-Brelot method of solving the Dirichlet problem on* $\partial\mathrm{U}$.

Proof. –

1/ Will be established if we can show that any relatively open set $\theta \subset \partial\mathrm{U}$ is σ-compact. If this is the case then any compact $\mathrm{K} \subset \partial\mathrm{U}$ is Baire. Since any closed subset of $\partial\mathrm{U}$ is σ-compact all closed sets in $\partial\mathrm{U}$ are Baire hence all Borel sets in $\partial\mathrm{U}$ are Baire.

Let θ be open in $\partial\mathrm{U}$. Let W be an open set in V with $\mathrm{W} \cap \partial\mathrm{U} = \theta$. Let W′ be the component of $\mathrm{U} \cup \mathrm{W}$ containing U. W′ is a domain of type $\mathcal{H}$ and $\mathrm{W}' \cap \partial\mathrm{U} = \theta$. Since W′ is σ-compact, θ is seen to be σ-compact.

2/ Any point x in $\partial\mathrm{U}$ is a $\mathcal{G}_\delta$ in $\partial\mathrm{U}$ by 1/. Let $\{\omega_n : n \in \mathrm{Z}^+\}$ be a sequence of relatively open sets in $\overline{\mathrm{U}}$ with $\bigcap_{n=1}^{\infty} (\partial\mathrm{U} \cap \omega_n) = \{x\}$. Since U is a domain of type $\mathcal{H}$ it possesses an exhaustion $\{\mathrm{A}_n : n \in \mathrm{Z}^+\}$. $\{\omega_n - \overline{\mathrm{A}}_n : n \in \mathrm{Z}^+\}$ is a sequence of relatively open sets in $\overline{\mathrm{U}}$ with intersection $\{x\}$. Since $\{x\}$ is a $\mathcal{G}_\delta$ in $\overline{\mathrm{U}}$ it has a countable base of neighborhoods in $\overline{\mathrm{U}}$.

3/ We have established that any relatively open $\theta \subset \partial\mathrm{U}$ is σ-compact. Any compact K in $\partial\mathrm{U}$ meets at most finitely many disjoint clopen subsets of θ. As a result θ can have at most countably many disjoint relatively clopen subsets.

4/ $\overline{\mathrm{U}}$ is compact in V. By deleting a non-polar compact from $\mathrm{V}\text{-}\overline{\mathrm{U}}$ we may assume that V is of type $\mathcal{P}$. Theorem 19 of Brelot [3] guarantees that any continuous function on $\partial\mathrm{U}$ is resolutive hence that any bounded Baire function on $\partial\mathrm{U}$ is resolutive. From 1/ it follows that any bounded Borel function on $\partial\mathrm{U}$ is resolutive.

Remarks. – Constantinescu and Cornea [8, Exercise 3.1.8] give an example of a compact Brelot harmonic space such that one point does not possess a countable basis of neighborhoods. If one deletes another point from this space we obtain example of a non-compact Brelot harmonic space of type $\mathcal{H}$ with precisely one point not possessing a countable basis of neighborhoods. This point is easily seen, in the example, to be uncountably disconnecting in accordance with Proposition 2.

From Proposition 2 it follows that separable Brelot harmonic spaces are first countable. When are such spaces second countable ?

From Proposition 3 it follows that if U is a domain such that $\overline{U}$ has a neighborhood of type $\mathcal{H}$ then every Borel set in ∂U is Baire. When is ∂U metrizable ?

Constantinescu and Cornea [8, Theorem 2.4.4] show that if U is a domain such that $\overline{U}$ has a neighborhood of type $\mathcal{H}$ then any positive upper semi-continuous function with compact support in ∂U is resolutive. From this it follows that, for any compact $K \subset \partial U$, χ_K is resolutive. This may be used to give an alternate proof of Proposition 3.4).

The rest of this paper deals with various applications of Propositions 1, 2 and 3. Most of these applications have to do with thinness. Thinness was first defined by Brelot and found to have a very natural interpretation in terms of Cartan's fine topology in classical potential theory. We refer the reader to Brelot [3], [4] or [5]. [5] includes a very exhaustive analysis of notions of thinness and their properties in axiomatic potential theories. Three distinct notions of thinness arise in axiomatic potential theory, all of them coincident in classical potential theory. The definition of these three types of thinness that we give below is adapted from Bauer [1]. We refer the reader to Brelot [3] for definitions and properties of the reduced function of a positive function f over a set E, R_f^E, and its lower semi-continuous regularization $\hat{R}_f^E$.

DEFINITION. – *Let* U *be a domain of type* $\mathcal{P}$. *Let* $x \in U$ *and* $E \subset U - \{x\}$. *Let* p *be a strictly positive continuous potential on* U.

i) E *is weakly thin at* x *iff* $\hat{R}_p^{E\cap\delta}(x) < p(x)$ *for some neighborhood* δ *of* x.

ii) E *is thin at* x *iff* $R_p^{E\cap\delta}(x) < p(x)$ *for some neighborhood of* x.

iii) E *is strongly thin at* x *iff* $\inf_{x\in\delta\in B} R_p^{E\cap\delta}(x) = 0$ *where* B *is a base of neighborhoods of* x *in* U.

Remarks. – None of these concepts depends on the choice of p or that p is a potential but only on the fact that p is continuous at x with $p(x) > 0$. Bauer [1] or Brelot [5] show that iii) $\Rightarrow$ ii) $\Rightarrow$ i). Note that if $x \notin \overline{E}$ then iii) always holds. Brelot [5] defines a set E to be hyper-thin at $x \in \overline{E}$-E iff there is an $s \in \mathcal{S}_U^+$ with

$$\liminf_{\substack{y\to x\\ y\in E}} [s(y) - s(x)] = \infty$$

and to be hyper-thin at any $x \notin \overline{E}$.

