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LEO SARIO

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COMPLETENESS AND EXISTENCE OF BOUNDED BIHARMONIC FUNCTIONS ON A RIEMANNIAN MANIFOLD

by **Leo SARIO** (1)

A.S. Galbraith has communicated to us the following intriguing problem : Does the completeness of a manifold imply, or is it implied by, the emptiness of the class H^2B of bounded nonharmonic biharmonic functions ? Among all manifolds considered thus far in biharmonic classification theory (cf. Bibliography), those that are complete fail to carry H^2B-functions, and one might suspect that this is always the case. We shall show, however, that there do exist complete manifolds of any dimension that carry H^2B-functions. Moreover, there exist both complete and incomplete manifolds not permitting these functions, and, trivially, incomplete manifolds possessing them.

We attach a Bibliography of recent work in the field.

1. Let C be the totality of complete Riemannian manifolds M, characterized by an infinite distance of any point of M to the ideal boundary. Denote by $\mathcal{O}^N_{H^2B}$ and $\widetilde{\mathcal{O}}^N_{H^2B}$ the classes of N-manifolds, $N \geqslant 2$, for which $H^2B = \emptyset$ or $H^2B \neq \emptyset$, respectively.

THEOREM 1. – $C \cap \widetilde{\mathcal{O}}^N_{H^2B} \neq \emptyset$ *for every* N.

Proof. – Take the N-cylinder

$$|x| < \infty, \qquad |y| \leqslant 1, \qquad i = 1,2,\ldots,N-1,$$

with each face $y_i = 1$ identified with $y_i = -1$, so as to obtain a covering space of the N-torus in the same manner as a conventional cylinder is a covering surface of the torus. Let T be this N-cylinder with the Riemannian metric

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$$ds^2 = \mu^{-2}(x)\,dx^2 + \mu^{4/(N-1)}(x) \sum_{i=1}^{N-1} dy_i^2$$

where

$$\mu(x) = (2 + x^2)^{\frac{1}{2}} \log(2 + x^2).$$

To see that $T \in C$, it suffices to show, in view of the symmetry, that $\int_0^\infty \mu^{-1}(x)\,dx = \infty$. The verification is immediate :

$$\int_0^\infty (2 + x^2)^{-\frac{1}{2}} \log^{-1}(2 + x^2)\,dx > \frac{1}{2} \int_0^\infty (2 + x)^{-1} \log^{-1}(2 + x)dx$$

$$= \frac{1}{2} \Big]_0^\infty \log\log(2 + x) = \infty.$$

We introduce the function

$$u(x) = \int_0^x \mu^{-3}(t) \int_0^t \mu(s) \int_0^s \mu^{-3}(r)\,dr\,ds\,dt.$$

The Laplace-Beltrami operator $\Delta = d\delta + \delta d$ gives

$$\Delta u = -g^{-\frac{1}{2}}(g^{\frac{1}{2}} g^{xx} u')' = -\mu^{-1}(\mu\mu^2 u')' = -\int_0^x \mu^{-3}(r)\,dr$$

and

$$\Delta^2 u = -\mu^{-1}(\mu\mu^2(-\mu^{-3}))' = 0.$$

Thus u is nonharmonic biharmonic.

To see that u is bounded it suffices to show that it is so for $x > 0$. For all $s > 0$,

$$\int_0^s \mu^{-3}(r)\,dr = \int_0^s (2 + r^2)^{-3/2} \log^{-3}(2 + r^2)\,dr = o(1),$$

and for all $t > 0$,

$$\int_0^t \mu(s) \int_0^s \mu^{-3}(r)\,dr\,ds < c \int_0^t (2 + s^2)^{\frac{1}{2}} \log(2 + s^2)\,ds$$

$$< 2c \int_0^t (2 + s) \log(2 + s)\,ds$$

$$= c\Big[(2 + t)^2 \log(2 + t)$$

$$- \frac{1}{2}(2 + t)^2 + \text{const.}\Big].$$

Here and later c is a constant, not always the same. We let [ ] stand for the expression in brackets and obtain

$$u(x) < c \int_0^x (2 + t^2)^{-3/2} \log^{-3}(2 + t^2) [\quad] \, dt.$$

The dominating term in the integrand is majorized by

$$\frac{1}{2} t^{-3} \log^{-3} t \times (2 + t)^2 \log(2 + t).$$

The integral from 1 to $x > 1$ is bounded, and consequently so is u for all x.

