ANNALES DE L'INSTITUT FOURIER

FERNANDO CARDOSO
FRANÇOIS TRÈVES

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Annales de l'institut Fourier, tome 24, nº 1 (1974), p. 225-292
<http://www.numdam.org/item?id=AIF_1974__24_1_225_0>

Ann. Inst. Fourier, Grenoble
24,1 (1974), 225-292

A NECESSARY CONDITION OF LOCAL SOLVABILITY FOR PSEUDO-DIFFERENTIAL EQUATIONS WITH DOUBLE CHARACTERISTICS

by Fernando CARDOSO and François TREVES *

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* The work of F. Cardoso was supported by a Guggenheim fellowship, and that of F. Treves was partially supported by National Science Foundation Grant 27671.

1. Introduction and statement of the theorem.

Throughout this article, Ω will be an open subset of $\mathbf{R}^N$ (we shall assume that N is $\geqq 2$). We are going to study a pseudodifferential operator of order m in Ω, of the kind

$$P = P(x, D) \sim \sum_{j=0}^{+\infty} P_{m-j}(x, D), \tag{1.1}$$

where, for each $j = 0, 1, \ldots, P_{m-j}(x, \xi) \in C^\infty(\Omega \times (\mathbf{R}_N \backslash \{0\}))$ and is positive-homogeneous of degree $m - j$ with respect to ξ. The equivalence in (1.1) is the standard one in the theory of ψ do's.

We shall argue under the hypothesis that the principal symbol $P_m(x, \xi)$ of P can be factorized as follows :

$$P_m(x, \xi) = Q(x, \xi)\{L(x, \xi)\}^2 \tag{1.2}$$

in a conic neighborhood $\mathcal{U}$ of a point (x_0, ξ^0) of $\dot{T}^*(\Omega)$, the complement of the zero section in the cotangent bundle $T^*(\Omega)$ over Ω. That $\mathcal{U}$ is conic means that it is invariant under the dilation $(x, \xi) \to (x, \rho\xi)$ whatever $\rho > 0$. The factors Q, L are C^∞ functions in $\mathcal{U}$, positive-homogeneous of degree $m - 2$ and 1 respectively with respect to ξ, and have the following properties :

$$L(x_0, \xi^0) = 0, \tag{1.3}$$

$$d_\xi L(x_0, \xi^0) \neq 0, \tag{1.4}$$

$$Q(x_0, \xi^0) \neq 0. \tag{1.5}$$

We use the notation A = Re L, B = Im L and denote by H_A the Hamiltonian field of A :

$$H_A = \sum_{j=1}^{N} \frac{\partial A}{\partial \xi_j}\frac{\partial}{\partial x^j} - \frac{\partial A}{\partial x^j}\frac{\partial}{\partial \xi_j}.$$

By virtue of (1.4) at least one of the differentials $d_\xi A$ or $d_\xi B$ does not vanish at (x_0, ξ^0). We shall assume that it is $d_\xi A$. Possibly after some shrinking of $\mathcal{U}$, we may assume that

$$d_\xi A \textit{ does not vanish at any point of } \mathcal{U}, \tag{1.6}$$

and also that

$$Q \textit{ does not vanish at any point of } \mathcal{U}. \tag{1.7}$$

A consequence of (1.6) is that the *bicharacteristic strip* of A through (x_0, ξ^0), that is to say, the integral curve of H_A through that point is a true curve (i.e., is not reduced to a point) and that its projection in the x-space (i.e., in Ω) is also a true curve through x_0.

NOTATION 1.1. – *We shall denote by Γ_0 the bicharacteristic strip of* A *through* (x_0, ξ^0).

Throughout the article we shall make the following "finiteness" assumption :

The restriction of B *to* Γ_0 *has a zero of finite order* k_0 *at* (x_0, ξ^0). (1.8)

Let us regard momentarily $L(x, \xi)$, suitably extended to the complement of $\mathcal{U}$, as the symbol of a first-order pseudodifferential operator, $L(x, D)$. We recall the following result (see [6]) :

Suppose that (1.8) *holds with an odd integer* k_0 *and that* (1.9)

the change of sign of B *along* Γ_0, *at the point* (x_0, ξ^0), *is from positive to negative.* (1.10)

Then the transpose ${}^tL(x, D)$ *of* $L(x, D)$ *is not locally solvable at* x_0.

For the notion of local solvability of pseudodifferential operators, see, e.g., [6].

In the present paper we prove the following result :

THEOREM 1.1. – *If* (1.8) *and* (1.10) *hold, the transpose* ${}^tP$ *of* P *is not locally solvable at* x_0.

The noteworthy feature in Th. 1.1 is that the lower-order terms of P do not influence the conclusion.

Note that Condition (1.8) can be restated as follows :

$$H_A^j(B)(x_0, \xi^0) = 0 \quad \textit{if} \quad j < k_0, \quad H_A^{k_0}(B)(x_0, \xi^0) \neq 0, \tag{1.11}$$

whereas (1.10) says that

$$H_A^{k_0}(B)(x_0, \xi^0) < 0. \tag{1.12}$$

In [6] it has been proved that conditions such as (1.8) or (1.10) (i.e., (1.11) or (1.12)) are invariant under multiplication of L by a complex valued C^∞ function of (x, ξ) in a neighborhood of (x_0, ξ^0) which does not vanish there. This fact frees the above statements of the ambiguity as to the meaning of $A = \text{Re } L$ and $B = \text{Im } L$.

When P is a *differential* operator, $P_m(x, \xi)$ is homogeneous, and not merely positive-homogeneous, with respect to ξ, and we may take Q and L also to be homogeneous in ξ. In such case, if k_0 is odd and if $H_A^{k_0}(B)$ is > 0 at (x_0, ξ^0) it must necessarily be < 0 at $(x_0, -\xi^0)$. This implies at once the following :

COROLLARY 1.1. – *Suppose that* P *is a differential operator in* Ω *and that* (1.8) *holds with an odd integer* k_0. *Then, neither* P *nor* ${}^tP$ *is locally solvable at* x_0.

When $k_0 = 1$, Th. 1.1 can be more easily proved and we leave it to the reader. Also the case $k_0 = 1$ follows from recent results of J. Sjöstrand [7]. We shall therefore concentrate our attention to prove Th. 1.1 for $k_0 > 1$.

Although dealing with a rather special situation, the article is largely devoted to establishing the base for an investigation of the solvability problem in the general case of multiple real characteristics. A starting point for such an investigation is an asymptotic expansion in powers of ρ, about $\rho \sim +\infty$, of

$$P(e^{i\rho w}\varphi),$$

where w is a *complex* C^∞ function in Ω, whose gradient at x_0 is equal to ξ^0, and where $\varphi \in C_c^\infty(\Omega)$. Special, and generally cruder, versions of such an expansion are of customary use in pseudodifferential operators theory (see [3], [4], [6]). The techniques we use to establish the asymptotic formula which we need here are fairly standard. However, dealing with multiple characteristics demands a greater precision in the estimate of the remainder, in the asymptotic expansion, than what was needed in the study of ψdo's of principal type (i.e., with simple real characteristics), and this entails a few, not completely self-evident, modifications.

The proof of Th. 1.1 subdivides into two parts, according to whether the lower order terms have or do not have a "strong influence".

The precise measurement of this influence is achieved by means of the subprincipal part, used in [5] in the study of the Cauchy problem for equations with double characteristics (more precisely, it is the imaginary part of the squareroot of the invariant in question, defined in the characteristic variety, which is relevant).

Under Hypothesis (1.10) the subprincipal part plays a role in the proof but not in the statement of the theorem. We hope to show, in a forthcoming article, that this is not so when (1.10) does not hold (while (1.8) still does) : then the subprincipal part actually determines whether the operator is locally solvable or not.

2. The basic asymptotic formula : the ingredients.

Consider two C^∞ functions φ, w in the open set $\Omega \subset \mathbf{R}^N$. We assume that φ has compact support. We shall use an asymptotic expansion of the kind :

$$P(e^{i\rho w}\varphi) \sim e^{i\rho w} \sum_{j=0}^{+\infty} \rho^{m-j} \mathcal{P}_j^w \varphi, \quad \rho \sim +\infty, \tag{2.1}$$

where, for each $j = 0, 1, \ldots, \mathcal{P}_j^w$ is a *differential* operator of order $\leqq j$ in Ω, whose coefficients (which are C^∞ functions in Ω) depend on the derivatives of w (of order $\geqq 1$ but not exceeding $j + 1$). The asymptotic formula (2.1) is of standard use in pseudodifferential operators theory.

When w is real-valued, (2.1) is given a precise meaning, and is proved, in [4]. We shall be interested in the case when w is complex, i.e., nonreal. The expansion (2.1) has been established in [3] when w satisfies an inequality

$$c_0 |x - x_0|^2 \leqq \operatorname{Im} w(x) \tag{2.2}$$

in an open neighborhood $U \subset \Omega$ of x_0 (c_0 is a constant > 0). However, the estimates of the remainder and of its derivatives in [3] are not precise enough for our present needs. In the present paper, we plan to use the same asymptotic expansion, where the inequality (2.2) is replaced by a weaker one. In many of these more general cases the expansion (2.1) has been established (in [6]), but again the estimates of the remainder are not precise enough. For these reasons we shall give here the exact statement and its complete proof.

Let us say right away that, formally, the differential operators $\mathcal{P}_j^w$ are exactly those which occur in the standard interpretation of Formula (2.1). The first two are fairly easy to compute :

$$\mathcal{P}_0^w \varphi = \mathrm{P}_m(x\,,\,\partial w)\varphi\,, \tag{2.3}$$

where we have used the notation ∂w = grad w,

$$\mathcal{P}_1^w \varphi = \sum_{|\alpha|=1} \mathrm{P}_m^{(\alpha)}(x\,,\,\partial w)\;\mathrm{D}^\alpha\varphi + \widetilde{\mathrm{P}}_{m-1}(x\,,\,\partial w\,,\,\partial^2 w)\varphi, \tag{2.4}$$

where

$$\widetilde{\mathrm{P}}_{m-1}(x\,,\,\partial w\,,\,\partial^2 w) = \mathrm{P}_{m-1}(x\,,\,\partial w) + \sqrt{-1}\sum_{|\alpha|=2}\frac{1}{\alpha\,!}\,\mathrm{P}_m^{(\alpha)}(x, \partial w)\mathrm{D}^\alpha w. \tag{2.5}$$

We shall also need to know that $\mathcal{P}_2^w$ is equal to

$$\sum_{|\alpha|=2}\frac{1}{\alpha\,!}\,\mathrm{P}_m^{(\alpha)}(x\,,\,\partial w)\,\mathrm{D}^\alpha +$$

$$+ \sum_{|\alpha|=1}\Big(\mathrm{P}_{m-1}^{(\alpha)}(x\,,\,\partial w) + \sqrt{-1}\sum_{|\beta|=2}\frac{1}{\beta\,!}\,\mathrm{P}_m^{(\alpha+\beta)}(x\,,\,\partial w)\,\mathrm{D}^\beta w\Big)\,\mathrm{D}^\alpha +$$

$$+ \text{ zero order term.} \tag{2.6}$$

Of course, all such formulas as (2.3), (2.4), (2.5), (2.6), etc. , are objectionable, since w will be complex-valued and the $\mathrm{P}_{m-j}\,(x\,,\,\xi)$ might not be defined for complex ξ' s. Momentarily we may accept the validity of the above formulas either when w is real valued or when the $\mathrm{P}_{m-j}\,(x\,,\,\xi)$ can be holomorphically extended to complex values of ξ. We shall avoid this type of difficulty by replacing each homogeneous symbol $\mathrm{P}_{m-j}\,(x\,,\,\xi)$ by a suitable approximation of it which will be analytic with respect to ξ. For later reference, we shall introduce now the subprincipal part of P. Suppose that w has been chosen so as to satisfy

$$\mathrm{P}_m^{(\alpha)}(x\,,\,\partial w) = 0 \quad in \quad \Omega\,, \tag*{(2.5)$_*$}$$

for every $(n+1)$-tuple α of length 1. Notice that this implies that w satisfies the *characteristic equation*

$$P_m(x, \partial w) = 0 \quad in \quad \Omega.$$

It then follows that the differential operator $\mathcal{P}_1^w$ reduces to its zero-order term, which can be written :

$$\mathfrak{M}(x, \partial w) = P_{m-1}(x, \partial w) - \frac{1}{2} \sum_{|\alpha|=1} D_x^\alpha P_m^{(\alpha)}(x, \partial w), \quad (2.5)_{**}$$

and which is a well-defined function of $(x, \partial w)$ on the subset $(2.5)_*$ of the cotangent bundle. Under the form $(2.5)_{**}$ it has been used in the study of the Cauchy problem [5]. The subprincipal part of P is by definition :

$$\mathfrak{M}(x, \xi) = P_{m-1}(x, \xi) - \frac{1}{2} \sum_{|\alpha|=1} D_x^\alpha P_m^{(\alpha)}(x, \xi). \qquad (2.5)_{***}$$

Our only assumptions on the *phase function* w will be the following :

$$w(x) = \langle \xi^0, x - x_0 \rangle + w_2(x) \qquad (2.7)$$

with

$$|w_2(x)| \leqq \text{const. } |x - x_0|^2. \qquad (2.8)$$

In order to simplify our notation we shall take x_0 to be the origin in $\mathbf{R}^N$ and ξ^0 to be a *unit vector.* We shall write :

$$u = e^{-i\rho<\xi^0, x>} e^{i\rho w} \varphi = e^{i\rho w_2} \varphi.$$

We note that, for each $j = 0, 1, \ldots, \mathcal{P}_j^w$ depends *linearly* on P and, as a matter of fact, depends only on the homogeneous parts $P_{m-j'}$, $0 \leqq j' \leqq j$. This suggests that we handle separately each term P_{m-j} and add up the corresponding results. Such a procedure raises the problem of estimating the remainder, if we stop adding the results when j becomes sufficiently large. This problem can easily be settled, as we now show.