In Proposition 11.10 of [5] Brelot demonstrates the equivalence of strong thinness and hyper-thinness. It is easy to see that E is thin at an $x \in \overline{E}$-E iff there is an $s \in \mathcal{S}_U^+$ with

$$\liminf_{\substack{y\to x\\ y\in E}} [s(y) - s(x)] > 0.$$

One application of our countability results consists in showing that a polar set is strongly thin at any point of its complement.

PROPOSITION 4. –

1/ *Let* U *be a domain of type* $\mathfrak{L}$. *If* $Z \in U$ *is polar and* $f \geqslant 0$ *is any function on* Z, *then* $R_f^Z(x) = 0$ *if* $x \in$ U-Z.

2/ Z *is strongly thin at all points of* U-Z.

Proof. –

1/ Brelot [3] shows that if p is any non-zero potential then R_p^Z vanishes at at least one point in U-Z. $\hat{R}_p^Z$ is superharmonic and vanishes at at least one point of U hence vanishes everywhere. As a result if $\omega \subset\subset U$ is regular then R_p^Z vanishes ρ_x^ω almost everywhere for any $x \in \omega$. It follows that R_p^Z vanishes at at least one point of each component of U-$\overline{Z}$ hence that $R_p^Z \equiv 0$ in U-$\overline{Z}$.

Let $x \in \overline{Z}\text{-}Z$ and let p be a non-zero continuous potential on U. If $R_p^Z(x) > 0$ we assert that $\{x\}$ is polar. To see this we let $y \in U$ with $R_p^Z(y) = 0$. For any $n \in Z^+$ let $s_n \in \mathcal{S}_U^+$ satisfy $s_n \geqslant f$ on Z and $s_n(y) \leqslant 2^{-n}$. If $s = \sum_{n=1}^{\infty} s_n$ then $s \in \mathcal{S}_U^+$. Since each $s_n \geqslant R_p^Z$,

$$s(x) \geqslant \sum_{n=1}^{\infty} R_p^Z(x) = \infty.$$

This shows that $\{x\}$ is polar.

We now show that $R_p^Z(x) > 0$ implies that $x \in Z$. Since $\{x\}$ is polar it is a $\mathcal{G}_\delta$. Let $\{\omega_n : n \in Z^+\}$ be a sequence of neighborhoods of x strongly decreasing to $\{x\}$. Since $Z\text{-}\omega_n$ is polar and since $x \notin \overline{Z\text{-}\omega_n}$, $R_f^{Z-\omega_n}(x) = 0$. We assume now that $x \notin Z$ and will show that $R_p^Z(x) = 0$. Let $\epsilon > 0$. Choose an $s_n \in \mathcal{S}_U^+$ so that $s_n \geqslant p$ on $Z\text{-}\omega_n$ and $s_n(x) < \epsilon \cdot 2^{-n}$ for any $n \in Z^+$. If $s = \sum_{n=1}^{\infty} s_n$ then $s \in \mathcal{S}_U^+$. If $z \in Z\text{-}\omega_m$ then $s(z) \geqslant \sum_{n=m+1}^{\infty} s_n(z) \geqslant \sum_{n=m+1}^{\infty} p(z) > p(z)$. We see that $\epsilon \geqslant s(x) \geqslant R_p^Z(x) > 0$ for any $\epsilon \geqslant 0$. This contradiction shows that if $R_p^Z(x) > 0$, then $x \in Z$. We deduce that $R_p^Z(x) = 0$ for any $x \in U\text{-}Z$. We may deduce that $R_\infty^Z(x) = 0$ for any $x \in U\text{-}Z$, hence that $R_f^Z \equiv 0$ in U-Z for any function $f \geqslant 0$ on Z.

2/ The assertion that any polar Z is strongly thin at all points of U-Z is now immediate.

Boboc, Constantinescu and Cornea establish as Corollary 3.1 in [2] that if U is a domain of type $\mathfrak{P}$, if x is a $\mathcal{G}_\delta$ point of U, if $E \subset U\text{-}\{x\}$, and if s is in $\mathcal{S}_U^+$ then $R_s^E(x) = \hat{R}_s^E(x)$. From this it immediately follows that if x is a $\mathcal{G}_\delta$ point in U-E then E is thin at x iff it is weakly thin at x. We may also use this corollary to give a different proof of Proposition 4. The following proposition, by Proposition 2, includes the corollary 3.1 of [2] as a special case. In essence, we have removed restrictions on the point x and put them on the set E.

Proposition 5. – *Let* U *be a domain of type* $\mathfrak{P}$, $x \in U$ *and* $E \subset U\text{-}\{x\}$. *If either* i) *or* ii) *holds then* $R_s^E(x) = \hat{R}_s^E(x)$ *for any* $s \in \mathcal{S}_U^+$.

i) E *meets only countably many components of* U-$\{x\}$.

ii) E *is* $\mathcal{K}$*-analytic.*

Proof. – If V is the union of $\{x\}$ with a number of components of U-$\{x\}$ then V is a locally compact Hausdorff space with the induced topology from U. If V is the union of $\{x\}$ with countably many components $\{U_n : n \in Z^+\}$ then $\{x\}$ is a $\mathcal{G}_\delta$ in V since each U_n is σ-compact.

We wish to construct a union, $\hat{U}$, of $\{x\}$ with countably many components of U-$\{x\}$, such that $\hat{U}$ has a Brelot potential theory induced on it from U such that if $f \geqslant 0$ is any function on a set $A \subset \hat{U}$-$\{x\}$ then $R_f^A|_{\hat{U}} = (R_f^A)_{\hat{U}}$ and $\hat{R}_f^A|_{\hat{U}} = \widehat{(R_f^A)}_{\hat{U}}$. We furthermore want to ensure that $E \subset \hat{U}$. The main obstacle in constructing $\hat{U}$ is ensuring that x has a basis of regular neighborhoods in $\hat{U}$. We will construct such a basis by ensuring that there is a strongly decreasing sequence $\{\omega_n : n \in Z^+\}$ of regular neighborhoods of x in U with $\{\omega_n \cap \hat{U} : n \in Z^+\}$ forming the required basis of regular neighborhoods at x in $\hat{U}$. We guarantee that $\partial\omega_n \subset \hat{U}$ for all n, and that the original harmonic measures $\rho_y^{\omega_n}$ for $y \in \hat{U} \cap \omega_n$ are the harmonic measures for the potential theory on $\hat{U}$.

