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STRASSEN'S LAW OF THE ITERATED LOGARITHM

by James D. KUELBS (*).

1. Introduction.

Throughout E is a real locally convex Hausdorff topological vector space. If the topology on E is metrizable and complete, then E is called a Frechet space and it is well known that the topology on E is generated by an increasing sequence of semi-norms $\|\cdot\|_j$ ($j = 1, 2, \dots$) such that

$$\|\cdot\|_E = \sum_{j=1}^{\infty} \frac{\|\cdot\|_j}{2^j(1 + \|\cdot\|_j)}$$

gives an invariant metric on E which generates the topology on E . A semi-norm $\|\cdot\|$ is called a *Hilbert semi-norm* if $\|x\|^2 = (x, x)$ where (\cdot, \cdot) is an inner product which may possibly vanish at some $x \neq 0$. We say a Frechet space E is of *Hilbert space type* if we can choose an increasing sequence of Hilbert semi-norms which generate the topology of E .

Let E be a vector space over the reals and assume $E = \bigcup_{n=1}^{\infty} E_n$ where $\{E_n : n \geq 1\}$ is an increasing sequence of linear subspaces of E . Further, assume each E_n is a Frechet space such that the topology induced by E_{n+1} on E_n is identical to the topology initially given on E_n . Given the

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sequence of subspaces $\{E_n\}$ we define a locally convex Hausdorff topology on E called the strict inductive limit topology by saying a convex subset V of E is a neighbourhood of zero if $V \cap E_n$ is a neighbourhood of zero in E_n for $n = 1, 2, \dots$. When we provide E with this topology we call E a *strict inductive limit of real Frechet space* $\{E_n\}$.

By the Borel subsets of a topological vector space we mean the minimal sigma algebra containing the open sets. All probability measures are assumed to be defined on the Borel sets. The Borel probability measure μ is *tight* if for each $\epsilon > 0$ there exists a compact subset K_ϵ such that

$$\mu(K_\epsilon) > 1 - \epsilon.$$

We say μ is *regular* if for each $\epsilon > 0$ and each Borel set A there is a compact set $K \subseteq A$ such that $\mu(A \cap K^c) < \epsilon$.

If E is a Frechet space it is not difficult to show that if μ is a tight Borel probability measure on E , then there exists a closed separable subspace M of E such that $\mu(M) = 1$. Furthermore, if E is a separable Frechet space then it is well known that all Borel probability measures on E are regular, and hence a tight probability measure on a Frechet space is regular.

A Borel probability measure on a locally convex topological vector space is said to be a *mean-zero Gaussian measure* if each continuous linear functional on E has a Gaussian distribution with mean zero.

The basic structure of a mean-zero Gaussian measure has been the object of much study and is given, for example, in [2, Theorem 4]. In [4] slightly more detailed information is provided when E is a strict inductive limit of Frechet spaces, and it is this we turn to now.

Let μ be a tight mean-zero Gaussian measure on E where E is a strict inductive limit of Frechet spaces $\{E_n\}$. Then it is known (see [4] for details) that there exists a unique separable Hilbert space H_μ with norm $\|\cdot\|_\mu$ such that

- 1) $H_\mu \subseteq E_N$ for some N ;
- 2) the closure of H_μ in E_N (and hence in E) is a separable subspace of E_N of μ -measure one;
- 3) the identity map of H_μ into E_N is continuous. If Γ

is the map which restricts an element in E' (the topological dual of E) to H_μ , then Γ is linear and $\Gamma(E')$ is a dense linear subspace of H'_μ (the topological dual of H_μ). Hence if we identify H'_μ and H_μ (as we do) then $\Gamma(E')$ can be viewed as a dense subset of H_μ . Further for $f \in E'$ we have

$$\|\Gamma(f)\|_\mu^2 = \int_E |f(x)|^2 \mu(dx)$$

4) If m is the canonical Gaussian cylinder set measure on H_μ restricted to the cylinder sets of E induced by E' then m extends to a unique regular mean zero Gaussian measure $\tilde{\mu}$ on E and, in fact, $\tilde{\mu} = \mu$.

Here by the canonical Gaussian cylinder set measure on H_μ we mean the cylinder set measure on H_μ determined by specifying that every linear functional f on H_μ is Gaussian with mean-zero and variance $\|f\|_\mu^2$ where we have identified H_μ and H'_μ in the usual way to compute the norm of f .