Let $S(x, \xi)$ be a symbol of degree s, i.e., $S \in C^\infty(\Omega \times \mathbf{R}_N)$ and for any pair of N-typles α, β,

$$|\partial_\xi^\alpha \partial_x^\beta S(x, \xi)| \leqq C_{\alpha,\beta}(x) (1 + |\xi|)^{s-|\alpha|}, \; x \in \Omega, \; \xi \in \mathbf{R}_N, \quad (2.9)$$

where $C_{\alpha,\beta}$ is a continuous function in Ω, everywhere > 0. We have :

$$S(x , D) (e^{i\rho w}\varphi) = (2\pi)^{-N} \int\int e^{i<\xi,x-y>} S(x , \xi) e^{i\rho<\xi^0,y>} u(y) dy d\xi . \tag{2.10}$$

Let us set $\xi = \eta + \rho\,\xi^0$:

$$S(x , D) (e^{i\rho w}\varphi) =$$

$$= (2\pi)^{-N} \int\int e^{i<\eta+\rho\xi^0,x>-i<\eta,y>} S(x , \eta + \rho\xi^0) u(y) dy d\eta. \tag{2.11}$$

Let us denote by ν the smallest positive integer such that

$$2\nu > N. \tag{2.12}$$

Then, by integration by parts with respect to y, we see that

$$S(x , D) (e^{i\rho w}\varphi) =$$

$$= (2\pi)^{-N} \int\int e^{i<\eta+\rho\xi^0,x>-i<\eta,y>} S(x , \eta + \rho\xi^0) v(y) dy \frac{d\eta}{(1 + |\eta|^2)^{\nu}} , \tag{2.13}$$

where :

$$v = (1 - \Delta)^{\nu} u, \quad \Delta = \text{Laplace operator}. \tag{2.14}$$

Using the expression (2.13) we may prove the following :

LEMMA 2.1. – *Under the preceding hypotheses, to every integer* $J \geqq 0$ *there are positive integers* J', M' *such that, if the order* s *of* $S(x , D)$ *is* $\leqq - J'$, *the following is true* :

To every compact subset K *of* Ω *there is a constant* $C(K) > 0$ *such that, for every* $\rho > 1$, $\varphi \in C_c^\infty(K)$ *and* $w \in C^\infty(\Omega)$ *satisfying* (2.7),

$$\sup_K | S(x , D) (e^{i\rho w}\varphi) | \leqq C(K) \rho^{-J} \sup \sum_{|\alpha| \leqq M'} \rho^{-|\alpha|/2} |D^\alpha (e^{i\rho w_2}\varphi)|. \tag{2.15}$$

One may take $J' = J + \nu$ *and* $M' = 2(J + 2\nu)$, *where* ν *is the smallest integer* $> N/2$.

Proof. – In the right-hand side of (2.13) we subdivide the domain of integration into two regions : a region $\mathcal{R}_0$, where $|\eta + \rho\xi^0| \leqq \rho/2$; a region $\mathcal{R}_1$, where $|\eta + \rho\xi^0| > \rho/2$. We have :

$$|(2\pi)^{-N}\iint_{\mathcal{R}_1} e^{i<\eta+\rho\xi^0,x>-i<\eta,y>}\, S(x\,,\eta+\rho\xi^0)\,v(y)\,dy\,(1+|\eta|^2)^{-\nu}d\eta|$$

$$\leqq C_N C_{0,0}(x) \sup_{\mathcal{R}_1} (1+|\eta+\rho\xi^0|)^s \int |v(y)|\,dy$$

$$\leqq C_N C_{0,0}(x)\,(1+\rho/2)^s \int |v(y)|\,dy \quad (\text{if } s\leqq 0)$$

$$\leqq C(N\,,s\,,K)\,\rho^{-J-\nu} \sup |(1-\Delta)^\nu u| \text{ if } s\leqq -J-\nu \text{ and } x\in K.$$

The function $C_{0,0}$ is the one introduced in (2.9).

Let k be an integer $\geqq J+\nu$. Observing that, in $\mathcal{R}_0$, $|\eta|\geqq \rho/2$, we have :

$$|(2\pi)^{-N}\int\int_{\mathcal{R}_0.} e^{i<\eta+\rho\xi^0,x>-i<\eta,y>}\, S(x\,,\eta+\rho\xi^0)\,v(y)\,dy\,(1+|\eta|^2)^{-\nu}\,d\eta| =$$

$$|(2\pi)^{-N}\int\int_{\mathcal{R}_0} e^{i<\eta+\rho\xi^0,x>-i<\eta,y>}\, S(x\,,\eta+\rho\xi^0)\,v_k(y)\,dy\,(1+|\eta|^2)^{-\nu-k}\,d\eta|$$

$$\leqq C_N C_{0,0}(x)\,(1+\rho^2/4)^{-k}\int |v_k(y)|\,dy$$

$$\leqq C(N\,,k\,,K)\rho^{-J-k-\nu}\int |v_k(y)|\,dy,$$

where $$v_k=(1-\Delta)^k\,v=(1-\Delta)^{k+\nu}\,u.$$

Combining the two inequalities we have obtained shows that (2.15) is valid with $J'=J+\nu$ and $M'=2(\nu+k)$, as stated.

We shall be interested in the following consequence of Lemma 2.1:

COROLLARY 2.1. – *Same hypotheses as in Lemma 2.1. To every pair of integers* J, M $\geqq 0$ *there are positive integers* J$'$, M$'$ *such that, if the order s of* $S(x\,,D)$ *is* $\leqq -J'$, *the following is true* :

To every compact subset K *of* Ω *there is a constant* $C(K)>0$ *such that, for every* $\rho>1$, $\varphi\in C_c^\infty(K)$ *and* $w\in C^\infty(\Omega)$ *satisfying* (2.7),

$$\sup_K \sum_{|\alpha|\leqq M} |D^\alpha\{S(x\,,D)\,(e^{i\rho w}\varphi)\}| \leqq$$

$$C(K)\,\rho^{-J} \sup \sum_{|\alpha|\leqq M'} \rho^{-|\alpha|/2}\,|D^\alpha(e^{i\rho w_2}\,\varphi)|\,.$$

One may take $J' = J + M + \nu$ *and* $M' = 2(J + 2\nu)$, *where* ν *is the smallest integer* $> N/2$.

Proof. – It suffices to apply Lemma 2.1 to every pseudodifferential operator $D^\alpha S(x, D)$, $|\alpha| \leqq M$, and add up the corresponding inequalities.

Lemma 2.1 and its corollary enables us to replace $P(x, D)$ by a finite sum

$$\sum_{j=0}^{J'} P_{m-j}(x, D)\, \chi(D) \tag{2.17}$$

where $\chi(\xi) = \chi_0(|\xi|)$, with $\chi_0 \in C^\infty(\mathbf{R}^1)$, $\chi_0 = 0$ for $t < 1/3$ and $\chi_0 = 1$ for $t > 2/3$ (t denotes the variable in $\mathbf{R}^1$). Let us denote momentarily by $P_{(J')}(x, D)$ the finite sum (2.17). We apply Cor. 2.1 with

$$S(x, D) = P(x, D) - P_{(J')}(x, D).$$

Estimate (2.16) reads

$$\sup_K \sum_{|\alpha| \leqq M} |D^\alpha\{P(x, D) - \sum_{j=0}^{J'} P_{m-j}(x, D)\, \chi(D)\}\, (e^{i\rho w}\varphi)| \leqq$$

$$\leqq C(K)\, \rho^{-J} \sup \sum_{|\alpha| \leqq M'} \rho^{-|\alpha|/2}\, |D^\alpha(e^{i\rho w_2}\varphi)|. \tag{2.18}$$

It is valid provided that J', M' (and $C(K)$) be large enough.

For the remainder of the argument and until its conclusion, we shall assume that $P(x, D)$ is indeed equal to the finite sum (2.17) ; in other words, we shall omit the subscript (J') in $P_{(J')}(x, D)$. From the type of estimates which we shall ultimately establish, it will be obvious, thanks to the inequality (2.18), that such a substitution is permitted.

We shall make an additional assumption about $P(x, D)$, which will be removed at the end. We are going to assume that $P(x, \xi)$ can be extended as a *holomorphic* function of ξ in an open subset of $\mathbf{C}_N$ of the following type :

$$\xi \in \mathbf{C}_N \ ; \textit{ for some } \rho > 1,\ |\xi - \rho\xi^0| < 2\varepsilon\rho \tag{2.19}$$

where ε is a number such that $0 < \varepsilon < 1/6$ (the 1/6 is due to the presence of the cut-off function χ in (2.17) ; the 2ε, instead of ε, in (2.19) is there for technical convenience ; we recall that $|\xi^0| = 1$).

In view of (2.7) and (2.8) we may find an open neighborhood U of $x_0 = 0$ in Ω such that

$$x \in \mathrm{U} \Rightarrow |\partial w(x) - \xi^0| < \varepsilon/2. \tag{2.20}$$

This means that, for $\rho > 1$, $\rho \partial w(x)$ remains in the set (2.19) as long as x remains in U. Because of this we may form the functions

$$\mathrm{P}^{(\alpha)}(x, \rho\partial w) = \sum_{j=0}^{\mathrm{J}'} \rho^{m-j-|\alpha|} \mathrm{P}^{(\alpha)}_{m-j}(x, \partial w). \tag{2.20'}$$

There are no factors $\chi(\partial w)$ since ∂w (and $\rho\partial w$) remains in the region where $\chi(\xi) = 1$ (we have of course extended χ to the set (2.19) as the function equal to one).

Under the preceding analyticity assumption about $\mathrm{P}(x, \xi)$ we are now going to define the differential operators $\mathcal{R}^w_j$ entering in the asymptotic formula (2.1). Let us set

$$\mathrm{H}(x, y) = w(x) - w(y) - (x - y) \cdot \partial w(x). \tag{2.21}$$

We observe that

$$|\mathrm{H}(x, y)| \leqq \text{const. } |x - y|^2. \tag{2.22}$$

Then, if $u \in \mathrm{C}^\infty_c(\mathrm{U})$, where U is the open neighborhood of x_0 in (2.20),

If $j \leqq \mathrm{J}'$, $\mathcal{R}^w_j \varphi$ *is equal to the coefficient of* ρ^{m-j} *in the expression* (2.23)

$$\sum_{|\alpha| \leqq \mathrm{J}_0} \frac{1}{\alpha!} \mathrm{P}^{(\alpha)}(x, \rho\partial w) \; \mathrm{D}^\alpha_y\{\varphi(y)\, e^{-i\rho \mathrm{H}(x,y)}\}|_{y=x}, \tag{2.24}$$

where J_0 *is any integer* $\geqq 2\mathrm{J}'$.

The following remarks are in order : because of (2.22),

$$\mathrm{D}^\alpha_y\{\varphi(y)\, e^{-i\rho \mathrm{H}(x,y)}\}|_{y=x}$$

is a polynomial with respect to ρ of degree $\leqq |\alpha|/2$. If we multiply it by $\mathrm{P}^{(\alpha)}(x, \rho\partial w)$, we obtain (in view of (2.20')) a finite linear combination of powers of ρ which are $\leqq m - |\alpha|/2$. This linear combination does not involve ρ^{m-j} if $j \leqq J'$ and $|\alpha| > 2J'$. It follows from this that the coefficient of ρ^{m-j} in (2.24) remains unchanged if we increase J_0 beyond $2J'$ (if $j \leqq J'$).

It is also obvious that, if we had included in our expression of $P(x, D)$ (which presently is given in (2.17)) homogeneous parts $P_{m-j}(x, D)$ with $j \geqq J' + 1$, this would not have affected the $\mathcal{P}_i^w$, $i \leqq J'$.

It is clear, in our definition (2.23), that the cut-off function χ does not enter in the expressions of the $\mathcal{P}_j^w$. This is, of course, due to the fact that (2.20) holds and that ∂w remains in the region where $\chi(\xi) = 1$.

3. The basic asymptotic formula (cont'd) : The case of symbols which are analytic in the ξ variables.

Throughout this section, we assume that $P(x, D)$ can be continued analytically into the complex region (2.19). The function w verifies (2.7) and (2.8), the open neighborhood U of $x_0 = 0$ is chosen so as to have (2.20). The amplitude function φ will have compact support contained in U. We set :

$$R_{J'}^{w,\rho}(x, D)\varphi = e^{-i\rho w} P(x, D)(e^{i\rho w}\varphi) - \sum_{j=0}^{J'} \rho^{m-j}\mathcal{P}_j^w \varphi. \tag{3.1}$$

LEMMA 3.1. – *Suppose that each homogeneous symbol* $P_{m-j}(x, \xi)$ *can be holomorphically continued to the complex region* (2.19). *Let* $w \in C^\infty(\Omega)$ *satisfy* (2.7) *and* (2.8). *Let* U *be an open neighborhood of* $x_0 = 0$ *in* Ω *such that* (2.20) *holds. Then, to every pair of integers* $J, M \geqq 0$ *there are integers* $J', M' \geqq 0$ *such that the following is true* :

To every compact subset K *of* Ω *there is a constant* $C(K) > 0$ *such that, for every* $\rho > 1$ *and every* $\varphi \in C_c^\infty(U)$, *we have* :

$$\sup_K \sum_{|\alpha| \leqq M} |D^\alpha\{e^{i\rho w} R_{J'}^{w,\rho}(x, D)\varphi\}| \leqq$$

$$\leqq C(K)\rho^{-J} \sup \sum_{|\alpha| \leqq M'} \rho^{-|\alpha|/2}\{|D^\alpha(e^{i\rho w_2}\varphi)| + |e^{i\rho w_2} D^\alpha \varphi|\}. \quad (3.2)$$

Proof. – We shall begin by requiring

$$J' \geqq J + M + \nu \quad (\nu : \text{smallest integer} > N/2). \quad (3.3)$$

This enables us to apply Cor. 2.1 and assume that $P(x, D)$ is equal to the finite sum (2.17).

We select an open neighborhood of $\overline{U}$ whose closure $\overline{V}$ is a compact subset of Ω, such that

$$|\partial w(x) - \xi^0| < \varepsilon \quad \textit{for every} \quad x \in V.$$

Such a neighborhood V exists in view of (2.20). Let g, $h \in C_c^\infty(V)$ with $g = 1$ in U, $h = 1$ in a neighborhood of the support of g. We have :

$$P(x, D)(e^{i\rho w}\varphi) = h(x) P(x, D)(e^{i\rho w}\varphi) + S(x, D)(e^{i\rho w}\varphi)$$

where, in view of the fact that $\operatorname{supp}\varphi \subset U$,

$$S(x, D)(e^{i\rho w}\varphi) = (1 - h(x)) P(x, D)\{g e^{i\rho w}\varphi\}.$$

Since the supports of g and $1 - h$ are disjoint, the order of $S(x, D)$ is $-\infty$, and we may apply (2.16). This shows that we may replace $P(x, D)$ by $h(x) P(x, D)$ or, rather, assume that the support of $P(x, \xi)$ lies in the cylinder $\{(x, \xi) ; x \in V\}$.

We have :

$$e^{-i\rho w} P(x, D)(e^{i\rho w}\varphi)$$

$$= (2\pi)^{-N} \int\int e^{i\langle x-y, \xi - \rho\partial w(x)\rangle - i\rho H(x,y)} P(x, \xi)\varphi(y)\, dy d\xi.$$

We set

$$R(x, \xi) = P(x, \xi) - \sum_{|\alpha| \leqq J_0} \frac{1}{\alpha!} P^{(\alpha)}(x, \rho\partial w(x))\eta^\alpha, \quad (3.4)$$

where

$$\eta = \xi - \rho\,\partial w(x).$$

The "remainder" R depends on ρ and on the integer J_0 ; the latter will be chosen eventually. Let us also underline the fact that the definition of $R(x, \xi)$ makes sense since $R \equiv 0$ when $x \notin V$ and, when $x \in V$, the generally complex factor $\rho \partial w(x)$ belongs to the region (2.19).