Let V_0 be the countable union of all components of U-$\{x\}$ containing points of E. Let $\{K_n^0 : n \in Z^+\}$ be a strongly increasing exhaustion of V_0 by compacts. Let $\omega_0 \subset\subset U\text{-}K_0^0$ be a regular neighborhood of x. Let V_1 be the union of V_0 with the finite number of components of U-$\{x\}$ meeting $\partial\omega_0$. Let $\{K_n^1 : n \in Z^+\}$ be a strongly increasing exhaustion of the σ-compact V_1 by compacts. Suppose, furthermore, that $K_n^1 \cap V_0 = K_n^0$ for all $n \in Z^+$. Let $\omega_1 \subset\subset \omega_0 - K_1^1$ be a regular neighborhood of x. Suppose now that we have constructed the sequence $\{V_j\}_{j=0}^n$ consisting of countable unions of components of U-$\{x\}$, the strongly increasing exhaustion of V_j by compacts $\{K_m^j : m \in Z^+\}$ for each $j = 1, \ldots, n$ and the regular open sets $\{\omega_j : j = 0, \ldots, n\}$ such that

i) If $1 \leqslant j \leqslant n$ then V_j is the union of V_{j-1} with the finite number of components of U-$\{x\}$ meeting $\partial\omega_{j-1}$.

ii) If $1 \leqslant j \leqslant n$ then $K_m^j \cap V_{j-1} = K_m^{j-1}$ for all $m \in Z^+$.

iii) If $1 \leqslant j \leqslant n$ then $\omega_j \subset\subset \omega_{j-1} - K_j^j$.

Let V_{n+1} be the union of all components of U-$\{x\}$ meeting $\partial\omega_n$ with V_n. Let $\{K_m^{n+1}\ ; m \in Z^+\}$ be an exhaustion of V_{n+1} with $K_m^{n+1} \cap V_n = K_m^n$ for all $m \in Z^+$. Finally let $\omega_{n+1} \subset\subset \omega_n - K_{n+1}^{n+1}$. We now have inductively defined all of these sequences.

Let $V = \bigcup_{n=1} V_n$ and let $\hat{U} = V \cup \{x\}$. V has only countably many components. Consequently $\hat{U}$ is a locally compact Hausdorff space in which x has a countable basis of neighborhoods.

Let $K \subset V$ be compact. K is a subset of some V_n and consequently $K \subset K_m^m$ for some $m \geqslant n$. For this m, $\omega_m \subset\subset$ U-K. Consequently, $[\omega_m \cap \hat{U}] \cap K = \emptyset$. This shows that $\{\omega_m \cap \hat{U} : m \in Z^+\}$ forms a basis of neighborhoods of x in $\hat{U}$ which is strongly decreasing. Let W be a component of U-$\{x\}$ not in V. For any $n \in Z^+$, $W \cap \partial\omega_n = \emptyset$ and $x \in W$, so $W \cap \omega_n \neq \emptyset$. Since W is a domain $\partial\omega_n$ cannot disconnect W, so $W \subset \omega_n$. Since n is arbitrary $W \subset \bigcap_{n=1}^{\infty} \omega_n - \{x\}$. We let

$$U' = U\text{-}\hat{U} = \bigcap_{n=1}^{\infty} \omega_n - \{x\}.$$

If $\hat{V}$ is open in $\hat{U}$ and does not contain x it is open in U. Let $\hat{\mathcal{H}}_{\hat{V}} = \mathcal{H}_{\hat{V}}$. If $\hat{V}$ is open in $\hat{U}$ and contains x then $V = \hat{V} \cup U'$ is open in U and $V \cap \hat{U} = \hat{V}$. Let $\hat{\mathcal{H}}_{\hat{V}} = \{h|_{\hat{V}} : h \in \mathcal{H}_V\}$. If $h' \in \hat{\mathcal{H}}_{\hat{V}}$ there is an n with $\overline{\omega}_n \cap \hat{U} \subset \hat{V}$. Since $\partial\omega_n = \partial(\omega_n \cap V)$, any h in $\mathcal{H}_V$ with $h|_{\hat{V}} = h'$ must satisfy $h(y) = \int h'\, d\rho_y^{\omega_n}$ for all $y \in \omega_n$. This holds in particular for all $y \in U'$. That is, h is uniquely determined by h'. Therefore the mapping $h \to h|_{\hat{V}}$ is 1-1, onto, linear and is an order isomorphism for the natural orders on $\hat{\mathcal{H}}_{\hat{V}}$ and $\mathcal{H}_V$. The sheaf $\{\hat{\mathcal{H}}_{\hat{V}} : \hat{V}$ open in $\hat{U}\}$ satisfies Axiom I of Brelot. Axiom II is easily verified at all points y of $\hat{U}$-$\{x\}$ At x, $\mathcal{B} = \{\omega_n \cap \hat{U} : n \in Z^+\}$ is seen to be a regular basis system for x in $\hat{U}$. In fact the harmonic measures on $\partial(\omega_n \cap \hat{U}) = \partial\omega_n$ are the original harmonic measures on $\partial\omega_n$. Axiom III is easily verified for any domain $\hat{V} \subset \hat{U}$. The only possible question is if $x \in \hat{V}$. Here we use the I-I correspondence between $\hat{\mathcal{H}}_{\hat{V}}$ and $\mathcal{H}_V$. If $\{h'_\alpha : \alpha \in \Gamma\}$ is an increasing family in $\hat{\mathcal{H}}_{\hat{V}}$ the corresponding family $\{h_\alpha : \alpha \in \Gamma\} \subset \mathcal{H}_V$ is also increasing. If $y \in \hat{V}$ with $\sup_{\alpha\in\Gamma} h'_\alpha(y) < \infty$ then $\sup_{\alpha\in\Gamma} h_\alpha(y) < \infty$ so $\sup_{\alpha\in\Gamma} h_\alpha$ is harmonic in the component of V containing y. Since $\hat{V}$ is a domain so is V, hence if $\sup_{\alpha\in\Gamma} h'_\alpha(y) < \infty$ then $h = \sup_{\alpha\in\Gamma} h_\alpha$ is in $\mathcal{H}_V$ and $h' = h|_V \in \hat{\mathcal{H}}_{\hat{V}}$.