Thus if μ is a tight mean zero Gaussian measure on a strict inductive limit of Frechet spaces E then by the previous remarks we see μ is *uniquely determined* on E by the unique Hilbert space H_μ and we denote this relationship by saying μ is *generated by* H_μ .

Let Ω_E denote the continuous functions ω from $[0, \infty)$ into the strict inductive limit E of the Frechet spaces $\{E_n\}$ such that $\omega(0) = 0$, and let \mathcal{F} be the sigma-algebra of Ω_E generated by the functions $\omega \rightarrow \omega(t)$. Let μ be a tight mean-zero Gaussian measure on E generated by H_μ , and define for each Borel set $A \subseteq E$ and $t \geq 0$

$$\mu_t(A) = \begin{cases} \mu(A/\sqrt{t}) & t > 0 \\ \delta_0(A) & t = 0 \end{cases}$$

where δ_0 denotes the unit mass at zero. Then by [4, Theorem 4] there is a unique probability measure P on \mathcal{F} such that if $0 = t_0 < t_1 < \dots < t_n$ then $\omega(t_j) - \omega(t_{j-1})$ ($j = 1, \dots, n$) are independent and $\omega(t_j) - \omega(t_{j-1})$ has distribution $\mu_{t_j - t_{j-1}}$ on E . The stochastic process $\{W_t : t \geq 0\}$ defined on $(\Omega_E, \mathcal{F}, P)$ by $W_t(\omega) = \omega(t)$ has stationary independent mean-zero Gaussian increments and we call it the *Brownian motion in E generated by μ* or, for simplicity, *μ -Brownian motion in E* . In the case E is a real separable Banach space

the existence of a μ -Brownian motion in E was first considered in [3].

Let F be a topological vector space, and let C_F denote the continuous functions on $[0, 1]$ into F which vanish at zero.

If F is a locally convex Hausdorff topological vector space whose topology is generated by the semi-norms $\{\|\cdot\|_\alpha : \alpha \in A\}$, then we make C_F into a locally convex Hausdorff space in the topology generated by the semi-norms $\{\|f\|_{\alpha, \infty} : \alpha \in A\}$ where $\|f\|_{\alpha, \infty} = \sup_{0 \leq t \leq 1} \|f(t)\|_\alpha$. It is easy to see that the topology on C_F is independent of the family of semi-norms used to generate the given topology on F . Further, if F is a Frechet space whose topology is generated by the increasing sequence of semi-norms $\|\cdot\|_j$ then we make C_F into a Frechet space in the locally convex topology generated by the sequence of semi-norms

$$\|f\|_{j, \infty} = \sup_{0 \leq t \leq 1} \|f(t)\|_j,$$

and if N is a closed subspace of F , then C_N is a closed subspace of C_F . If E is a strict inductive limit of the Fre-

chet spaces $\{E_n\}$ then we have $C_E = \bigcup_{n=1}^{\infty} C_{E_n}$ since we know each compact set in E (in particular, every continuous image of $[0, 1]$ into E) must be a compact subset of some E_n . Further, the topology induced on C_{E_n} by $C_{E_{n+1}}$ is that originally given for C_{E_n} since E_{n+1} induces on E_n the given topology on E_n . Hence we make C_E into a strict inductive limit of Frechet spaces $\{C_{E_n}\}$.

If $\|\cdot\|$ is a continuous semi-norm on E we define the semi-norm $\|f\|_\infty$ on C_E by $\|f\|_\infty = \sup_{0 \leq t \leq 1} \|f(t)\|$.

If μ is a mean-zero Gaussian measure on E (the strict inductive limit of Frechet spaces $\{E_n\}$) then the probability measure P induced on C_E by μ -Brownian motion in E up to time $t = 1$ is called the *Wiener measure on C_E generated by μ* or the μ -*Wiener measure on C_E* . Further, since C_E is a strict inductive limit of the Frechet spaces $\{C_{E_n}\}$ the μ -Wiener measure P on C_E is, as one might expect, a tight mean-zero Gaussian measure on C_E and hence is generated