We observe that

$$(2\pi)^{-N} \int\int e^{i<x-y,\xi-\rho\partial w(x)>-i\rho H(x,y)} \eta^\alpha \varphi(y)\, dy\, d\xi =$$

$$= (2\pi)^{-N} \int\int e^{i<x-y,\eta>} D_y^\alpha\{e^{-i\rho H(x,y)} \varphi(y)\} dy\, d\eta =$$

$$D_y^\alpha\{e^{-i\rho H(x,y)} \varphi(y)\}|_{x=y},$$

as one sees by applying the Cauchy Integral theorem to the integral with respect to η (possibly after introducing a convergence factor of the kind $\exp(-\delta(\eta_1^2 + \cdots + \eta_N^2))$ with $\delta \to +0$). Thus we obtain :

$$R(x, D)(e^{i\rho w}\varphi) = P(x, D)(e^{i\rho w}\varphi) -$$

$$- e^{i\rho w} \sum_{|\alpha| \leqq J_0} \frac{1}{\alpha!} P^{(\alpha)}(x, \rho\partial w(x))\, D_y^\alpha\{\varphi(y)\, e^{-i\rho H(x,y)}\}|_{y=x} \quad (3.5)$$

We introduce two cut-off functions $g, h \in C^\infty(\mathbf{R}_N)$ (these have no relation with the functions so denoted in the first paragraph of the present proof ; we shall not use the latter again). We suppose that $g + h \equiv 1$, that the support of g lies in the region $\{\xi ; |\xi - \rho\xi^0| < \varepsilon\rho\}$, whereas that of h lies in the region $\{\xi ; |\xi - \rho\xi^0| > \varepsilon\rho/2\}$.

We note that, when $x \in V$ and $\xi \in \operatorname{supp} g$, $\eta = \xi - \rho\partial w(x)$ belongs to the complex region (2.19). We may then use the remainder formula for the Taylor expansion :

$$R(x, \xi) = (J_0!)^{-1} \int_0^1 (1-t)^{J_0} \left(\frac{\partial}{\partial t}\right)^{J_0+1} P(x, \rho\partial w(x) + t\eta)\, dt$$

$$= (J_0 + 1) \sum_{|\alpha| = J_0+1} \frac{\eta^\alpha}{\alpha!} \int_0^1 (1-t)^{J_0} P^{(\alpha)}(x, \rho\partial w(x) + t\eta) dt.$$

Let us write :

$$I_g(x) = (2\pi)^{-N} \int\int e^{i<x-y,\eta>-i\rho H(x,y)} R(x, \xi) g(\xi) \varphi(y)\, dy\, d\xi,$$

and define $I_h(x)$ similarly, by substituting h for g. We see that $I_g(x)$

can be written as an integral with respect to the measure $(1-t)^{J_0}dt$ over the unit interval $(0,1)$ of a linear combination of terms of the form

$$T^0_\alpha(x,t)=\int\int e^{i<x-y,\eta>-i\rho H(x,y)}\,P^{(\alpha)}(x,\rho\partial w(x)+t\eta)\eta^\alpha g(\xi)\varphi(y)dy\,d\xi$$

$$=\int\int e^{i<x-y,\eta>}\,P^{(\alpha)}(x,\rho\partial w(x)+t\eta)g(\xi)D^\alpha_y\{\varphi(y)\,e^{-i\rho H(x,y)}\}\,dy\,d\xi$$

$$=\int\int e^{i<x-y,\eta>-i\rho H(x,y)}\,P^{(\alpha)}(x,\rho\partial w(x)+t\eta)g(\xi)$$

$$\left\{\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}c_{\alpha,\beta}(x,y,\rho)\,D^\beta\varphi(y)\right\}dy\,d\xi,$$

where $\beta\leqslant\alpha$ means $\beta_j\leqq\alpha_j$, $j=1,\ldots,N$, and

$$c_{\alpha,\beta}(x,y,\rho)=e^{i\rho H(x,y)}\,D^{\alpha-\beta}_y\{e^{-i\rho H(x,y)}\}.$$

One of the crucial observations in the proof is the following one : by virtue of (2.22),

$$c_{\alpha,\beta}(x,y,\rho)=\sum_{|\gamma|+2d\leqq|\alpha-\beta|}c_{\alpha,\beta,\gamma;d}(x,y)\,\rho^d\{\rho(x-y)\}^\gamma.$$

Consequently, $T^0_\alpha(x,t)$ is a linear combination of terms of the form

$$\rho^{|\gamma|+d}\,T^0_{\alpha,\beta,\gamma;d}(x,t),$$

setting

$$T^0_{\alpha,\beta,\gamma;d}(x,t)=\int\int e^{i<x-y,\eta>-i\rho H(x,y)}\,D^\gamma_\xi\{P^{(\alpha)}(x,\rho\partial w(x)+t\eta)g(\xi)\}$$

$$D^\beta\varphi(y)\,dy\,d\xi.$$

Such a term can be decomposed, in turn, into a sum

$$T^{00}_{\alpha,\beta,\gamma;d}(x,t)+T^{0.1}_{\alpha,\beta,\gamma;d}(x,t),$$

where

$$T^{00}_{\alpha,\beta,\gamma;d}(x,t)=\int\int e^{i<x-y,\eta>-i\rho H(x,y)}\,P^{(\alpha+\gamma)}(x,\rho\partial w(x)+t\eta)g(\xi)$$

$$D^\beta\varphi(y)dy\,d\xi,$$

whereas $T^{0,1}_{\alpha,\beta,\gamma;d}(x,t)$ is a linear combination of terms of the form

$$T^{0,1}_{\alpha,\beta,\gamma',\gamma'';d}(x,t)=\int\int e^{i<x-y,\eta>-i\rho H(x,y)}$$

$$P^{(\alpha+\gamma')}(x,\rho\partial w(x)+t\eta)\,g^{(\gamma'')}(\xi)\,D^\beta\varphi(y)dy\,d\xi,$$

with $$\gamma' + \gamma'' = \gamma\,,\ |\gamma''| > 0.$$

We are going to set

$$\widetilde{\mathrm{I}}_h = e^{i\rho w}\,\mathrm{I}_h\,,\ \widetilde{\mathrm{T}}^{00}_{\alpha,\beta,\gamma;d} = e^{i\rho w}\,\mathrm{T}^{00}_{\alpha,\beta,\gamma;d}\,,\ \mathrm{T}^{0,1}_{\alpha,\beta,\gamma',\gamma'';d} = e^{i\rho w}\,\mathrm{T}^{0,1}_{\alpha,\beta,\gamma',\gamma'';d}\ .$$

1. *Estimate of* $\widetilde{\mathrm{I}}_h$.

Writing $u = e^{i\rho w_2}\varphi$, we have :

$$(2\pi)^{\mathrm{N}}\,\widetilde{\mathrm{I}}_h = \int\int e^{i<x,\xi>-i<\xi-\rho\xi^0,y>}\ \mathrm{R}(x\,,\xi)\,h(\xi)\,u(y)\,dy\,d\xi$$

$$= \int\int e^{i<x,\xi>-i<\xi-\rho\xi^0,y>}\ |\xi-\rho\xi^0|^{2k}\ |\xi-\rho\xi^0|^{-2k}\,\mathrm{R}(x\,,\xi)\,h(\xi)u(y)dyd\xi$$

$$= \int\int e^{i<x,\xi>-i<\xi-\rho\xi^0,y>}\ |\xi-\rho\xi^0|^{-2k}\ \mathrm{R}(x\,,\xi)\,h(\xi)\,(-\Delta)^k\,u(y)dyd\xi.$$

We observe that, in the support of h,

$$\rho \leqq 2\varepsilon^{-1}\,|\xi-\rho\xi^0|\,,\ |\xi| \leqq (2+\varepsilon)\varepsilon^{-1}\,|\xi-\rho\xi^0|\,.$$

Note that, by (2.4), $|\mathrm{R}(x\,,\xi)| \leqq$ const. $(1+|\xi|)^{\mathrm{J}_0}$ (assuming that $\mathrm{J}_0 \geqq m$). Let us therefore require

$$k \geqq \mathrm{J} + \mathrm{J}_0 + \mathrm{N} + 1. \tag{3.6}$$

We obtain at once :

$$|\widetilde{\mathrm{I}}_h| \leqq \mathrm{C}\rho^{-\mathrm{J}-k}\,\sup|\Delta^k(e^{i\rho w_2}\varphi)|\,. \tag{3.7}$$

We could have estimated, in similar fashion, the derivatives of order $\leqq \mathrm{M}$ of $\widetilde{\mathrm{I}}_h$. We would have required :

$$k \geqq \mathrm{J} + \mathrm{M} + \mathrm{J}_0 + \mathrm{N} + 1\,, \tag{3.8}$$

and obtained :

$$\sum_{|\alpha|\leqq \mathrm{M}} |\mathrm{D}^\alpha\,\widetilde{\mathrm{I}}_h| \leqq \mathrm{C}\rho^{-\mathrm{J}-k}\,\sup|\Delta^k(e^{i\rho w_2}\varphi)|. \tag{3.9}$$

2. *Estimate of* $\widetilde{\mathrm{T}}^{00}_{\alpha,\beta,\gamma;d}$.

We have :

$$|\widetilde{\mathrm{T}}^{00}_{\alpha,\beta,\gamma;d}| \leqq \mathrm{C}\,\sup_{x,\xi}\,(1+|\rho\,\partial w(x)+t\eta|)^{m-|\alpha+\gamma|}\,(\int g(\xi)d\xi)\int|e^{i\rho w}\mathrm{D}^\beta\varphi|dy,$$

where the supremum with respect to x, ξ is taken for x ranging over V and ξ ranging over the support of g. We have then

$$|\rho\,\partial w(x) - \rho\xi^0| < \varepsilon\rho\,,\ |\xi - \rho\xi^0| < \varepsilon\rho,$$

hence

$$|\eta| = |\rho\,\partial w(x) - \xi| < 2\varepsilon\rho < \rho/3.$$

From this follows at once that, if $0 \leqq t \leqq 1$,

$$|\rho\,\partial w(x) + t\eta| > \rho/3.$$

Since $\int g(\xi)\,d\xi \leqq C_N\rho^N$, we obtain

$$\rho^{|\gamma|+d}\,|\widetilde{T}^{00}_{\alpha,\beta,\gamma;d}| \leqq C'\rho^{m+N+d-|\alpha|+|\beta|/2}\,\rho^{-|\beta|/2}\,\sup|e^{i\rho w_2}\,D^\beta\varphi|.$$

We recall that $d \leqq \frac{1}{2}\,|\alpha - \beta|$, and that $|\alpha| = J_0 + 1$. Therefore, if

$$J + m + N \leqq \frac{1}{2}\,J_0, \tag{3.10}$$

we conclude that

$$\rho^{|\gamma|+d}\,|\widetilde{T}^{00}_{\alpha,\beta,\gamma;d}| \leqq C'^{-J-|\beta|/2}\,\sup|e^{i\rho w_2}\,D^\beta\varphi|. \tag{3.11}$$

Similarly, if we require

$$J + M + m + N \leqq \frac{1}{2}\,J_0, \tag{3.12}$$

we can achieve that

$$\rho^{|\gamma|+d}\sum_{|\tilde\alpha|\leqq M}|D^{\tilde\alpha}\,\widetilde{T}^{00}_{\alpha,\beta,\gamma;d}| \leqq C'\,\rho^{-J-|\beta|/2}\,\sup|e^{i\rho w_2}\,D^\beta\varphi|. \tag{3.13}$$

3. *Estimate of* $\widetilde{T}^{0,1}_{\alpha,\beta,\gamma',\gamma'';d}$.

This is very similar to the estimate of $\widetilde{I}_h$ derived in Part I. We have :

$$\widetilde{T}^{0,1}_{\alpha,\beta,\gamma',\gamma'';d} = \int\int e^{i\langle x,\xi\rangle - i\langle \xi-\rho\xi^0,y\rangle}\,P^{(\alpha+\gamma')}(x\,,\rho\,\partial w(x) + t\eta)$$
$$g^{(\gamma'')}(\xi)\,e^{i\rho w_2(y)}\quad D^\beta\,\varphi(y)\,dy\,d\xi.$$

Since $|\gamma''| > 0$, the domain of integration with respect to ξ is contained in the region $|\xi - \rho\xi^0| > \varepsilon\rho/2$. We multiply and divide the integrand by $|\xi - \rho\xi^0|^{-2k}$. By arguing exactly as in Part I, but observing here that

$$|P^{(\alpha+\gamma')}(x\,,\rho\,\partial w(x) + t\eta)| \leqq \text{const.}$$

since $|\alpha| = J_0 + 1 > m$, we obtain easily :

$$\rho^{|\gamma|+d}\,|\widetilde{T}^{0,1}_{\alpha,\beta,\gamma',\gamma'';d}| \leqq C'\,\rho^{N+d+|\gamma|-2k} \sup |\Delta^k (e^{i\rho w_2} D^\beta \varphi)|$$

$$\leqq C''\,\rho^{N+d+|\gamma|+|\beta|-2k} \sup \sum_{|\beta'| \leqq |\beta|+2k} |D^{\beta'} (e^{i\rho w_2} \varphi)|\,.$$

We note that $|\beta + \gamma| + d \leqq |\alpha| = J_0 + 1$ and that $|\beta| \leqq J_0$, hence, if we require

$$k > J + N + 3J_0/2, \tag{3.14}$$

we obtain

$$\rho^{|\gamma|+d}\,|\widetilde{T}^{0,1}_{\alpha,\beta,\gamma',\gamma'';d}| \leqq C''\,\rho^{-J} \sup \sum_{|\theta| \leqq J_0+2k} \rho^{-|\theta|/2}\,|D^\theta (e^{i\rho w_2} \varphi)| \tag{3.15}$$

Similarly, if we require

$$k > J + M + N + 3J_0/2, \tag{3.16}$$

we obtain

$$\rho^{|\gamma|+d} \sum_{|\tilde{\alpha}| \leqq M} |D^{\tilde{\alpha}}\,\widetilde{T}^{0,1}_{\alpha,\beta,\gamma',\gamma'';d}| \leqq$$

$$C''\,\rho^{-J} \sup \sum_{|\theta| \leqq J_0+2k} \rho^{-|\theta|/2}\,|D^\theta (e^{i\rho w_2} \varphi)| \tag{3.17}$$