This establishes that we have defined a Brelot potential theory on $\hat{U}$. We now wish to establish the connection between reduced functions on U and those on $\hat{U}$. To do this we need to characterize superharmonic functions on open subsets $\hat{V}$ of $\hat{U}$. If $\hat{V}$ does not contain x then $\hat{\mathcal{S}}_{\hat{V}}$, the superharmonic function on $\hat{V}$ for the potential theory on $\hat{U}$, is seen to be $\mathcal{S}_{\hat{V}}$. If $x \in \hat{V}$ and $s \in \hat{\mathcal{S}}_{\hat{V}}$ then s must be in $\mathcal{S}_{V\text{-}\{x\}}$ when restricted to $\hat{V}\text{-}\{x\}$ and $s(x) = \sup\{\int s\, d\rho_x^{\omega_n} : n \in Z^+\}$. Conversely, a lower semi-continuous s on $\hat{V}$ satisfying these two conditions is seen to be in $\hat{\mathcal{S}}_{\hat{V}}$. If $x \in \hat{V}$ we wish to identify $\hat{\mathcal{S}}_{\hat{V}}$ with a certain subset of $\mathcal{S}_V$ where $V = \hat{V} \cup [U\text{-}\hat{U}]$. It may happen that for an $s \in \mathcal{S}_V$, $s(x) > \sup\limits_{n \in Z^+} \int s\, d\rho_x^{\omega_n}$. This is the only thing which prevents s from lying in $\hat{\mathcal{S}}_{\hat{V}}$ for $s|_{\hat{V}\text{-}\{x\}} \in \mathcal{S}_{\hat{V}\text{-}\{x\}}$. Let s be superharmonic on the open set $W \subset U$ and let $\delta \subset\subset W$ be a regular open set. If we let

$$s^\delta = \begin{cases} s \text{ on } W\text{-}\delta \\ H_s^\delta \text{ in } \delta \end{cases}, \text{ then } s \in \mathcal{S}_W.$$

For any s in $\mathcal{S}_V$ we let

$$\hat{s} = \sup\{s^{\omega_n} : n \in Z^+\}.\ \hat{s} \equiv s \text{ on } \hat{V}\text{-}\{x\} \text{ and } \hat{s}(x) \leqslant s(x).$$

We note that $\hat{s} \in \mathcal{S}_V$ and $\hat{s}(x) = \sup\{s^{\omega_n}(x) : n \in Z^+\}$. Consequently $\hat{s}|_{\hat{V}} \in \hat{\mathcal{S}}_{\hat{V}}$. If $\hat{s}$ is some function in $\hat{\mathcal{S}}_{\hat{V}}$ we may extend $\hat{s}$ to $V = \hat{V} \cup (U\text{-}\hat{U})$ by setting $\hat{s}(y) = \sup\{\int \hat{s}\, d\rho_y^{\omega_n} : n \in Z^+\}$ for any $y \in U\text{-}\hat{U}$. $\hat{s}$ is in $\mathcal{S}_V$, is harmonic on each component of $U\text{-}\hat{U}$ and satisfies

$$s(x) = \sup_n \{s^{\omega_n}(x) : n \in Z^+\}.$$

Consequently we may identify

$$\hat{\mathcal{S}}_{\hat{V}} \text{ with } \{s|_{\hat{V}} : s \in \mathcal{S}_V \ ;\ s(x) = \sup_{n \in Z^+} s^{\omega_n}(x)\} =$$

$$= \{\sup(s^{\omega_n}|_V : n \in Z^+) : s \in \mathcal{S}_V\}.$$

We are now ready to find the reduced functions on U. Let $A \subset U\text{-}\{x\}$, $f \geqslant 0$ a function on A. We assert that the reduced function for f over A in $\hat{U}$, $(R_f^A)_{\hat{U}}$, is equal to $R_f^A|_{\hat{U}}$. Note that

$$(R_f^A)_{\hat{U}} = \inf\{s : s \in \hat{\mathcal{S}}_{\hat{U}}^+ : s \geqslant f \text{ on } A\}.$$

If $\hat{s} \in \hat{\mathfrak{S}}_{\hat{U}}$ and $\hat{s} \geqslant f$ on A its corresponding function s in $\mathfrak{S}_U^+$ satisfies $s \geqslant f$ on A since $s \equiv \hat{s}$ on $\hat{U}$. Consequently $R_f^A|_{\hat{U}} \leqslant (R_f^A)_{\hat{U}}$. Now let $s \in \mathfrak{S}_U^+$ and $s \geqslant f$ on A. $s^{\omega_n} \geqslant f$ on A-ω_n for any n so

$$\hat{s} = \sup\{s^{\omega_n} : n \in Z^+\}$$

satisfies $\hat{s} \geqslant f$ on A and also satisfies $\hat{s} \leqslant s$. Since

$$\hat{s}|_{\hat{U}} \in \hat{\mathfrak{S}}_{\hat{U}}^+ \ , \ R_f^A|_{\hat{U}} \geqslant (R_f^A)_{\hat{U}}.$$

This establishes that $R_f^A|_{\hat{U}} \equiv (R_f^A)_{\hat{U}}$.