by a Hilbert space \mathcal{H} in C_E . Indeed, Theorem 4 of [4] shows that

$$(1.1) \quad \mathcal{H} = \left\{ f \in C_E : f(t) = \sum_{j \geq 1} \int_0^t \frac{d}{ds} e_j(f(s)) ds \quad e_j, 0 \leq t \leq 1 \right\} \\ \text{and } \|f\|_{\mathcal{H}} < \infty$$

where the convergence of the series is in the H_μ norm for each t , $\{e_j : j \geq 1\}$ is any complete orthonormal set in H_μ which is a subset of E' (recall E' is viewed as densely embedded in H_μ), and the norm on \mathcal{H} is given by

$$(1.2) \quad \|f\|_{\mathcal{H}}^2 = \sum_{j \geq 1} \int_0^1 \left[\frac{d}{ds} e_j(f(s)) \right]^2 ds.$$

Here, of course, $e_j(x)$ denotes the value of the linear functional corresponding to e_j at x .

2. Strassen's Log Log result.

Let E be a strict inductive limit of Frechet spaces $\{E_n\}$ of Hilbert space type. Assume X_1, X_2, \dots are independent identically distributed E -valued random variables such that $E(f(X_1)) = 0$ for all $f \in E'$ and satisfying:

(A) The Borel measure λ induced by each X_k on E is tight and the distribution of each f in E' with respect to λ is degenerate at a point or absolutely continuous with respect to Lebesgue measure, and

$$(B) \quad E\|X_1\|_{j,n}^3 < \infty \quad (j, n = 1, 2, \dots)$$

where $\{\|\cdot\|_{j,n} : j \geq 1\}$ denotes the increasing family of Hilbert semi-norms generating the topology on E_1 ($n = 1, 2, \dots$).

Further, let $S_0 = 0$, $S_n = X_1 + \dots + X_n$ for $n \geq 1$, and define the random polygonal functions $\{f_n\}$ to be

$$(2.1) \quad f_n\left(\frac{k}{n}, \omega\right) = \frac{S_k(\omega)}{\sqrt{2nLLn}} \quad (k = 0, \dots, n; n \geq 3)$$

where $L \log n$ denotes $\log \log n$, and $f_n(t, \omega)$ is linear over subintervals $\frac{k}{n} \leq t \leq \frac{(k+1)}{n}$ for $0 \leq k \leq n-1$.

If $\|\cdot\|$ is a semi-norm on E and F is any subset of E we define for each x in E

$$(2.2) \quad \|x - F\| = \inf_{y \in F} \|x - y\|$$

THEOREM 1. — *Let E be a strict inductive limit of Frechet spaces of Hilbert space type and assume $\{X_k\}$ is a sequence of independent identically distributed E -valued random variables satisfying conditions (A) and (B). Then there exists a unique regular mean zero Gaussian measure μ on E determined by the covariance function*

$$(2.3) \quad T(f, g) = E(f(X_1)g(X_1)) \quad (f, g \in E').$$

Further, if \mathcal{K} is the unit ball of the Hilbert space \mathcal{H} which generates μ -Wiener measure on C_E then, for every continuous semi-norm $\|\cdot\|$ on E we have

$$(2.4) \quad P\left(\lim_n \|f_n - \mathcal{K}\|_\infty = 0\right) = 1,$$

and

$$(2.5) \quad P(C(\{f_n : n \geq 3\}) = \mathcal{K}) = 1$$

where $C(\{g_n\})$ denotes the cluster points of the sequence $\{g_n\}$ in C_E with respect to the semi-norm $\|\cdot\|_\infty$.

COROLLARY 1. — *Under the assumptions of Theorem 1, if μ is the unique regular mean-zero Gaussian measure on E determined by the covariance function (2.3) and K is the unit ball of the Hilbert space H_μ which generates μ on E , then for every continuous semi-norm $\|\cdot\|$ on E we have*

$$(2.6) \quad P\left(\lim_n \left\| \frac{S_n}{\sqrt{2nL \log n}} - K \right\| = 0\right) = 1$$

and

$$(2.7) \quad P\left(C\left(\left\{ \frac{S_n}{\sqrt{2nL \log n}} : n \geq 3 \right\}\right) = K\right) = 1$$

where $C(\{x_n\})$ denotes the cluster points of the sequence $\{x_n\}$ in E with respect to the semi-norm $\|\cdot\|$.