We recall that $R(x, D)(e^{i\rho w}\varphi) = \widetilde{I}_g + \widetilde{I}_h$, where $\widetilde{I}_g = e^{i\rho w} I_g$. If we go back to the definitions of $\widetilde{T}^{00}_{\alpha,\beta,\gamma;d}$ and $\widetilde{T}^{0,1}_{\alpha,\beta,\gamma',\gamma'';d}$ and if we combine the estimates (3.9), (3.13) and (3.17), we see that if we require

$$J_0 \geqq 2(J + M + m + N), \tag{3.18}$$

$$M' \geqq 2(J + M + N + 1 + 2J_0), \tag{3.19}$$

we shall have :

$$\sum_{|\alpha| \leqq M} |D^\alpha \{R(x, D)(e^{i\rho w} \varphi)\}| \leqq$$

$$\text{const.}\ \rho^{-J} \sup \sum_{|\beta| \leqq M'} \rho^{-|\beta|/2} \{|D^\beta (e^{i\rho w_2} \varphi)| + |e^{i\rho w_2} D^\beta \varphi|\}. \tag{3.20}$$

Finally, we must consider

$$\Phi = \sum_{|\alpha| \leqq J_0} \frac{1}{\alpha\,!} P^{(\alpha)}(x, \rho \partial w(x)) D_y^\alpha \{\varphi(y)\, e^{-i\rho H(x,y)}\}|_{y=x} - \sum_{j=0}^{J'} \rho^{m-j} \mathcal{R}_j^w \varphi,$$

where J_0 satisfies (3.18) and also, $J_0 \geqq 2J'$. By definition of the $\mathcal{R}_j^w$ (see (2.23)) and by virtue of (2.22),

$$\Phi = \sum_{j,\alpha,\beta,d} c_{j,\alpha,\beta,d}(x)\, \rho^{m-j-|\alpha|+d}\, D^\beta \varphi,$$

where the summation is performed over the integers j such that $j \leqq J' + J_0$ (see (2.17)), over the N-tuples α, β and the integers d such that

$$\beta_i \leqq \alpha_i,\ i = 1, \ldots, N\,;\ 2d \leqq |\alpha - \beta|\,; \qquad (3.21)$$
$$J' < j + |\alpha| - d\ ;\ |\alpha| \leqq J_0.$$

We have therefore

$$|\Phi| \leqq C \rho^{m-j-|\alpha|/2} \sum_{|\beta| \leqq J_0} \rho^{-|\beta|/2}\, |D^\beta \varphi|. \qquad (3.22)$$

Observing that $j + |\alpha|/2 > J'/2$, and requiring

$$J' \geqq 2(J + m), \qquad (3.23)$$

we derive from (3.22) :

$$|e^{i\rho w} \Phi| \leqq C \rho^{-J} \sum_{|\beta| \leqq J_0} \rho^{-|\beta|/2}\, |e^{i\rho w} D^\beta \varphi|. \qquad (3.24)$$

Similarly, by requiring

$$J' \geqq 2(J + M + m), \qquad (3.25)$$

we obtain

$$\sum_{|\alpha| \leqq M} |D^\alpha(e^{i\rho w} \Phi)| \leqq C \rho^{-J} \sum_{|\beta| \leqq J_0} \rho^{-|\beta|/2}\, |e^{i\rho w_2} D^\beta \varphi|. \qquad (3.26)$$

In order to complete the proof of Lemma 3.1, it suffices to combine the estimates (3.20) and (3.26), with the identity (3.5) and with the definition of Φ. The reader will observe that the integer J' is determined by (3.3) and (3.25) ; we must then have (3.18) and also $J_0 \geqq 2J'$. This determines J_0, which then enables us to choose M' according to (3.19).

4. The basic asymptotic formula (end) : Approximation by symbols analytic in the ξ variables.

In this section we shall drop the precondition that the symbol $P(x, \xi)$ should be analytic with respect to ξ (in a complex region such as (2.19)). In fact, we shall assume that P is given by (1.1), Introduction. As before, we are given two integers M, $J \geqq 0$ and select two other integers J', M' fulfilling the requirements of Cor. 2.1 and Lemma 3.1. In virtue of Cor. 2.1, we may focus upon the finite sum (2.17), which we denote by $P_{(J')}(x, D)$, as in (2.18). It is clear that, given any integer $N' \geqq 0$, we may find a symbol $P_{(J', N')}(x, \xi)$, which can be extended as a holomorphic function of ξ in the set (2.19) (for a suitable choice of $\varepsilon > 0$; as we shall see, this choice can be made independently of J' and N'), such that the "remainder"

$$R_{(J', N')}(x, \xi) = P_{(J')}(x, \xi) - P_{(J', N')}(x, \xi) \tag{4.1}$$

satisfies the following inequality :

$$|R_{(J', N')}(x, \xi)| \leqq C_{J', N'}(x) \left| \frac{\xi}{|\xi|} - \xi^0 \right|^{N'} (1 + |\xi|)^m \tag{4.2}$$

for all $x \in \Omega$, $\xi \in \mathbf{R}_N$; $C_{J', N'}$ is a continuous positive function in Ω.

One way of achieving this is by using finite Taylor expansions :

$$P_{m-j, N'}(x, \xi) = |\xi|^{m-j} \sum_{|\alpha| \leqq N'} \frac{1}{\alpha!} P_{m-j}^{(\alpha)}(x, \xi^0) \left(\frac{\xi}{|\xi|} - \xi^0 \right)^{\alpha}$$

$$= \sum_{|\alpha| \leqq N'} \frac{1}{\alpha!} P_{m-j}^{(\alpha)}(x, |\xi| \xi^0) (\xi - |\xi| \xi^0)^{\alpha},$$

and by setting

$$P_{(J',N')}(x, \xi) = \sum_{j=0} P_{m-j,N'}(x, \xi)\, \chi(\xi).$$

There might be other ways. In the application, in the next sections, we shall use a modification of the finite Taylor expansion method.

Let us consider a symbol $S(x, \xi) \in C^\infty(\Omega \times \mathbf{R}_N)$ satisfying an inequality of the kind (4.2) :

$$|S(x, \xi)| \leqq C(x) \left| \frac{\xi}{|\xi|} - \xi^0 \right|^{N'} (1 + |\xi|)^m \quad , \ x \in \Omega,\ \xi \in \mathbf{R}_N, \quad (4.3)$$

with $C(x)$ a positive continuous function in Ω.

LEMMA 4.1. – *To every pair of integers* J, $M \geqq 0$ *there are integers* M', $N' \geqq 0$ *such that, if* (4.3) *holds, the following is true* :

To every compact subset K *of* Ω *there is a constant* $C(K) > 0$ *such that, for every* $\rho > 1$, $\varphi \in C_c^\infty(K)$ *and* $w \in C^\infty(\Omega)$ *satisfying* (2.7) *and* (2.8),

$$\sup_K \sum_{|\alpha| \leqq M} |D^\alpha S(x, D)\, (e^{i\rho w}\varphi)| \leqq$$

$$\leqq C(K)\, \rho^{-J} \sup \sum_{|\alpha| \leqq M'} \rho^{-|\alpha|/2}\, |D^\alpha (e^{i\rho w_2}\varphi)| \qquad (4.4)$$

Proof. – We have :

$$S(x, D)\,(e^{i\rho w}\varphi) = (2\pi)^{-N} \int\int e^{i<x,\xi> - i<\xi - \rho\xi^0, y>} (1 + |\xi - \rho\xi^0|^2)^k$$

$$(1 + |\xi - \rho\xi^0|^2)^{-k}\, S(x, \xi)\, u(y)\, dy\, d\xi$$

$$= (2\pi)^{-N} \int\int e^{i<x,\xi> - i<\xi - \rho\xi^0, y>} (1 + |\xi - \rho\xi^0|^2)^{-k}$$

$$S(x, \xi)\, (1 - \Delta)^k\, u(y)\, dy\, d\xi,$$

where $u = e^{i\rho w_2}\varphi$. We further have, according to (4.3),

$$(1 + |\xi - \rho\xi^0|^2)^{-k}\, |S(x, \xi)| \leqq$$

$$\leqq C(K)\, (1 + |\xi|)^{-N'}\, |\xi - \rho\xi^0|^{N'}\, (1 + |\xi - \rho\xi^0|^2)^{-k},\ x \in K,\ \xi \in \mathbf{R}_N .$$

This inequality is obvious when $|\xi| \leqq 1$; it follows at once from (4.3) for $|\xi| > 1$ and $|\xi|^{-N'}$ substituted for $(1 + |\xi|)^{-N'}$. Next we observe that

$$|\xi - |\xi|\xi^0| \leqq |\xi - \rho\xi^0| + ||\rho\xi^0| - |\xi|| \leqq 2|\xi - \rho\xi^0|.$$

Also $\rho \leqq (1 + |\xi|)(1 + |\xi - \rho\xi^0|)$. We reach at once the conclusion that we can choose N′ and k so as to have, for x in K,

$$(1 + |\xi - \rho\xi^0|^2)^{-k} |S(x, \xi)| \leqq \text{const.} (1 + |\xi|)^{-N-1} \rho^{-J-k}.$$

From this and from the above expression of $S(x, D)(e^{i\rho w}\varphi)$ we derive at once

$$\sup_K |S(x, D)(e^{i\rho w}\varphi)| \leqq \text{const.} \rho^{-J-k} \sup |(1 - \Delta)^k u|.$$

The derivatives of order $\leqq$ M of $S(x, D)(e^{i\rho w}\varphi)$ can be estimated in a similar fashion.

Q.E.D.

To establish the asymptotic expansion for a pseudodifferential operator P of the kind (1.1), Introduction, it now suffices to combine Inequality (2.18) and Lemma 4.1, where $S = R_{(J',N')}$, with Lemma 3.1 applied to the analytic approximation $P_{(J',N')}$. We may summarize :

THEOREM 4.1. – *Let* $w \in C^\infty(\Omega)$ *satisfy* (2.7) *and* (2.8) *and* U *be an open neighborhood of* x_0 *in* Ω *such that* (2.20) *holds. To every pair of integers* J, M $\geqq 0$ *there are integers* J′, M′, N′ $\geqq 0$ *such that the following is true :*

Let

$$P_{(J',N')}(x, \xi) = \sum_{j=0} P_{(J',N'),m-j}(x, \xi)\, \chi(\xi) \tag{4.5}$$

be a symbol satisfying the hypotheses in Lemma 3.1, *such, furthermore, that* (4.1) *and* (4.2) *hold. Let us denote by* $\mathcal{P}^w_{(J',N'),j}$, $j = 0,1, \ldots,$ *the differential operators associated to* $P_{(J',N')}(x, D)$ *in the manner described in* (2.23).

Then to every compact subset K *of* Ω *there is a constant* $C(K) > 0$ *such that, for every* $\rho > 1$ *and every* $\varphi \in C_c^\infty(U)$, *we have :*

$$\sup_{K} \sum_{|\alpha| \leqq M} \left| D^\alpha \left\{ P(x, D)(e^{i\rho w}\varphi) - e^{i\rho w} \sum_{j=0} \rho^{m-j} \mathcal{P}^w_{(J',N'),j}\, \varphi \right\} \right| \leqq$$

$$\leqq C(K)\rho^{-J} \sup \sum_{|\alpha| \leqq M'} \rho^{-|\alpha|/2} \{ |D^\alpha(e^{i\rho w_2}\varphi)| + |e^{i\rho w_2} D^\alpha \varphi| \}. \quad (4.6)$$

5. Beginning of the proof of theorem 1.1.

We start with some simplifying remarks about the problem. Possibly after renaming the coordinates, we may assume, by (1.4), that

$$(\partial/\partial \xi_N)\, L(x_0, \xi^0) \neq 0. \quad (5.1)$$

We shall write t instead of y^N, τ instead of η_N. Setting $n = N - 1$, we shall reserve the notation y for $(y^1, \dots, y^n)$, η for $(\eta_1, \dots, \eta_n)$. We may apply the implicit function theorem : possibly after a redefinition of the "elliptic factor" Q, shrinking of the conic neighborhood $\mathcal{U}$ around its axis and a *canonical* change of variables $(x, \xi) \to (y, t, \eta, \tau)$ in the cotangent bundle $T^*(\Omega)$ to straighten up the bicharacteristics of A, we may assume that

$$\tilde{L} = \tau - ib(y, t, \eta) \quad in \quad \mathcal{U}, \quad (5.2)$$

where $\mathcal{U}$ is now a conic neighborhood of the point (x_0, ξ^0) in the new coordinates, $b \in C^\infty(\mathcal{U}')$ is a *real* positive-homogeneous function of degree *one* with respect to η and $\mathcal{U}'$ is the τ-projection of $\mathcal{U}$. We may also assume that

$$|\tilde{Q}(y, t, \eta, \tau)| \geqq c\, |(\eta, \tau)|^{m-2} \quad in \quad \mathcal{U}, \quad (5.3)$$

and that in the new coordinates, (x_0, ξ^0) has become $(0, 0, \eta^0, 0)$ (note that (1.3) implies that τ^0, the τ-coordinate of (x_0, ξ^0), must be equal to zero). We have denoted by $\tilde{L}(y, t, \eta, \tau)$ and $\tilde{Q}(y, t, \eta, \tau)$ the transforms of $L(x, \xi)$ and $Q(x, \xi)$ respectively. We have also the right to assume that $\mathcal{U}$ is contained in a cone :

$$|\tau| < \text{const. } |\eta|. \quad (5.4)$$

Because of the invariance [6] of the hypotheses of Th. 1.1, not only under multiplication of L by an elliptic symbol, but also under canonical change of coordinates in $T^*(\Omega)$, they continue to hold when L has the form (5.2). We also remark that the property under study, namely, local solvability, is invariant under such transformations, which means to say that P is locally solvable at x_0 if and only if $U^{-1}PU$ is locally solvable at (0, 0), where U is an elliptic Fourier integral operator associated with the canonical transformation.

It is classical that Poisson brackets such as $\{A, B\} = H_A B$ are invariant under changes of coordinates in the base space and associated changes of coordinates in the fibres, in $T^*(\Omega)$, and more generally, under *canonical changes* of coordinates in $T^*(\Omega)$. By virtue of (5.2), the bicharacteristic strips of Re L (including the one through $(0, 0, \eta^0, 0)$ which we have denoted by Γ_0) are straight lines parallel to the t-axis and the Hamiltonian of Re L is $\partial/\partial t$. In view of this, (1.11) and (1.12) can be restated as

$$(\partial/\partial t)^j\, b(0, t, \eta^0) \begin{cases} = 0 & \text{if } j < k_0 \\ > 0 & \text{if } j = k_0 \end{cases} \quad \textit{at} \quad t = 0. \tag{5.5}$$

For convenience, we shall choose $\mathcal{U}' = \mathcal{O} \times \{t \in \mathbf{R}^1, |t| < t_1\}$ where $\mathcal{O}$ is an open conic neighborhood of $(0, \eta^0)$ and $t_1 > 0$. We shall apply the Weierstrass-Malgrange preparation theorem. By shrinking of $\mathcal{U}'$, if necessary, we may assume that there exist two C^∞ real-valued functions $E(y, t, \eta)$ and $f(y, t, \eta)$, positive homogeneous with respect to η, of degrees 1 and 0, respectively, such that

$$b = Ef \quad \textit{in} \quad \mathcal{U}', \tag{5.6}$$

$$E(y, t, \eta) > 0 \quad \textit{in} \quad \mathcal{U}' \tag{5.7}$$

$$f = t^{k_0} + a_1(y, \eta)\, t^{k_0 - 1} + \ldots + a_{k_0}(y, \eta), \tag{5.8}$$

where the $a_j(y, \eta)$ are real-valued and C^∞ in $\mathcal{O}$ and vanish at $(0, \eta^0)$.