We next assert that $\hat{R}_f^A|_{\hat{U}} \equiv \widehat{(R_f^A)_{\hat{U}}}$. This identity holds at all points of the open set $\hat{U}$-$\{x\}$ since $R_f^A|_{\hat{U}} \equiv (R_f^A)_{\hat{U}}$ on $\hat{U}$-$\{x\}$. We need only show that $\hat{R}_f^A(x) = \widehat{(R_f^A)_{\hat{U}}}(x)$. Since $\hat{U} \subset U$,

$$\widehat{(R_f^A)_{\hat{U}}}(x) = \liminf_{\substack{y \to x \\ y \in \hat{U}\text{-}\{x\}}} (R_f^A)_{\hat{U}}(x) \geqslant \liminf_{\substack{y \to x \\ y \in U\text{-}\{x\}}} R_f^A(x) = \hat{R}_f^A(x).$$

Also

$$\widehat{(R_f^A)_{\hat{U}}}(x) = \sup_{n \in Z^+} \{\int (R_f^A)_{\hat{U}}\, d\rho_x^{\omega_n}\} =$$

$$= \sup_{n \in Z^+} \{\int R_f^A\, d\rho_x^{\omega_n}\} \leqslant \sup_{x \in \delta \in B} \{\int R_f^A\, d\rho_x^{\delta}\} = \hat{R}_f^A(x).$$

(B is a basis of regular neighborhoods of x in U). This establishes that $\hat{R}_f^A|_{\hat{U}} \equiv \widehat{(R_f^A)_{\hat{U}}}$.

We are now in a position to complete the proof of this theorem.

$$R_s^E(x) = (R_s^E)_{\hat{U}}(x) = \widehat{(R_s^E)_{\hat{U}}}(x) = \hat{R}_s^E(x)$$

since $E \subset \hat{U}$-$\{x\}$ and since $\{x\}$ is a $\mathcal{G}_\delta$ in U. This establishes i).

ii) Let $E \subset U$-$\{x\}$ be $\mathcal{K}$-analytic. Let F be a σ-compact in E with $R_s^E(x) = R_s^F(x)$. F meets only countably many components of U-$\{x\}$ so

$$R_s^E(x) = R_s^F(x) = \hat{R}_s^F(x) \leqslant \hat{R}_s^E(x) \leqslant R_s^E(x).$$

This establishes ii).

PROPOSITION 6. – *Let* U *be a domain of type* $\mathfrak{P}$, $x \in \mathrm{U}$ *and* $\mathrm{E} \subset \mathrm{U}\text{-}\{x\}$. *If* E *is weakly thin at* x *it is thin at* x *in either of the following cases :*

i) E *meets only countably many components of* U-$\{x\}$.

ii) E *is* $\mathcal{K}$*-analytic.*

Proof. – Let E be weakly thin at x. Let $\delta \subset\subset \mathrm{U}$ be a neighborhood of x such that $\hat{\mathrm{R}}_p^{\mathrm{E}\cap\overline{\delta}}(x) < p(x)$ for some strictly positive continuous potential p on U. $\mathrm{E} \cap \overline{\delta}$ meets only countably many components of U-$\{x\}$ if i) holds and $\mathrm{E} \cap \overline{\delta}$ is $\mathcal{K}$-analytic if ii) is satisfied. Consequently if i) or ii) hold then $\mathrm{R}_p^{\mathrm{E}\cap\overline{\delta}}(x) = \mathrm{R}_p^{\mathrm{E}\cap\overline{\delta}}(x) < p(x)$. This establishes the proposition.

For the sake of completeness we wish to establish Bauer's sufficient conditions for the equivalence of thinness and strong thinness, Bauer [1, Theorem 4]. He establishes this theorem in the case where U has a countable basis of neighborhoods. We remove this restriction. Our theorem depends on the Riesz-Martin representation theorem for the cone of positive superharmonic functions $\mathcal{S}_{\mathrm{U}}^+$ established in full generality by Constantinescu in [6]. This proposition is essentially independent of the rest of the results of this paper.

PROPOSITION 7. – *Let* U *be a domain of type* $\mathfrak{P}$, *let* $x \in \mathrm{U}$ *and* E U-$\{x\}$. *If* E *is thin at* x *it is strongly thin at* x *if either* i) *or* ii) *hold.*

i) $\{x\}$ *is polar*

ii) *Every locally bounded potential* p *in* U *with support* $\phi(p) \subset \{x\}$ *is continuous at* x.

Proof. – If the theorem is true when ii) holds it immediately follows if i) holds, for if $\{x\}$ is polar and p is a locally bounded potential with $\phi(p) \subset \{x\}$ then p is in fact harmonic in U hence, is continuous at x. In order to prove this theorem for ii) we shall need to establish the following lemma.

LEMMA 8. – *Let* U *be a domain of type* $\mathfrak{P}$, $x \in \mathrm{U}$ *and* $\mathrm{E} \subset \mathrm{U}\text{-}\{x\}$. *Let* E *be thin at* x *but not strongly thin. Let* x *be an element of* $\mathcal{S}_{\mathrm{U}}^+$ *not specifically dominating any potential* p *with* $\phi(p) = \{x\}$. *Then* $\liminf\limits_{\substack{y \to x \\ y \in \mathrm{E}}} s(y) = s(x)$.