This completes the proof of Theorem 1.

2. The following simple example, valid for $N \geqslant 3$, is perhaps also of interest. Let

$$T: \quad |x| < \infty, \qquad |y| \leqslant \pi, \qquad |z_i| \leqslant 1, \qquad i = 1, \ldots, N-2,$$

with the metric

$$ds^2 = dx^2 + e^{-x} dy^2 + e^{(2e^x - x)/(N-2)} \sum_{i=1}^{N-2} dz_i^2 ,$$

the opposite faces again identified by pairs. Clearly $T \in C$.

The function

$$u = \cos y$$

belongs to H^2B. In fact,

$$\begin{aligned} \Delta u &= - e^{-e^x + x} (e^{e^x - x} e^x)(-\cos y) \\ &= e^x \cos y, \end{aligned}$$

and

$$\Delta^2 u = - e^{-e^x + x} [(e^{e^x - x} e^x)' \cos y + e^{e^x - x} e^x e^x (-\cos y)] = 0.$$

Thus $T \in C \cap \widetilde{\mathfrak{O}}_{H^2B}$.

3. The reason that we are only interested in nonharmonic biharmonic functions is, of course, that completeness is known not

to exclude bounded harmonic functions (Nakai-Sario [6]). For $N \geqslant 3$, we insert here a simple proof of this fact.

Take the N-cylinder

$$T: \ |x| < \infty, \quad |y| \leqslant 1, \quad |z_i| \leqslant 1, \quad i = 1, \dots, N-2,$$

with the metric

$$ds^2 = dx^2 + e^{2x^2} dy^2 + \sum_{i=1}^{N-2} dz_i^2 .$$

Trivially $T \in C$. The function

$$h(x) = \int_0^x e^{-t^2} dt$$

is harmonic,

$$\Delta h = - e^{-x^2} (e^{x^2} e^{-x^2})' = 0.$$

It also is bounded and, in fact, even Dirichlet finite :

$$D(h) = c \int_{-\infty}^{\infty} e^{-2x^2} e^{x^2} dx < \infty.$$

4. We return to nonharmonic biharmonic functions.

THEOREM 2. – $C \cap \mathcal{O}^N_{H^2B} \neq \emptyset$ *for every* N.

Proof. – The Euclidean N-space $E^N \in C$. Every biharmonic function u has an expansion in spherical harmonics S_{nm}

$$u = \sum_{u=0}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{n+2} + b_{nm} r^n) S_{nm} .$$

If $u \in H^2B$, then

$$\int_{|x|=r} u S_{nm} d\omega = c(a_{nm} r^{n+2} + b_{nm} r^n)$$

is bounded in r, hence $a_{nm} = b_{nm} = 0$ for all n, except for b_{01}. Therefore u is constant.

5. In view of $u = r^2 \in H^2B$ on the Euclidean N-ball, we have trivially $\widetilde{C} \cap \widetilde{\mathcal{O}}^N_{H^2B} \neq \emptyset$ for every N, with $\widetilde{C}$ the totality of incomplete Riemannian manifolds. It remains to show :

THEOREM 3. – $C \cap \mathcal{O}^N_{H^2B} \neq \emptyset$ *for every* N.

Proof. – Let E^N_α be the N-space $0 < r < \infty$ with the metric

$$ds = r^\alpha \, |dx| ,$$

α a constant. It is known (Sario-Wang [19, 21]) that if $N \geqslant 4$, $E^N_\alpha \in \mathcal{O}_{H^2B}$ for every α ; $E^2_\alpha \in \mathcal{O}_{H^2B}$ if and only if $\alpha \neq -1 \mp n/2$, $n = 1,2,\dots$; $E^3_\alpha \in \mathcal{O}_{H^2B}$ if and only if $\alpha \neq -1 \mp \left[\frac{1}{2}\, n(n+1)\right]^{\frac{1}{2}}$. On the other hand, $E^N_\alpha \in \widetilde{C}$ for every α, hence the theorem.

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Leo SARIO
Department of Mathematics
University of California
Los Angeles, Calif. (USA)