We shall need the following :

LEMMA 5.1. — *Let f be a real polynomial satisfying* (5.8). *Then, given any open conic subset $\mathcal{O}'$ of $\mathcal{O}$, containing $(0, \eta^0)$ and any $\varepsilon > 0$, there is an open subset $\mathcal{O}''$ of $\mathcal{O}'$ and a real C^∞ function $\varphi(y, \eta)$ defined in $\mathcal{O}''$, such that, $|\varphi(y, \eta)| < \varepsilon$ in $\mathcal{O}''$, and f changes sign, from minus to plus, in the t-direction, across*

$$\Sigma = \{(y, t, \eta)\ ;\ (y, \eta) \in \mathcal{O}''\ ;\ t = \varphi(y, \eta)\}.$$

Proof. – For every $(y, \eta) \in \mathcal{O}$, there is a set (possibly empty) of real numbers

$$T_\epsilon(y, \eta) = \{t_1(y, \eta), \ldots, t_r(y, \eta)\}$$

such that $t \mapsto f(y, t, \eta)$ changes sign at $t_j(y, \eta)$, from minus to plus, and $|t_j(y, \eta)| < \varepsilon$ for all $j = 1, \ldots, r$. Since k_0 is *odd*, $T_\epsilon(y, \eta) \neq \phi$ as soon as (y, η) is close enough to $(0, \eta^0)$. Let $k_j (\leqslant k_0)$ be the order of the root $t_j(y, \eta)$ of $f(y, t, \eta)$ and $k = \inf k_j$, when (y, η) varies over $\mathcal{O}'$ and $t_j(y, \eta)$ over $T_\epsilon(y, \eta)$. Because k is odd, we have $0 < k \leqslant k_0$. Choose $(y', \eta') \in \mathcal{O}'$ such that $f(y', t, \eta')$ has a root $t' = t(y', \eta')$ of order k, $|t'| < \varepsilon$, and changes sign there from minus to plus. Since $(\partial/\partial t)^{k-1} f(y', t, \eta')$ has a *simple root* at t', we know by the implicit function theorem, that the set Σ defined by

$$(\partial/\partial t)^{k-1} f(y, t, \eta) = 0$$

can be represented, in a neighborhood $\mathcal{O}''$ of (y', η') in $\mathcal{O}'$, by $t = \varphi(y, \eta)$, where φ is a real C^∞ function. On the other hand, we know that every line parallel to the t axis, through a point $(y, 0, \eta)$ near $(y', 0, \eta')$, contains a point $(y, t(y, \eta), \eta)$, with $t(y, \eta)$ near t', where f changes sign from minus to plus and all derivatives $(\partial/\partial t)^\ell f$, $\ell \leqslant k-1$, vanish. This shows that $(y, t(y, \eta), \eta) \in \Sigma$. Furthermore, by shrinking $\mathcal{O}''$, we may assume that

$$\frac{f(y, t, \eta)}{[t - \varphi(y, \eta)]^k} \neq 0 \text{ in a neighborhood of } (y', t', \eta'). \tag{5.9}$$

Q.E.D.

We shall apply the lemma with $\varepsilon = t_2 < t_1$ and then perform another canonical change of variables to flatten the surface Σ, around some point $(y_1, \eta^1) \in \mathcal{O}''$, perpendicular to the bicharacteristic t-lines. The upshot of all this is that (continuing to denote the new variables by (y, t, η, τ)) we may assume, by further shrinking of $\mathcal{O}''$ and t_2, that

$$b(y, t, \eta) = t^k \beta(y, t, \eta) \text{ in } \mathcal{U}'' = \mathcal{O}'' \times \{t \in \mathbf{R}^1 ; |t| < t_2\} \text{(1)}$$

where $\beta > 0$ in $\mathcal{U}''$ and $k \leqslant k_0$ odd. (5.10)

(1) In the application of the lemma, we assumed that $\mathcal{O}''$ is a conic subset, by extending φ to be a positive-homogeneous function of degree zero, with respect to η.

Next we consider the subprincipal part of P (see $(2.5)_{***}$) :

$$\Phi(y, t, \eta) = \mathcal{M}(y, t, \eta, it^k \beta(y, t, \eta)), \tag{5.11}$$

in the conic subset $\mathcal{U}_0$ of $\mathcal{U}$ whose τ-projection coincides with $\mathcal{U}''$. We shall restrict Φ to Σ and choose a point $(y_2, \eta^2) \in \mathcal{O}''$ such that $t \to \Phi(y, t, \eta)$ vanishes of minimum order there. Let q be this minimum order ; it may very well be zero. Also, we do not exclude the case $q = +\infty$ which simply means that Φ has a zero of infinite order on Σ. We may then write (decreasing t_2 if necessary) :

$$\Phi(y, t, \eta) = t^q \psi(y, t, \eta) \text{ in } \mathcal{U}'', \ \psi(y_2, 0, \eta^2) \neq 0. \tag{5.12}$$

It is also evident that

$$\operatorname*{grad}_{y,\eta} \Phi(y, t, \eta) = O(|t|^q) \text{ in } \mathcal{U}''. \tag{5.13}$$

We shall make the following remark :

(5.14) Since local solvability is an open property, ${}^tP$ will not be locally solvable at the origin if it is not locally solvable at arbitrarily close points. Therefore, since $\mathcal{O}'$ in Lemma 5.1 is arbitrary, it is enough for the conclusion of Th. 1.1, to prove that ${}^tP$ is not locally solvable at the point $(y_2, 0)$. We may then take advantage of the decomposition (5.10) and of the considerations that follow it. From now on, we assume that by a translation of the coordinates, $(y_2, 0)$ becomes $(0, 0)$.

The starting point in the proof of Th. 1.1 is the same as always in this kind of question : the remark of Hörmander as to the functional-analytic consequence of local solvability, here of the pseudodifferential operator ${}^tP$, at the origin (see [6], pp. 1, 2) : if ${}^tP$ were locally solvable at the origin, there would be two open neighborhoods $V \subset U$ of $(0, 0)$ in Ω, a compact subset K of U, an integer $M \geqq 0$, a constant $C > 0$ such that

$$\left|\int fv \, dy dt\right| \leqq C \left\{\sup \sum_{|\alpha| \leqq M} |D^\alpha f|\right\} \left\{\sup_K \sum_{|\alpha| \leqq M} |D^\alpha (Pv)|\right\}, \tag{5.15}$$

for every $f, v \in C_c^\infty(V)$. The proof of Th. 1.1 will consist in proving that, in the present situation, (5.15) cannot hold--whatever the choice of U, V, K, M, C. In order to show this one takes

$$v = e^{i\rho w}\varphi \ , \ \rho \sim +\infty, \tag{5.16}$$

with $w \in C^\infty(\Omega)$, $\varphi \in C_c^\infty(V)$ chosen in such a way that v is an approximate solution, in a sense that we are now going to make precise, of the homogeneous equation $Pv = 0$.

First of all we wish to apply Th. 4.1. For this we need a good analytic approximation of $P^\natural(y, t, \eta, \tau)$, the symbol of P in the new coordinate system (y, t), of the kind (4.5). We shall describe it below, in all details. If we choose J', M', N' according to the requirements of Th. 4.1 and if we combine (5.15) with (4.6), we obtain :

$$|\int fv\, dydt| / \sup \sum_{|\alpha| \leq M} |D^\alpha f| \leq$$

$$\leq C \sup \sum_{|\alpha| \leq M} \left| D^\alpha \left\{ e^{i\rho w} \sum_{j=0}^{J'} \rho^{m-j} \mathcal{P}^{w}_{(J',N')} \varphi \right\} \right| +$$

$$+ CC(K)\rho^{-J} \sup \sum_{|\alpha| \leq M'} \rho^{-|\alpha|/2} \{ |D^\alpha(e^{i\rho w_2}\varphi)| + |e^{i\rho w_2} D^\alpha \varphi| \}. \tag{5.17}$$

The choice of J will be made later ; of course, it helps to determine J', M', N' as stated in Th. 4.1.

We now proceed with the construction of the analytic approximation to $P^\natural(y, t, \eta, \tau)$. As we have already said, N' is chosen in accordance with the requirements of Th. 4.1. Let us then set

$$\beta_{N'}(y, t, \eta) = \sum_{|\alpha| \leq N'} \frac{1}{\alpha!} \beta^{(\alpha)}(y, t, |\eta|\, \eta^2)\, (\eta - |\eta|\, \eta^2)^\alpha, \tag{5.18}$$

$$b_{N'}(y, t, \eta) = t^k \beta_{N'}(y, t, \eta), \tag{5.19}$$

$$\widetilde{L}_{N'}(y, t, \eta, \tau) = \tau - i b_{N'}(y, t, \eta). \tag{5.20}$$

Also :

$$\widetilde{Q}_{N'}(y, t, \eta, \tau) = \sum_{|\alpha| \leq N'} \frac{1}{\alpha!} \widetilde{Q}^{(\alpha)}(y, t, r\eta^2, 0)\, ((\eta, \tau) - r(\eta^2, 0))^\alpha, \tag{5.21}$$

where we have used the notation :

$$r = (|\eta|^2 + \tau^2)^{1/2} . \tag{5.22}$$

We have :

$$|(\tilde{L} - \tilde{L}_{N'}) (y , t , \eta , \tau)| \leqq \text{const. } |\eta| \left| \frac{\eta}{|\eta|} - \eta^2 \right|^{N'+1} . \tag{5.23}$$

But $|\eta - |\eta|\eta^2| \leqq 2\ |\eta - r\eta^2|$ and $|\eta|^{-N'} \leqq$ const. $r^{-N'}$ by (5.4) (recalling that the whole argument takes places in $\mathcal{U}_0$). Thus :

$$|\eta| \left| \frac{\eta}{|\eta|} - \eta^2 \right|^{N'+1} = |\eta|^{-N'}\ |\eta - |\eta|\eta^2|^{N'+1} \leqq \text{const. } r^{-N'}\ |\eta - r\eta^2|^{N'+1},$$

and we see that (5.23) implies :

$$|(\tilde{L} - \tilde{L}_{N'}) (y , t , \eta , \tau)| \leqq \text{const. } r \left| \frac{1}{2} (\eta , \tau) - (\eta^2, 0) \right|^{N'+1} . \tag{5.24}$$

We have immediately :

$$|(\tilde{Q} - \tilde{Q}_{N'}) (y , t , \eta , \tau)| \leqq \text{const. } r^{m-2} \left| \frac{1}{r} (\eta , \tau) - (\eta^2, 0) \right|^{N'+1} . \tag{5.25}$$

Let us denote by $\tilde{P}_m(y , t , \eta , \tau)$ the principal symbol of P in the new coordinates $y , t,$ which is the same as the transform of $P_m(x , \xi)$ under the change of coordinates $(x , \xi) \to (y , t , \eta , \tau)$. Let $g(y , t)$ be a $\mathcal{C}^\infty$ function with compact support in the projection of $\mathcal{U}_0$ in Ω, equal to 1 in a neighborhood of the origin, $h(\eta , \tau)$ a $\mathcal{C}^\infty$ function in $\mathbf{R}_{n+1} \backslash \{0\}$, positive-homogeneous of degree zero with respect to (η , τ), equal to 1 in a neighborhood of $(\eta^2, 0)$, such that the support of $g(y , t)\ h(\eta , \tau)$ is contained in $\mathcal{U}_0$. We set

$$P^{\natural}_{m,N'}(y , t , \eta , \tau) = g(y , t)\ h(\eta , \tau)\ (\tilde{Q}_{N'} \tilde{L}^2_{N'})\ (y , t , \eta , \tau) + \tag{5.26}$$

$$+ (1 - g(y , t))\ h(\eta , \tau) \sum_{|\alpha| \leqq N'} \frac{1}{\alpha!} \tilde{P}^{(\alpha)}_m (y , t , r\eta^2 , 0)\ ((\eta , \tau) - $$

$$- r(\eta^2 , 0))^\alpha,\ (y , t) \in \Omega,\ (\eta , \tau) \in \mathbf{R}_{n+1} \backslash \{0\}.$$

We observe that $P_{m,N'}(y, t, \eta, \tau)$ can be extended as a holomorphic function of (η, τ) in a region of the kind

$$(\eta, \tau) \in C_{n+1} \; ; \; \exists \rho > 1 \textit{ such that } |(\eta, \tau) - \rho(\eta^2, 0)| < \epsilon\rho \tag{5.27}$$

that is to say, of the kind (2.19). We derive from (5.24) and (5.25) :

$$\left|\widetilde{P}_m(y, t, \eta, \tau) - P^{\natural}_{m,N'}(y, t, \eta, \tau)\right| \leqq C(y, t)\, r^m \left|\frac{1}{r}(\eta, \tau) - (\eta^1, 0)\right|^{N'+1}, \tag{5.28}$$

where $C(y, t)$ is a positive continuous function in Ω.