Proof. – We first note that since E is not strongly thin at x, $x \in \mathrm{E}$. We next recall that if s_1, s_2 are in $\mathcal{S}^+_{\mathrm{U}}$, $s_1 \succ s_2$ means that s_1 specifically dominates s_2 and this is true iff $s_1 = s_2 + s$ for some $s \in \mathcal{S}^+_{\mathrm{U}}$. By Corollary 3.2 of Constantinescu [6], $\mathcal{S}^+_{\mathrm{U}}$ possesses a compact base Δ for a locally convex vector space topology, T, on $\mathcal{S}^+_{\mathrm{U}}$. (This topology is a generalization of Herve's topology). $\mathcal{S}^+_{\mathrm{U}} - \mathcal{S}^+_{\mathrm{U}}$ is a completely reticulated vector lattice for the specific order induced by the cone $\mathcal{S}^+_{\mathrm{U}}$. Δ is therefore a Choquet simplex. Every element $s \in \mathcal{S}^+_{\mathrm{U}}$ has a representation as the barycenter of a unique positive maximal measure "carried" by the extreme points, $\xi(\Delta)$, of Δ. If s_1 and s_2 are in $\mathcal{S}^+_{\mathrm{U}}$ with corresponding maximal measures μ_1 and μ_2 then $s_1 \succ s_2$ iff $\mu_1 \geqslant \mu_2$ as measures. The elements of $\xi(\Delta)$ are either minimal harmonic functions in Δ or are potentials with point support, Herve [10, Theorem 16.2]. If $\mathcal{P}^{\{x\}}_{\mathrm{U}}$ to the cone

$$\{p \in \mathcal{P}_{\mathrm{U}} : \phi(p) = \{x\}\}$$

then $p \in \xi(\Delta)$ iff $p \in \xi(\mathcal{P}^{\{x\}}_{\mathrm{U}} \cap \Delta)$. On $\xi(\Delta)$-$\mathcal{H}^+_{\mathrm{U}}$ the mapping $p \to \phi(p)$ is a continuous projection onto U such that U possesses the quotient topology under ϕ. $p \in \mathcal{S}^+_{\mathrm{U}}$ is in $\mathcal{P}^{\{x\}}_{\mathrm{U}}$ iff μ_p, the representing measure on $\xi(\Delta)$ for p, satisfies $\mu_p(\xi(\Delta)\text{-}\phi^{-1}(x)) = 0$. Consequently, $s \in \mathcal{S}^+_{\mathrm{U}}$ specifically dominates no p in $\mathcal{P}^{\{x\}}_{\mathrm{U}}$ iff $\mu_s(\phi^{-1}\{x\}) = 0$. This is the case iff $\inf\{\mu_s[\phi^{-1}(\delta)] : \delta \in \mathrm{B} = 0\}$ for some basis B of regular neighborhoods of $\{x\}$ strongly contained in U.

Now let s be an element of $\mathcal{S}^+_{\mathrm{U}}$ specifically dominating no element of $\mathcal{P}^{\{x\}}_{\mathrm{U}}$. Let $\{\omega_n : n \in \mathrm{Z}^+\}$ be a strongly decreasing sequence of regular neighborhoods of x with $\omega_1 = \mathrm{U}$ and $\sum\limits_{n=1}^{\infty} \mu_s[\phi^{-1}(\omega_n)] < \infty$.

For any $n \in \mathrm{Z}^+$, let $\mu_n = \chi_{\phi^{-1}(\omega_n)}\mu_s$. Then $\mu_s = \sum\limits_{n=1}^{\infty} (\mu_n - \mu_{n+1})$ and $s = \sum\limits_{n=1}^{\infty} \int_{\xi(\Delta)} p d(\mu_n - \mu_{n+1})$. We assume that

$$\liminf_{\substack{y \to x \\ y \in \mathrm{E}}} [s(y) - s(x)] = \delta > 0$$

and show that E is strongly thin at $\{x\}$. This will establish the lemma.

Since $s(x) < \infty$ we must have

$$\sum_{n=1}^{\infty} \int_{\xi(\Delta)} p(x)\, d(\mu_n - \mu_{n+1}) < \infty.$$

If $s_n = \int_{\xi(\Delta)} p d\mu_n$ then $\lim_{n\to\infty} s_n(x) = 0$. Let $\{\nu_n : n \in Z^+\}$ be a subsequence of $\{\mu_n : n \in Z^+\}$ such that $\sum_{n=1}^{\infty} \int_{\xi(\Delta)} p(x) d\nu_n < \infty$. Let

$$\widetilde{s}_n = \int_{\xi(\Delta)} p d\nu_n \qquad \text{and let} \qquad \widetilde{s} = \sum_{n=1}^{\infty} \widetilde{s}_n = \int_{\xi(\Delta)} pd\Big(\sum_{n=1}^{\infty} \nu_n\Big).$$

Then $\widetilde{s}_n \in \mathfrak{S}_U^+$ for all $n \in Z^+$ and $\widetilde{s} \in \mathfrak{S}_U^+$. $\widetilde{s} \leqslant s$ so $\widetilde{s}(x) < \infty$. We assert that $\liminf_{\substack{y\to x\\ y\in E}} [\widetilde{s}(y) - \widetilde{s}(x)] = \infty$. To see this we first pick n_0 so that $\sum_{n=n_0+1}^{\infty} \widetilde{s}_n(x) < \delta$. For any $n \in Z^+$, $s - s_n = \int_{\xi(\Delta)-\phi^{-1}(\omega_n)} pd\mu_s$ is harmonic in ω_n hence is continuous at x. As a result,

$$\liminf_{\substack{y\to x\\ y\in E}} [s_n(y) - s_n(x)] \geqslant \delta.$$

Thus, for each $n \in Z^+$,

$$\liminf_{\substack{y\to x\\ y\in E}} [\widetilde{s}_n(y) - \widetilde{s}_n(x)] \geqslant \delta.$$