Remark. — (1) In case E is a real Frechet space of Hilbert space type then (2.6) and (2.7) hold in the topology of E since convergence in E is equivalent to the convergence in countably many continuous semi-norms. Further, since C_E is also a Frechet space when E is a Frechet space we have $\{f_n: n \geq 3\}$ converging and clustering to \mathcal{K} with probability one in C_E .

(2) If we assume the existence of a tight mean zero Gaussian measure μ on E with covariance function as given in (2.3), then Theorem 1 and its corollary are valid under the weaker assumptions that $\{X_k\}$ is a sequence of independent identically distributed E -valued random variables with covariance as given in (2.3), $E(f(X_1)) = 0$ for each $f \in E'$, and property (B) holds.

3. Sketch of the proof of Theorem 1.

Since (A) holds we know by arguing as in Theorem 1 of [4] that $\lambda(E_N) = 1$ for some integer N . Further, using the ideas of [4, Theorem 1] we produce a sequence of positive constants a_j such that $\sum_{j=1}^{\infty} \max \{a_j, a_j^{1/2}(E\|X_1\|_{j,N}^3)^{1/3}\} < \infty$ and an inner product

$$(3.1) \quad (x, x)_0 = \sum_{j=1}^{\infty} a_j (x, x)_{j,N}$$

which converges on a subset of E_N yielding a subspace H of E_N satisfying:

(3.2) H is a real separable Hilbert space in the norm $\|\cdot\|_0$ given by the inner product (3.1).

(3.3) The identity map of H into E_N is continuous and maps Borel subsets of H to Borel subsets of E_N . In fact, the Borel subsets of H are precisely the Borel subsets of E_N intersected with H .

(3.4) $\lambda(H) = 1$ and hence λ determines a unique tight probability measure on the Borel subsets of H .

Now $\lambda(H) = 1$ implies $X_1 + \dots + X_n$ is in H with pro-

bability one for all $n \geq 1$ so we need only work in H . Since $\sum_{j=1}^{\infty} a_j^{1/2} (E \|X_1\|_{j,N}^3)^{1/3} < \infty$ we have from (3.1) that

$$E \|X_1\|_0^3 < \infty.$$

Further, since E' separates points of H we have E' dense in H' under the mapping which restricts an f in E' to H , and hence $E(f(X_1)) = 0$ for each $f \in E'$ implies $E(X_1) = 0$ where $E(X_1)$ is the Bochner integral of X_1 in H (note that X_1 is H -valued with probability one).

The covariance function $T(f, g)$ defined in (2.3) can thus be viewed as being defined on a dense subspace of H' . Now $E \|X_1\|_0^2 < \infty$ implies T can be extended to a symmetric non-negative continuous bilinear form on H' and thus T determines a symmetric non-negative bounded operator S on H' which is also of trace class. Hence S determines a unique mean zero Gaussian measure on H which is, of course, regular. Using (3.3) we now get μ regular on E_N and hence on E since the identity map of E_N into E is continuous. That μ is a mean zero Gaussian measure on E follows easily and it is unique by [2].

Since $\mu(H) = 1$ the μ -Brownian motion in E assigns probability one to the subset of continuous paths from $[0, \infty)$ into H and the μ -Wiener measure assigns probability one to C_H which is a Borel subset of C_E . We also know that $H_\mu \subseteq H$ and that H_μ is the Hilbert space in H which generates μ on H . Furthermore, \mathcal{K} is the unit ball of the Hilbert space which generates the μ -Wiener measure on C_H .

Thus it suffices to prove the theorem when X_1, X_2, \dots are H -valued and $\|\cdot\|_\infty$ is replaced by $\|\cdot\|_{0,\infty}$ since any continuous semi-norm on E restricted to H is weaker than the norm $\|\cdot\|_0$. To do this we use a combination of the ideas developed in [5] where we proved Strassen's Log Log Law for Brownian motion in a Banach space, and those of [1] which exploited the use of Berry-Essen estimates to give a proof of Strassen's result [8] for one dimensional Brownian motion. Of course, we need Berry-Essen estimates for Hilbert space valued random variables and these are obtained in [9] extending the results of [6] and [7]. In fact, our results in the Hilbert space case require only that X_1, X_2, \dots be independent

with mean zero, common covariance, and $\sup_k E\|X_1\|^3 < \infty$ where $\|\cdot\|$ is the Hilbert space norm. Hence Theorem 1 now follows from Theorem 3.2 of [9].

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