At this point we introduce the homogeneous parts $P^{\natural}_{m-j}(y, t, \eta, \tau)$ of degree $m - j, j > 0$, of the symbol of P in the new coordinates system (y, t). The reader should be careful not to think that $P^{\natural}_{m-j}(y, t, \eta, \tau)$ is the transform of $P_{m-j}(x, \xi)$ under the transformation $(x, \xi) \to (y, t, \eta, \tau)$; lower-order terms have no invariant meaning and their expressions in new coordinates depend on the terms which have a higher order than theirs. By (5.11) we may write :

$$\mathcal{M}(y, t, \eta, \tau) = \Phi(y, t, \eta, \tau) + (\tau - \mathrm{i}t^k \beta(y, t, \eta))\, \Phi_1(y, t, \eta, \tau) \tag{5.29}$$

where Φ_1 is positive-homogeneous of degree $m - 2$ with respect to (η, τ). Next we set :

$$\psi_{N'}(y, t, \eta) = \sum_{|\alpha| \leqq N'} \frac{1}{\alpha!} \psi^{(\alpha)}(y, t, |\eta|\eta^2)\, (\eta - |\eta|\eta^2)^\alpha, \tag{5.30}$$

$$\Phi_{N'}(y, t, \eta) = t^q \psi_{N'}(y, t, \eta), \tag{5.31}$$

$$\Phi_{1,N'}(y, t, \eta, \tau) = \sum_{|\alpha| \leqq N'} \frac{1}{\alpha!} \Phi_1^{(\alpha)}(y, t, r\eta^2, 0)\, ((\eta, \tau) - r(\eta^2, 0))^\alpha, \tag{5.32}$$

$$\begin{aligned}\mathcal{M}_{N'}(y, t, \eta, \tau) &= \Phi_{N'}(y, t, \eta) + \\ &+ g(y, t)\, h(\eta, \tau)\, (\tau - \mathrm{i}t^k \beta_{N'}(y, t, \eta))\, \Phi_{1,N'}(y, t, \eta, \tau) + \\ &+ (1 - g(y, t))\, h(\eta, \tau) \sum_{|\alpha| \leqq N'} \frac{1}{\alpha!} \mathcal{M}^{(\alpha)}(y, t, r\eta^2, 0)\, ((\eta, \tau) - r(\eta^2, 0))^\alpha,\end{aligned}$$

$$(y, t) \in \Omega, \; (\eta, \tau) \in \mathbf{R}_{n+1} \backslash \{0\}.$$

Using $(2.5)_{***}$ we then write :

$$P^{\natural}_{m-1,N'}(y, t, \eta, \tau) = \mathcal{M}_{N'}(y, t, \eta, \tau) + \frac{1}{2} \sum_{|\alpha|=1} D^{\alpha}\widetilde{P}^{(\alpha)}_{m,N'}(y, t, \eta, \tau). \tag{5.35}$$

We derive at once :

$$\left|(P^{\natural}_{m-1} - P^{\natural}_{m-1,N'})(y, t, \eta, \tau)\right| \leqq C_1(y, t)\, r^{m-1} \left|\frac{1}{r}(\eta, \tau) - (\eta^2, 0)\right|^{N'+1}, \tag{5.35}$$

where $C_1(y, t)$ is a positive continuous function in Ω. It is easily verified that $P^{\natural}_{m-1,N'}(y, t, \eta, \tau)$ can be extended holomorphically to the region (5.27). We set for $j > 1$:

$$P^{\natural}_{m-j,N'}(y, t, \eta, \tau) = \sum_{|\alpha| \leqq N'} \frac{1}{\alpha!} P^{(\alpha)}_{m-j}(y, t, r\eta^2, 0)\,((\eta, \tau) - r(\eta^2, 0))^{\alpha}, \tag{5.36}$$

whence, for $j = 2, 3, \ldots,$

$$\left|(P^{\natural}_{m-j} - P^{\natural}_{m-j,N'})(y, t, \eta, \tau)\right| \leqq C_j(y, t)\, r^{m-j} \left|\frac{1}{r}(\eta, \tau) - (\eta^2, 0)\right|^{N'+1}. \tag{5.37}$$

The $P_{m-j,N'}(y, t, \eta, \tau)$ can also be extended holomorphically to the region (5.27). We now set :

$$P^{\natural}_{(J',N')}(y, t, \eta, \tau) = \sum_{j=0}^{J'} P^{\natural}_{m-j,N'}(y, t, \eta, \tau)\,\chi(\eta, \tau), \tag{5.38}$$

where $\chi(\eta, \tau) = \chi_0(r) = 0$ for $r < 1/3$, $= 1$ for $r > 2/3$ (r given by (5.22)). It is seen at once that $P_{(J',N')}(y, t, \eta, \tau)$ is an approximation of $P^{\natural}(y, t, \eta, \tau)$, the symbol of P in the coordinates (y, t), which fulfills all the requirements of Th. 4.1.

It is obvious that we have :

$$(\partial/\partial t)^j b_{N'} = (\partial/\partial t)^j b \textit{ at } (0, 0, \eta^2) \textit{ for all } j,\ 0 \leqq j \leqq N' - 1. \tag{5.39}$$

It follows at once from this that if we take $N' > k$, the basic hypotheses in Th. 1.1, fulfilled by b, must also be fulfilled by $b_{N'}$:

$$(\partial/\partial t)^j b_{N'}(0, t, \eta^2) \begin{cases} = 0 \ \textit{if} \ j < k \\ > 0 \ \textit{if} \ j = k \end{cases} \quad \textit{at} \ t = 0. \tag{5.40}$$

We have reached the following conclusion :

(5.41) *If we take Inequality (5.17) as the starting point, we have the right to replace throughout the argument the symbol of* P *in the coordinates* (y, t) *by* $P_{(J',N')}(y, t, \eta, \tau)$ *defined in* (5.38).

In view of this and in order to simplify the notation, we shall reason by contradiction, assuming that (5.17) holds and that the symbol of P in the coordinates $y, t,$ is exactly equal to $P_{(J',N')}(y, t, \eta, \tau)$. We shall therefore omit all superscript tildes $\sim$ and sharps $\natural$ and all subscripts (J', N'). In particular, we shall write $\mathcal{R}_j^w$ instead of $\mathcal{R}^w_{(J',N'),j}$.

6. The principal part of the phase-function

The function w entering in (4.6) will play the role of a "phase-function" or rather, of the *principal part of the phase function* : in certain instances we shall add to it a "perturbation term" of the form $\rho^{-1/2} w_1$ (see Sections 8, 9). In all cases we take w to be a $\mathcal{C}^\infty$ function in Ω, satisfying

$$L(y, t, w_y, w_t) \sim 0, \tag{6.1}$$

$$w|_{t=0} = \langle \eta^2, y \rangle + i|y|^2/2. \tag{6.2}$$

We have used (and we shall use, further on) the notation ~ 0 to mean *vanishing of infinite order with respect to y, at y* $= 0$. Note that (6.1) can be rewritten :

$$w_t - i t^k \beta(y, t, w_y) \sim 0. \tag{6.3}$$

We recall that b is equal to $b_{N'}$ given by (5.19). Let us set

$$w_2 = w - \langle \eta^2, y \rangle.$$

We have :

$$w_{2t} - it^k \beta(y, t, \eta^2 + w_{2y}) \sim 0, \tag{6.4}$$

$$w_2|_{t=0} = i|y|^2/2. \tag{6.5}$$

Let us differentiate (6.4) with respect to y and put $y = 0$ in the result. This yields :

$$w^0_{2yt} - it^k \beta_y(0, t, \eta^2 + w^0_{2y}) - iw^0_{2yy} t^k \beta_\eta(0, t, \eta^2 + w^0_{2y}) = 0 \tag{6.6}$$

where the upper indices zero indicate that one should take $y = 0$. Equation (6.6) may be viewed as a system of n nonlinear ordinary differential equations in the unknown functions w^0_{2y} (t is the variable). From (6.5) we derive :

$$w^0_{2y}|_{t=0} = 0. \tag{6.7}$$

The Picard iteration theorem yields easily the estimate :

$$|w_{2y}(0, t)| \leqq \text{const.} \left| \int_0^t |t'|^k |\partial_{y,\eta} \beta(0, t', \eta^2)| \, dt' \right|. \tag{6.8}$$

We return to (6.4) and (6.5). We derive from it :

$$\begin{aligned} w_{2t}(0, t) &= it^k \beta(0, t, \eta^2 + w_{2y}(0, t)) \\ &= it^k \beta(0, t, \eta^2) + it^k \beta_\eta(0, t, \eta^2) w_{2y}(0, t) + \qquad (6.9) \\ &\qquad + O(t^k |w_{2y}(0, t)|^2), \end{aligned}$$

hence :

$$\left| w_2(0, t) - i \int_0^t t'^k \beta(0, t', \eta^2) \, dt' \right| \leqq \text{const.} \left(\int_0^t |t'|^k |\partial_{y,\eta} \beta(0, t', \eta^2)| dt' \right)^2. \tag{6.10}$$

Since, for $|t| < t_2$ sufficiently small, we have $\beta(0, t, \eta^2) > 0$, the function

$$B(t) = \int_0^t t'^k \beta(0, t', \eta^2) \, dt' \tag{6.11}$$

is $\geqslant 0$. It follows at once from (6.10) that

$$|w(0, t) - iB(t)| \leqslant \text{const. } B(t)^2, \tag{6.12}$$

and by the same token, from (6.8), that

$$|w_y(0\,,t) - \eta^2| \leqslant \text{const. } B(t). \tag{6.13}$$

We have

$$w(y\,,t) = w(0\,,t) + \langle w_y(0\,,t), y\rangle + \frac{1}{2}\, i\,|y|^2 + 0(|y|^2(|y|+|t|)), \tag{6.14}$$

and therefore, by virtue of (6.12) and (6.13),

$$\left| w(y\,,t) - \langle \eta^2\,, y\rangle - \frac{1}{2}\, i\,|y|^2 - i B(t)\right| \leqslant$$

$$\text{const. } \{B(t)^2 + |y|\, B(t) + |y|^2(|y| + |t|)\} \tag{6.15}$$

An obvious consequence of (6.15) (and of the definition of w) is that

$$|w(y\,,t) - \langle \eta^2\,, y\rangle| \leqslant C\{|y|^2 + B(t)\} \leqslant C' \operatorname{Im} w(y\,,t) \tag{6.16}$$

for all $(y\,,t)$ in a sufficiently small neighborhood U of the origin in $\mathbf{R}^{n+1}$.

Incidentally note that, from the definition of B, we have

$$B(t) = b_0 \frac{t^{k+1}}{k+1}(1 + 0(t)) \quad , \quad b_0 > 0. \tag{6.17}$$

Note that the estimate (6.16) reads :

$$|w_2(y\,,t)| \leqslant C\{|y|^2 + B(t)\} \leqslant C' \operatorname{Im} w_2(y\,,t), \tag{6.18}$$

whereas (6.13) is equivalent with

$$|w_{2y}(y\,,t)| \leqslant C''\{|y| + B(t)\}. \tag{6.19}$$

We also obtain, directly from (6.9),

$$|w_{2t}(y\,,t)| \leqslant C''\, |t|^k\,, \tag{6.20}$$

and thus, combining (6.18), (6.20), we get :

$$|\partial w_2(y\,,t)| \leqq C''' \{\operatorname{Im} w_2(y\,,t)\}^{1/2}. \tag{6.21}$$

Equation (6.1) implies :

$$P_m(y\,,t\,,w_y\,,w_t) \sim 0. \tag{6.22}$$

Next we must compute $\mathcal{P}_1^w$. If α is an $(n+1)$-tuple of length 1, we have (in $\mathcal{U}_0$)

$$P_m^{(\alpha)}(y, t, \eta, \tau) = (Q^{(\alpha)} L^2 + 2QLL^{(\alpha)})(y, t, \eta, \tau),$$

hence, in view of (6.1),

$$P_m^{(\alpha)}(y, t, w_y, w_t) \sim 0. \tag{6.23}$$

The zero-order term in $\mathcal{P}_1^w$ will be :

$$P_{m-1}(y, t, w_y, w_t, w_{yy}, w_{yt}, w_{tt}) = -Q(y, t, w_y, w_t)\,\sigma(y, t). \tag{6.24}$$

It should be recalled that we are interested in the differential operators $\mathcal{P}_j^w$ merely in some open neighborhood U of the origin in $\Omega \subset \mathbf{R}^{n+1}$ where the support of the "amplitude" function φ will lie. The neighborhood U is chosen small enough so that, if (y, t) remains in U, (w_y, w_t) remains in the region (5.27). We may and shall assume that

$Q(y, t, \eta, \tau)$ *does not vanish at any point of* (5.27) (6.25)

This has the consequence that $Q(y, t, w_y, w_t)^{-1}$ is a $\mathcal{C}^\infty$ function in U.

As we have said in Section 2, we shall also need to know the leading part of $\mathcal{P}_2^w$. It is given in (2.6). We shall describe it explicitly in the coordinates y, t. Let us set :

$$\mathcal{L} = D_t - it^k \sum_{j=1} \beta_{\eta_j}(y, t, w_y)\, D_{y^j}. \tag{6.26}$$

It is checked at once that

$\mathcal{P}_2^w - Q(y, t, w_y, w_t)\,\mathcal{L}^2$ *is a first-order differential operator* (6.27)

(which we shall denote by $Q(y, t, w_y, w_t)\,\mathcal{M}$).

We have reached the following conclusion :

$$Q(y, t, w_y, w_t)^{-1}(\rho^2 \mathcal{P}_0^w \varphi + \rho \mathcal{P}_1^w \varphi + \mathcal{P}_2^w \varphi) - (\mathcal{L}^2 \varphi + \mathcal{M}\varphi -$$

$$- \rho\sigma(y, t)\varphi) = \rho \sum_{|\alpha|=1} c_\alpha(y, t)\, D^\alpha \varphi + \rho^2 c_0(y, t)\varphi, \tag{6.28}$$

where

$$c_0(y,t) \sim 0,\ c_\alpha(y,t) \sim 0 \ \ for\ |\alpha| = 1. \tag{6.29}$$

Let us set

$$\mathcal{Q}_j = -\,Q(y,t,w_y,w_t)^{-1}\mathcal{P}^w_{j+2}. \tag{6.30}$$

We may rewrite the inequality (5.17) in the following manner :

$$|\textstyle\int\int f v\, dydt| / \sup \sum_{|\alpha|\leq M} |D^\alpha f| \leq$$

$$C\rho^{m-2} \sup \sum_{|\alpha|\leq M} \left| D^\alpha \left\{ e^{i\rho w}\left(\mathcal{L}^2 + \mathfrak{M} - \rho\sigma - \sum_{j=1}^{J'-2} \rho^{-j}\mathcal{Q}_j\right)\varphi \right\} \right| +$$

$$+ CC(K)\rho^{-J} \sup \sum_{|\alpha|\leq M'} \rho^{-|\alpha|/2} \{ |D^\alpha(e^{i\rho w_2}\varphi)| + |e^{i\rho w_2} D^\alpha \varphi| \} +$$

$$+ C'' \sup (|y| + |t|)^{J''} \sum_{\substack{|\alpha|\leq M \\ |\beta|\leq 1}} \rho^{m-|\beta|} |D^\alpha(e^{i\rho w} D^\beta \varphi)|, \tag{6.31}$$

where J″ is a large positive integer, to be chosen later. The last term in (6.31) originates with the right-hand side in (6.28). The α's and β's in (6.31) stand for $(n+1)$-tuples.