We see that

$$\liminf_{\substack{y\to x\\ y\in E}} [\widetilde{s}(y) - \widetilde{s}(x)] \geqslant \liminf_{\substack{y\to x\\ y\in E}} \left[\sum_{n=1}^{K} (\widetilde{s}_n(y) - \widetilde{s}_n(x))\right] - \sum_{n=K+1}^{\infty} \widetilde{s}_n(x).$$

If $K \geqslant n_0$ then

$$\liminf_{\substack{y\to x\\ y\in E}} [\widetilde{s}(y) - \widetilde{s}(x)] \geqslant \sum_{n=1}^{K} \liminf_{\substack{y\to x\\ y\in E}} [\widetilde{s}_n(y) - \widetilde{s}_n(x)]$$
$$- \delta \geqslant K \cdot \delta - \delta = (K - 1)\delta.$$

Since $K \geqslant n_0$ is arbitrary, $\liminf_{\substack{y\to x\\ y\in E}} [\widetilde{s}(y) - \widetilde{s}(x)] = \infty$. This is true only if E is strongly thin at x. This contradiction establishes that $\liminf_{\substack{y\to x\\ y\in E}} s(y) = s(x)$.

Proof of Proposition 7. – We assume that ii) holds and that E is thin at x. If $x \notin \overline{E}$ then E is strongly thin at x. Consequently we may assume that $x \in \overline{E}$. There is an $s \in \mathcal{S}_U^+$ with

$$\liminf_{\substack{y \to x \\ y \in E}} s(y) > s(x) > 0 .$$

Let $0 \neq h \in \mathcal{H}_U^+$ with $h(x) = \liminf_{\substack{y \to x \\ y \in E}} s(y)$. If $s' = s \wedge h$ then s' is locally bounded and $\liminf_{\substack{y \to x \\ y \in E}} s'(y) > s'(x)$. Let $s' = \int_{\xi(\Delta)} p d\nu$ represent s' as the barycenter of a positive measure ν on $\xi(\Delta)$ where Δ is a compact base of $\mathcal{S}_U^+$. Let

$$s_1 = \int_{\phi^{-1}\{x\}} p\, d\nu \leqslant s \quad , \quad s_2 = \int_{\xi(\Delta)-\phi^{-1}\{x\}} p\, d\nu \leqslant s .$$

Since s_1 is a potential with support $\{x\}$ and is locally bounded it is continuous at x. It follows that $\liminf_{\substack{y \to x \\ y \in E}} s_2(y) > s_2(x)$. s_2 does not specifically dominate any potential with support in $\{x\}$, consequently E must be strongly thin at x by the previous lemma. This establishes the proposition.

Our final result is an application of Proposition 3 to demonstrate the existence of a strong barrier at a regular boundary point of a domain whose closure has a neighborhood of type $\mathcal{R}$.

DEFINITION. – *Let* U *be an open set in* X *with* $z \in \partial U$. *A barrier in* U *at* z *is a superharmonic function* $s > 0$ *with domain* $\omega \cap U$, *for* ω *some neighborhood of* z *in* X, *such that* $\limsup_{\substack{x \to z \\ x \in \omega \cap U}} s(x) = 0$.

An ω*-barrier in* U *at* z, *for a neighborhood* ω *of* z, *is a barrier in* U *at* z, *whose domain includes* $\omega \cap U$, *which satisfies* $\liminf_{\substack{x \to y \\ x \in \omega \cap U}} s(x) \geqslant \rho$ *for all* $y \in \partial\omega \cap \overline{U}$ *for some real* $\rho > 0$.

A strong barrier in U *at* z *is a barrier* s *in* U *at* z *with domain* $\omega \cap U$, *for some neighborhood* ω *of* z, *such that* s *is a* δ*-barrier for any neighborhood* δ *of* z *with* $\delta \subset\subset \omega$. *Equivalently* s *is a strong* *barrier at z iff for some neighborhood ω of z, s is an ω-barrier, and, for all neighborhoods δ of z with $\delta \subset\subset \omega$, s is bounded away from* 0 *on* $(\omega - \delta) \cap \mathrm{U}$.

Remarks. – The existence of a barrier for an open set U at a boundary point is a necessary and sufficient condition for the regularity of z as a boundary point Brelot [3, Theorem 22]. The use of a strong barrier makes the proof of regularity somewhat simpler. The proof of the following proposition is simply an extension of the method of proof used by Brelot for that theorem.

PROPOSITION 9. – *Let* U *be a domain whose closure has a neighborhood of type* $\mathcal{H}$. *Let the point* $z \in \overline{\mathrm{U}}$ *possess a barrier in* U. *There is a strong barrier in* U *at* z.

Proof. – Let h be a barrier in U at z. Let ω be an open set containing z with $\omega \cap \mathrm{U} \subset \mathrm{domain}(h)$. By Proposition 3, z has a countable basis of neighborhoods in $\overline{\mathrm{U}}$. Let $\{\delta_n : n \in \mathrm{Z}^+\}$ be a strongly decreasing sequence of regular neighborhoods of z in X such that $\{\delta_n \cap \mathrm{U} : n \in \mathrm{Z}^+\}$ is a basis of neighborhoods of z in $\overline{\mathrm{U}}$, and such that $\delta_1 \subset\subset \omega$. The strong barrier we will construct will have domain $\delta_1 \cap \mathrm{U}$. We will inductively construct a decreasing sequence $\{s_n : n \in \mathrm{Z}^+\} \subset \mathcal{S}^+_{\delta_1 \cap \mathrm{U}}$, and a subsequence $\{\delta(m) : m \in \mathrm{Z}^+\}$ of $\{\delta_n : n \in \mathrm{Z}^+\}$ satisfying :

i) For any neighborhood δ of z with $\delta \subset\subset \delta_1$ and for any $m \in \mathrm{Z}^+$ there is a real $\rho(\delta, m) > 0$ such that $s_m \geqslant \rho(\delta, m)$ on $\mathrm{U} \cap (\delta_1 - \overline{\delta})$.

ii) $\limsup\limits_{\substack{x \to z \\ x \in \mathrm{U} \cap \delta_1}} s_m(x) < 2^{-m}$ for all $m \in \mathrm{Z}^+$.

iii) $s_m = s_{m-1}$ on $[\delta_1 - \delta(m)] \cap \mathrm{U}$ for all $m \in \mathrm{Z}^+$.