We are going to perform the last analytic approximation of the proof. Here, however, the analyticity will be in the variables y, t : we shall replace each coefficient in $\mathcal{L}$, $\mathfrak{M}$ and in the $\mathcal{Q}_j$, as well as σ, by its finite Taylor expansion of order $J'' + M$ about $(0,0)$. In order to make the notation lighter we shall continue to write $\mathcal{L}$, $\mathfrak{M}$, $\mathcal{Q}_j$, σ respectively. The last term in the right-hand side of (6.31) must be modified : the summation over β must range over all multi-indices of length $\leq J'$. At last we get :

$$|\textstyle\int\int f v\, dydt| / \sup \sum_{|\alpha|\leq M} |D^\alpha f| \leq \tag{6.32}$$

$$C\rho^{m-2} \sup \sum_{|\alpha|\leq M} \left| D^\alpha \left\{ e^{i\rho w}\left(\mathcal{L}^2 + \mathfrak{M} - \rho\sigma - \sum_{j=1}^{J'-2} \rho^{-j}\mathcal{Q}_j\right)\varphi \right\} \right| +$$

$$+ C\rho^{-J} \sup \sum_{|\alpha| \leq M'} \rho^{-|\alpha|/2} \{ |D^\alpha (e^{i\rho w_2} \varphi)| + |e^{i\rho w_2} D^\alpha \varphi| \} +$$

$$+ C \sup (|y| + |t|)^{J''} \sum_{\substack{|\alpha| \leq M \\ |\beta| \leq J'}} \rho^{m-|\beta|} |D^\alpha (e^{i\rho w} D^\beta \varphi)|.$$

The inequality (6.32) will be our starting point for the argument which follows. But first, we need to relate σ and $\mathcal{M}$. The expression (6.24) can be rewritten as :

$$\sigma(y, t) = - Q^{-1}(y, t, w_y, w_t)\, \mathcal{M}(y, t, w_y, w_t). \tag{6.33}$$

Because of the analytic approximation in the (y, t) variables, (6.3) reads now :

$$w_t - it^k \beta(y, t, w_y) = 0 \ \textit{in}\ U. \tag{6.34}$$

If we use (5.11) and the fact that $w_y = \eta^2 + O(|y| + |t|^k)$, we obtain :

$$\mathcal{M}(y, t, w_y, w_t) = \Phi(y, t, w_y) = \Phi(y, t, \eta^2 + O(|y| + |t|^k)) =$$

$$= \Phi(0, t, \eta^2) + y \cdot (\Phi_y(0, t, \eta^2) + \Phi_\eta(0, t, \eta^2)) + \tag{6.35}$$

$$+ O(|y|^2 + |t|^k) \ \textit{in}\ U.$$

By (5.12) and (5.13) we get :

$$\mathcal{M}(y, t, w_y, w_t) = t^q(\tilde{\sigma}_0(t) + y \cdot \tilde{\sigma}_1(t)) + O(|y|^2 + |t|^k) \tag{6.36}$$

where $\tilde{\sigma}_0(0) \neq 0$.

Since $Q(y, t, w_y, w_t)$ is a nonvanishing analytic function in U (6.25), we also have :

$$\sigma(y, t) = t^q(\sigma_0(t) + y \cdot \sigma_1(t)) + O(|y|^2 + |t|^k) \tag{6.37}$$

where $\sigma_0(0) \neq 0$.

7. Assessing the influence of the lower-order terms.

From here on the proof of Th. 1.1 will subdivide into two parts, according to whether the lower-order terms in our equation have or do not have an influence. What is to be understood by this will be defined below. The implications of either one of the two hypotheses will surface as we go along. In both cases we may take the inequality (6.32) as starting point. We recall that the phase-function must be chosen as described in Section 6. The main purpose here is to find a suitable approximate solution of the equation

$$(\mathcal{L}^2 + \mathcal{M} - \rho\sigma)\varphi = \sum_{j=1}^{J'-2} \rho^{-j}\mathcal{Q}_j\varphi. \tag{7.1}$$

We recall that the last step in Section 6 was to replace all the coefficients entering in our problem by their finite Taylor expansions of sufficiently high order with respect to (y, t). Thus the function $\beta(y, t, \eta)$, which prior to this substitution was analytic with respect to η, is now analytic with respect to all its arguments. Similarly (and that is what is relevant to our present concern) the "zero-order" coefficient $\sigma(y, t)$ is analytic and in fact a *polynomial* with respect to y, t. Incidentally, we note that the basic hypotheses on $w(y, t)$ only bear on the finite Taylor expansion of order $k + 1$ of $w(y, t)$ at the origin.

We shall say that *the lower-order terms in the pseudodifferential operator* (1.1) *have little influence at the origin* if the following is true :

(7.2) *There is an open neighborhood* (which we take to be U) *of the origin in* $\mathbf{R}^{n+1}$ *and a constant* $C > 0$ *sucht that, for all points* (y, t) *in* U,

$$\left|\int_0^t |\operatorname{Im}\sqrt{\sigma}(y, t')|\,dt'\right| \leqq C(|y| + |t|^{\frac{k+1}{2}}). \tag{7.3}$$

When (7.2) does not hold, we shall say that *the lower-order terms in* (1.1) *have a strong influence at the origin.* We shall establish some of the implications of this latter hypothesis in the next section.

We begin now the analysis of Hypothesis (7.2).

LEMMA 7.1. – *Suppose that* $\sigma(y, t)$ *satisfies* (7.2). *Then this is also true of* $\sigma(y, t) - \sigma_2(y, t)$, *where* $\sigma_2(y, t)$ *is any continuous function in a neighborhood* $V \supset U$ *of the origin, such that, for some constant* $C > 0$ *and all* (y, t) *in* U,

$$|\sigma_2(y, t)| \leqq C\,(|y|^2 + |t|^{k-1} + |y|\ |t|^{\frac{k-3}{2}}). \tag{7.4}$$

Proof. – For each y, let $J(y)$ denote the set of t's such that $(y, t) \in U$; we subdivide $J(y)$ into two parts : $J_1(y)$, in which $|\sigma(y, t)| > 2|\sigma_2(y, t)|$; $J_2(y)$, its complement. We write $J_j(y, t)$ for the intersection of $J_j(y)$ with the interval joining 0 to t ($j = 1,2$). We have, for $t' \in J_1(y, t)$,

$$|\mathrm{Im}\,\{\sigma(y, t') - \sigma_2(y, t')\}^{1/2}| = |\mathrm{Im}\,\sqrt{\sigma}(y, t')\,(1 - (\sigma_2/2\sigma)(y, t')$$
$$+ \ldots)| \leqq C_0\,|\mathrm{Im}\,\sqrt{\sigma}(y, t')| + C_1\,|\sigma_2(y, t')|\,/\,|\sigma(y, t')|^{1/2} \leqq$$
$$C_0\,|\mathrm{Im}\,\sqrt{\sigma}(y, t')| + C_2\,|\sigma_2(y, t')|^{1/2},$$

hence, by (7.3) and (7.4),

$$\left|\int_{J_1(y,t)} |\mathrm{Im}\,\sqrt{(\sigma - \sigma_2)(y, t')}\ |dt'\right| \leqq C_3\,(|y| + |t|^{\frac{k+1}{2}}) +$$
$$+ C_4\,(|y|\ |t| + |t|^{\frac{k+1}{2}} + |y|^{\frac{1}{2}}\ |t|^{\frac{k+1}{4}}) \leqq C_5\,(|y| + |t|^{\frac{k+1}{2}}).$$

On the other hand, if $(y, t') \in J_2(y, t)$,

$$|\mathrm{Im}\,\{\sigma(y, t') - \sigma_2(y, t')\}^{1/2}| \leqq C_6\,(|y| + |t|^{\frac{k-1}{2}} + |y|^{\frac{1}{2}}\ |t|^{\frac{k-3}{4}}).$$

The integral of the left-hand side over $J_2(y, t)$ is therefore also bounded by $C_5(|y| + |t|^{(k+1)/2})$, if $C_5 > 0$ is large enough, and this proves Lemma 7.1.

We apply Lemma 7.1 with

$$\sigma_2(y, t) = \sigma(y, t) - t^q\sigma_0(t) - t^q\sigma_1(t)\cdot y. \tag{7.5}$$

Thus we see that (7.2) is equivalent with the validity, for all (y, t) in the neighborhood U of the origin, of an estimate :

$$\left|\int_0^t |\mathrm{Im}\,\sqrt{t'^q(\sigma_0(t') + \sigma_1(t')\cdot y)}\ |dt'\right| \leqq \text{const.}\ (|y| + |t|^{p+1}), \tag{7.6}$$

where we have used the notation :

$$k = 2p + 1 \tag{7.7}$$

The following lemma is then almost evident :

LEMMA 7.2. – *For (7.2) to hold, it is necessary and sufficient, either that*

$$q \geqq 2p \tag{7.8}$$

or else that the following condition be satisfied :

$$t^q \sigma_0(t) = c_q t^q + c_{q+1} + \ldots, \tag{7.9}$$

where $q = 2r$ *is an even integer* $< 2p$, c_q *is* > 0 *and all the* $c_j (q < j < p + r)$ *are real.*

Proof. – We suppose that (7.8) does not hold. Then, for $t \sim 0$, $t^q \sigma_0(t) \sim c_q t^q$. We apply (7.6) with $y = 0$ and conclude at once that we must have q even and $c_q > 0$. Then note that

$$\sqrt{t^q \sigma_0(t)} = c_q^{1/2} t^{q/2} \left(1 + \frac{1}{2} f(t) + \ldots\right), \tag{7.10}$$

where $f(t) = c_q^{-1}(c_{q+1} t + \ldots)$. We must have

$$|\mathrm{Im} \sqrt{t^q \sigma_0(t)}| \leqq \text{const. } |t|^p, \tag{7.11}$$

hence $c_j t^{j-r}$ $(q < j)$ must either be real (for t real) or else we must have $j - r \geqq p$. This proves that (7.2) implies (7.9).

The proof of the sufficiency of our condition is essentially a reversal of the preceding reasoning and is left to the reader.

COROLLARY 7.1. – *The lower-order terms in* (1.1) *have a strong influence at the origin if and only if one of the mutually exclusive following condition holds :*

$$t^q \sigma_0(t) = c_q t^q + \ldots, \; q < 2p, \; c_q \neq 0, \tag{7.12}$$

and $c_q t^q$ *is not everywhere positive in any neighborhood of the origin ;*

$$t^q \sigma_0(t) = c_q t^q + \ldots, \ q = 2r < 2p, \ c_q > 0, \tag{7.13}$$

and c_j $(q < j < p + r)$ *not all real.*

8. Situations in which the lower-order terms have a strong influence : perturbation of the phase-function.

Our purpose is to find a suitable approximate solution of the equation (7.1) (cf. first term in the right-hand side of (6.32)). In this section and in the next we shall be working under the assumption that *the lower-order terms in* (1.1) *have a strong influence* (at the origin), which means that one of the conditions in Cor. 7.1 holds. In this case we are going to choose

$$\varphi = e^{i\tau w_1} \psi, \tag{8.1}$$

where we have written $\tau = \rho^{1/2}$, a notation systematically used from now on. We shall require that w_1 satisfy in a suitable open subset of $\mathbf{R}^{n+1}$ (to be determined later).

$$(\mathcal{L} w_1)^2 = - \sigma. \tag{8.2}$$

Recalling that the order of $\mathcal{Q}_j$ is $\leqq j + 2$, Equation (7.1) translates into

$$(\mathcal{L}^2 + \mathcal{M})\psi + \tau \{2(\mathcal{L} w_1) \mathcal{L} + \mathcal{L}^2 w_1 + \mathcal{M}_1 w\} \psi = \sum_{j=1}^{J'-2} \sum_{j'=0}^{j+2} \tau^{2-j-j'} \mathcal{R}_{j,j'} \psi, \tag{8.3}$$

where $\mathcal{M}_1$ denotes the principal part of the first-order operator $\mathcal{M}$ and $\mathcal{R}_{j,j'}$ denotes a differential operator of order $\leqq j'$, whose coefficients are polynomials in y, t and in the derivatives of order $\leqq j'$ of grad w_1.

It is convenient to rewrite (8.3), ordering the terms according to the powers of τ :

$$\{2(\mathcal{L} w_1) \mathcal{L} + \gamma\} \psi = \sum_{l=1}^{2J'-3} \tau^{-l} \widetilde{\mathcal{Q}}_l \psi. \tag{8.4}$$

where γ and the differential operators $\widetilde{\mathcal{P}}_l$ (of order $\leqq l + 1$ respectively) depend on the derivatives of w_1 (or order $\geqq 1$ but not exceeding 2 and $l + 2$ respectively). We shall choose

$$\psi = g \sum_{\nu=0}^{J'''} \tau^{-\nu} \psi_\nu, \tag{8.5}$$

where J''' is a large positive integer, and the g and ψ_ν suitable C^∞ functions to be chosen. The ψ_ν will be independent of τ, but not so g : the latter will have compact support, contained in a certain neighborhood U_τ of a point $(0, t_\tau)$ which will converge to the origin as $\tau \to +\infty$; moreover, g will be equal to one in a subneighborhood V_τ of $(0, t_\tau)$. All this is going to be described precisely below. For the moment let us say that the support of g will be contained in an open set where the problem (8.2) has a smooth solution (there might not be, of course, any such set which contains the origin). If we continue to reason formally, we may say that the ψ_ν are solutions of the following equations :

$$\{2(\mathcal{L}w_1)\mathcal{L} + \gamma\}\psi_\nu = \sum_{\nu'<\nu} \widetilde{\mathcal{P}}_{\nu-\nu'}\psi_{\nu'} \tag{8.6}_\nu$$

with the agreement that $\psi_\nu \equiv 0$ if $\nu < 0$. If this is satisfied, we shall be able to write

$$\left(\mathcal{L}^2 + \mathfrak{M} - \rho\sigma - \sum_{j=1}^{J'-2} \rho^{-j}\mathcal{Q}_j\right)\varphi = \tag{8.7}$$

$$= e^{i\tau w_1}\left[g \sum_{\substack{\nu=0 \\ \nu+\nu'>J'''}}^{J'''} \sum_{\nu'=1}^{2J'-2} \tau^{-\nu-\nu'}\widetilde{\mathcal{P}}_{\nu'}\psi_\nu + \mathcal{L}_g\left(\sum_{\nu=0}^{J'''} \tau^{-\nu}\psi_\nu\right)\right]$$

where

$$\mathcal{L}_g = -\sum_{l=1}^{2J'-3} \sum_{\substack{|\alpha+\beta|=k_l \\ |\alpha|\geqq 1}} c_{l,\alpha}\tau^{-l}(Dg^\alpha)D^\beta + 2(\mathcal{L}w_1)\mathcal{L}g, \tag{8.8}$$

$k_l \leqq l + 1$ being the order of $\widetilde{\mathcal{P}}_l$.