If we have constructed these sequences, the sequence $\{s_m : m \in \mathrm{Z}^+\}$ is eventually constant on any compact $\mathrm{K} \subset \mathrm{U} \cap \delta_1$. $s = \lim\limits_{m \to \infty} s_m$ is easily seen to be in $\mathcal{S}^+_{\mathrm{U} \cap \delta_1}$ and $s \leqslant s_m$ for all $m \in \mathrm{Z}^+$. If $\delta \subset\subset \delta_1$ is a neighborhood of z there is an m such that $\delta_{n(m)} \cap \mathrm{U} \subset \delta \cap \mathrm{U}$. $s = s_m$ on $[\delta_1 - \delta_{n(m)}] \cap \mathrm{U}$ so $s \geqslant \rho(\delta, m) > 0$ on $[\delta_1 - \delta] \cap \mathrm{U}$. Since $s \leqslant s_m$ for all m, $\limsup\limits_{\substack{x \to z \\ x \in \delta_1 \cap \mathrm{U}}} s(x) = 0$. Therefore s is the desired strong barrier in U at z.

To establish the theorem we only have to construct the sequences $\{s_m : m \in Z^+\}$ and $\{\delta_{n(m)} : m \in Z^+\}$.

Let K(1) be a compact subset of $(\partial\delta_1) \cap U$ such that if

$$\theta(1) = (\partial\delta_1) \cap U - K(1) \quad \text{then} \quad H^{\delta_1}_{\chi_{\theta(1)}}(z) < \frac{1}{2}.$$

For a large enough $\lambda_1 > 0$, $s_1 = t_1 = \lambda_1 h + H^{\delta_1}_{\chi_{\theta(1)}}$ has the property that $\liminf\limits_{\substack{y \to x \\ y \in U \cap \delta_1}} s_1(y) \geqslant 1$ for all $x \in (\partial\delta_1) \cap \overline{U}$. We assert that s_1 is bounded away from 0 on $[\delta_1 - \overline{\delta}] \cap U$ for any neighborhood δ of z with $\delta \subset\subset \delta_1$. This is because $H^{\delta_1}_{\chi_{\theta(1)}}$ is bounded away from 0 in $\delta_1 - W$, where W is a relatively open set in $\overline{\delta}_1$ such that $K(1) \subset W$ and $\overline{W}$ is compact in U. Since h must be bounded away from 0 on any compact in $\omega \cap U$ it is bounded away from 0 on $\overline{W}$. Consequently s_1 is bounded away from 0 on $[\delta_1 - \overline{\delta}] \cap U$ for any such δ. That is s_1 satisfies i). Since

$$\limsup_{\substack{x \to z \\ x \in \delta_1 \cap U}} s_1(x) = \limsup_{\substack{x \to z \\ x \in \delta_1 \cap U}} [\lambda_1 h(x) + H^{\delta_1}_{\chi_{\theta(1)}}(x)] < 0 + \frac{1}{2},$$

ii) is satisfies. iii) is vacuously satisfied when $\delta(1) = \delta_1$.

Assume, for the induction step, then s_m and $\delta(m)$ have been constructed satisfying i), ii) and iii). Choose $\delta(m+1)$ so that

$$\delta(m+1) \cap U \subset s_m^{-1}[0, 2^{-m}).$$

Choose $K(m+1)$ compact in $[\partial\delta(m+1)] \cap U$ so that if

$$\theta(m+1) = ([\partial\delta(m+1)] \cap U) - K(m+1)$$

then $H^{\delta_1}_{\chi_{\theta(m+1)}}(z) < \frac{1}{2}$. Pick $\lambda_{m+1} > 0$ so large that if

$$t_{m+1} = \lambda_{m+1} h + 2^{-m} H^{\delta(m+1)}_{\chi_{\theta(m+1)}}$$

then $\liminf\limits_{\substack{x \to y \\ x \in U \cap \delta(m+1)}} t_{m+1}(x) \geqslant 2^{-m}$ for all $y \in [\partial\delta(m+1)] \cap \overline{U}$. We note that $\limsup\limits_{\substack{x \to z \\ x \in \delta(m+1) \cap U}} t_{m+1}(x) < 2^{-(m+1)}$ and that t_{m+1} is bounded away from 0 on any $[\delta(m+1)-\overline{\delta}]\cap \mathrm{U}$ if $\delta \subset\subset \delta(m+1)$ is a neighborhood of z in X. Let

$$s_{m+1} = \begin{cases} s_m \text{ on } [\delta_1 - \delta(m+1)] \cap \mathrm{U} \\ t_{m+1} \wedge s_m \text{ in } \delta(m+1) \cap \mathrm{U} \end{cases} .$$

Then $s_{m+1} \in \mathcal{S}^+_{\delta_1 \cap \mathrm{U}}$, $s_{m+1} \leqslant s_m$ and s_{m+1} satisfies i), ii) and iii) of the induction hypotnesis. This completes the proof of the induction step hence establishes the theorem.

We remark that this proof did not depend on the fact that U was a domain but only on the fact that $\{z\}$ was a $\mathcal{G}_\delta$ in the topological space $\{z\} \cup \mathrm{U}$. This is true iff any compact neighborhood of z in X meets at most countably many components of U. Conversely if there is a strong barrier in U at z then $\{z\}$ is easily seen to be a $\mathcal{G}_\delta$ in $\{z\} \cup \mathrm{U}$.

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