We return now to (8.2) and show how to solve it. Application of the Cauchy-Kowalevskaya theorem will not be good enough for our purposes. Instead, we proceed as follows : because σ and the coefficients of $\mathcal{L}$ are analytic, and since we are looking for analytic solutions, we may consider (y, t) as complex variables, in some neighborhood $U^{\mathbf{C}}$ of the origin in $\mathbf{C}^{n+1}$, $U^{\mathbf{C}} \cap \mathbf{R}^{n+1} = U$, and perform a holomorphic change of coordinates (y, t) into (z, s), $s = t$, such that, in the new variables, the differential operator $\mathcal{L}$ becomes D_s. Observe that U becomes a piece of smooth $(n + 1)$-dimensional real manifold, which we call $U^{\natural}$. Let

$$U^+ = U \cap \{t > 0\} \quad \text{and} \quad U^- = U \cap \{t < 0\}. \tag{8.9}$$

By shrinking the (connected) neighborhood U, we may assume that in $U^+ \cup U^-$:

$$0 < |\sigma(y, t) - t^q \sigma_0(t)| \leqq 1/2 \; |t^q \sigma_0(t)| \tag{8.10}$$

and

$$|\sigma_1(t) \cdot y| \leqq 1/2 \; |\sigma_0(t)|. \tag{8.11}$$

Note that (8.10) assures that $\sigma(y, t)$ nowhere vanishes in $U^+ \cup U^-$. By the preceding holomorphism, $U^\pm$ are transformed into "simply connected" open subsets of $U^{\natural}$, which we call $U^{\pm\natural}$. In $U^{\pm\natural}$ we can trivially solve

$$w_{1s} = \pm \sqrt{\sigma(y(z, s), s)} \tag{8.12}$$

which is the transform of (8.2). According to Cor. 7.1, we have two cases to consider :

I) Assume that (7.12) holds. This together with (8.10) implies that $c_q t^q$ is everywhere negative in at least one of the two regions $U^\pm$. Without loss in generality we may suppose that

$$c_q t^q < 0 \;\; in \;\; U^+. \tag{8.13}$$

Then (8.11) yields :

$$\operatorname{Im} \sqrt{\sigma(y, t)} = c_0 t^{q/2} (1 + O(t)), \; c_0 \neq 0, \; in \; U^+. \tag{8.14}$$

II) Assume that (7.13) holds. Let $\operatorname{Im} t^q \sigma_0(t) = c_{j_0} t^{j_0}(1 + O(t))$, $c_{j_0} \neq 0$, $q < j_0 < p + r$. From (8.11) we get :

$$\operatorname{Im} \sqrt{\sigma(y, t)} = c_0 t^{j_0 - r} (1 + O(t)),\ c_0 \neq 0,\ in\ U^+. \tag{8.15}$$

From now on we set

$$l = \begin{cases} q \text{ if (7.12) holds} \\ 2(j_0 - r) \text{ if (7.13) holds.} \end{cases} \tag{8.16}$$

It is then true (see also (7.7)) that

$$\operatorname{Im} \sqrt{\sigma(y, t)} = c_0 t^{l/2} (1 + O(t))\ in\ U^+,\ c_0 \neq 0,\ l + 1 < k. \tag{8.17}$$

Next, we define in U^+, the function

$$W = w + (1/\tau) w_1. \tag{8.18}$$

Note that (8.2) implies :

$$w_{1t} = \pm \sqrt{\sigma(y, t)} + it^k \sum_{j=1}^{n} \beta_{\eta_j}(y, t, w_y) w_{1y^j}.$$

Set $\mathfrak{I} = \operatorname{Im} w_1$. It then follows that

$$\mathfrak{I}_t = \pm \operatorname{Im} \sqrt{\sigma(y, t)} + t^k \sum_{j=1}^{n} \beta_{\eta_j}(0, t, w_y(0, t)) w_{1y^j} + O(|y|).$$

Because of (8.17), we have

$$\mathfrak{I}_t = -|c_0| t^{l/2} (1 + O(t)) + O(|y|)\ in\ U^+, \tag{8.19}$$

where we choose the sign in the right-hand side so as to have $\mathfrak{I}_t \leqq 0$ for $y = 0$ and $t > 0$. We may require $w_1|_{t=0} = 0$, whence

$$\mathfrak{I} = -|c_0| \frac{t^{1+l/2}}{1 + l/2} (1 + O(t)) + t|y| O(1).$$

Let us set $h(\rho, t) = \rho \operatorname{Im} W|_{y=0}$. It follows at once from (6.15) and (6.17) that

$$h(\rho, t) = \rho b_0 \frac{t^{k+1}}{k+1} (1 + O(t)) - \rho^{\frac{1}{2}} |c_0| \frac{t^{1+l/2}}{1 + l/2} (1 + O(t)), \tag{8.20}$$

and also that

$$\left|\rho \operatorname{Im} W - \left\{ \frac{1}{2} \rho\, |y|^2 + h(\rho\,,t) \right\}\right| \leqslant$$

$$\text{const. } (\rho\,\{B(t)^2 + |y|\, B(t) + |y|^2\,(|y| + |t|)\} + \sqrt{\rho}\,|y|\,|t|\,. \quad (8.21)$$

From this and from (6.17) we derive that, if $|t| < t_2$ sufficiently small, there is a constant $K_0 > 0$, independent of ρ, such that

$$\left|\rho \operatorname{Im} W - \left\{ \frac{1}{2} \rho\, |y|^2 + h(\rho\,,t) \right\}\right| \leqslant \frac{1}{4} \rho |y|^2 + K_0(\rho B(t)^2 + t^2). \quad (8.22)$$

Let us set

$$\delta = \frac{k - l - 1}{2k - l}\,, \quad \epsilon = \frac{1}{2k - l}\,. \quad (8.23)$$

By (8.17) we have $0 < \delta < 1$, $0 < \epsilon < 1$. It is convenient to change variables and introduce $s = \rho^\epsilon t$. We may then write

$$h(\rho\,,t) = \rho^\delta\, h_1(\rho\,,s)\,,$$

where

$$h_1(\rho\,,s) = b_0 \frac{s^{k+1}}{k+1}(1 + O(s\rho^{-\epsilon})) - |c_0|\frac{s^{1+l/2}}{1 + l/2}(1 + O(s\rho^{-\epsilon})). \quad (8.24)$$

The function h_1 has a *minimum* for s equal to

$$s_m(\rho) = c_1 + o(1), \quad (8.25)$$

where $c_1 = |c_0/b_0|^{2\epsilon} > 0$ and $o(1)$, tends to zero with $1/\rho$. The value of h_1 at the minimum is equal to

$$-\, h_0(\rho) = -\, c_2 + o(1), \quad (8.26)$$

where $c_2 > 0$ is independent of ρ. Suppose now that s remains in a small interval

$$c_1 - \eta \leqslant s \leqslant c_1 + \eta \quad (0 < \eta < c_1/2)\,. \quad (8.27)$$

Then $|t| \leqslant 2\rho^{-\epsilon} c_1$ and $t^{2k} \leqslant 2c_1\rho^{-1}$. Since $B(t)^2$ is of the order of $t^{2(k+1)}$, we derive from (8.22) that, if (8.27) holds, we have

$$\left|\rho \operatorname{Im} W - \left\{ \frac{1}{2} \rho |y|^2 + h(\rho\,,t) \right\}\right| \leqslant \frac{1}{4} \rho |y|^2 + K\,t^2\,. \quad (8.28)$$

We sketch now, for use below, the graph of the function of $s \geqq 0$,

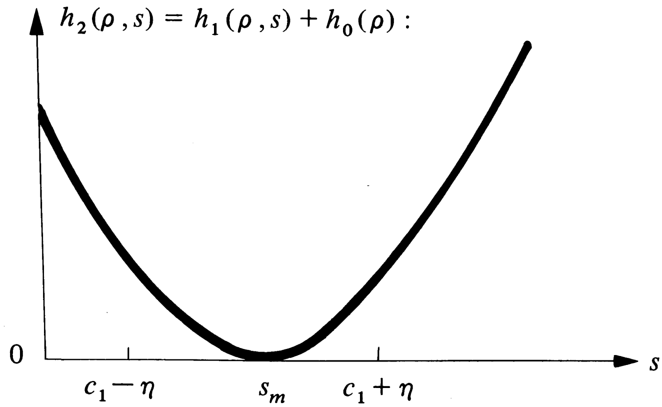

Fig. 8.01

We begin now the determination of the amplitude function ψ and, first of all, of the cut-off function g. We take

$$g(y, t) = g_0(\rho^{\epsilon} t)\, g_1(y) \ ;$$

g_1 is a C^∞ function defined in $\mathbf{R}^n$, with compact support contained in a sufficiently small neighborhood of the origin and equal to one in a subneighborhood of the origin. As for $g_0(s)$, it is also a C^∞ function on the real line ; but its support is contained in the interval (8.27). Moreover, $g_0(s)$ must be equal to *one* in the interval $\left(c_1 - \frac{\eta}{2}, c_1 + \frac{\eta}{2}\right)$ (see Fig. 8.01). Clearly, for ρ sufficiently large, $s_m(\rho)$ will be an interior point of this latter interval.

Next we shall study the terms ψ_ν in the expansion of ψ, by looking at the equations $(8.6)_\nu$ $(\nu = 0, 1, \ldots)$ which they satisfy. Once again, it is convenient to change variables (see page 266), (y, t) into (z, s), $s = t$, so that $\mathcal{L}$ becomes D_s. We shall reason in $U^{+\natural}$. All this, enables us to rewrite Eq. $(8.6)_\nu$ as follows :

$$-2w^{\natural}_{1s}\,\psi^{\natural}_{\nu s} + \gamma^{\natural}\,\psi^{\natural}_{\nu} = \sum_{\nu' - \nu} \widetilde{\mathcal{P}}^{\natural}_{\nu - \nu'}\,\psi^{\natural}_{\nu'} \qquad (8.29)_\nu$$

where $\gamma^{\natural}$ and the coefficients of the differential operator $\widetilde{\mathcal{P}}^{\natural}_{\nu-\nu'}$ (of order $\leqq \nu - \nu' + 1$) are polynomials in z, s and in the derivatives of grad $w_1^{\natural}$ of order $\leqq 1$ and $\leqq \nu - \nu' + 1$ respectively, hence holomorphic functions of (z, s) in $U^{+\natural}$. From $(8.29)_\nu$, we obtain :

$$\psi_{\nu s} + \lambda(z, s)\, \psi_\nu = \frac{1}{-2w_{1s}} \sum_{\nu'<\nu} \widetilde{\mathcal{P}}^{\natural}_{\nu-\nu'}\, \psi^{\natural}_{\nu'} \tag{8.30}$$

where $\lambda(z, s) = -\dfrac{\gamma^{\natural}}{2w^{\natural}_{1s}}$ is a holomorphic function of (z, s) in $U^{+\natural}$.
An easy computations shows that

$$\lambda(z, s) = q/4s + \lambda_0(z, s) \tag{8.31}$$

where $\lambda_0(z, s) = O(1)$.

We have used the fact that (see (8.3) and (8.12))

$$\gamma^{\natural} = -\, w^{\natural}_{1ss} + O(s^{q/2}) \tag{8.32}$$

and

$$w^{\natural}_{1s} = O(s^{q/2}). \tag{8.33}$$

Set

$$\mu(z, s) = \int_0^s \lambda_0(z, s')\, ds' \tag{8.34}$$

and

$$\chi_\nu = s^{q/4}\, e^{\mu(z,s)}\, \psi^{\natural}_\nu. \tag{8.35}$$

We derive from $(8.30)_\nu$:

$$\chi_{\nu s} = -\frac{1}{2w^{\natural}_{1s}} \sum_{\nu'<\nu} s^{q/4} e^{\mu(z,s)} \widetilde{\mathcal{P}}^{\natural}_{\nu-\nu'} (s^{-q/4} e^{-\mu(z,s)} \chi_{\nu'}). \qquad (8.36)_\nu$$

We solve $(8.36)_0$ by taking

$$\chi_0 \equiv 1 \tag{8.37}$$

and then recursively as ν increases, we obtain all χ_ν as holomorphic functions of (z, s) in $U^{+\natural}$ (recalling that $s = 0$ does not intersect $U^{+\natural}$).

We shall need the following :

LEMMA 8.1. – *For all pairs p_0, p_1 (p_0 an integer $\geqq 0$, p_1 an n-tuple) and for all z, s in* U^+ *(after shrinking U, if necessary) we obtain :*

$$|(\partial/\partial s)^{p_0}(\partial/\partial z)^{p_1}\chi_\nu(z\,,\,s)| \leqq C_{p_0,p_1}\,|s^{-p_0-\nu(1+q/2)}|. \qquad (8.38)_\nu$$

Proof. – First we observe that $(8.38)_0$ is true for all p_0, p_1. We then fix p_0 and $\underset{\sim}{p_1}$ and prove the lemma by induction on ν. The fact that the order of $\mathfrak{P}_{\nu-\nu'}$ is $\leqq \nu-\nu'+1$ and a combination of (8.31), (8.33), (8.34), $(8.36)_\nu$ and $(8.38)_{\nu'}$ with $\nu'<\nu$, yield :

$$(\partial/\partial s)\chi_\nu \;=\; O(s^{-1-\nu(1+q/2)}). \qquad (8.39)$$

This in turn implies $(8.38)_\nu$. Q.E.D.

If we revert to the functions ψ_ν and use the fact (see (8.16)) that $q \leqq l$, we obtain at once

$$\left|\left(\frac{\partial}{\partial s}\right)^{p_0}\left(\frac{\partial}{\partial z}\right)^{p_1}\psi_\nu(z\,,\,s)\right| \leqq C'_{p_0,p_1}\;|s^{-p_0-\nu(1+l/2)-l/4}|. \qquad (8.40)$$

Finally, it is easy to revert to the coordinates y, t (from now on and throughout Section 9, y and t will be *real*). We obtain at last :

$$\left|\left(\frac{\partial}{\partial t}\right)^{p_0}\left(\frac{\partial}{\partial y}\right)^{p_1}\psi_\nu(y\,,\,t)\right| \leqq C''_{p_0,p_1}\;t^{-p_0-\nu(1+l/2)-l/4} \;\; \textit{in } U^+. \qquad (8.41)$$

We are going to need a similar estimate for the derivatives of w_1. Notice that (8.12) implies (taking into account that $q \leqq l$) :

$$\left|\left(\frac{\partial}{\partial s}\right)^{p_0}\left(\frac{\partial}{\partial z}\right)^{p_1} w_1(y\,,t)\right| \leqq C'''_{p_0,p_1}\;t^{1-p_0} \;\; \textit{in } U^+. \qquad (8.42)$$

9. End of the proof of theorem 1.1 when the lower-order terms have a strong influence.

We are now in a position to show that if one of the conditions in Cor. 7.1 holds, an inequality such as (6.32) cannot be valid. As we have said, we choose

$$v = g e^{i\rho W} v_0 \tag{9.1}$$

where

$$g(y, t) = g_0(\rho^\epsilon t)\, g_1(y), \quad W = w + (1/\tau)\, w_1\,, \quad v_0 = \sum_{\nu=0}^{J'''} \tau^{-\nu} \psi_\nu .$$

We apply (8.7) and try to determine an upper bound for the *first* term in the right-hand side of (6.32). We recall that, on the support of $g(y, t)$, we have

$$\frac{1}{2}\, c_1 \rho^{-\epsilon} \leqq t \leqq \frac{3}{2}\, c_1 \rho^{-\epsilon}. \tag{9.2}$$

Let p_0, p_1 be any pair such that $p_0 + |p_1| \leqq M$. It is not difficult to see, by applying (8.41) and (8.42), that, on supp g,

$$\left(\frac{\partial}{\partial t}\right)^{p_0} \left(\frac{\partial}{\partial y}\right)^{p_1} \Big(g e^{i\rho W} \sum_{\substack{\nu=0 \\ \nu+\nu'>J'''}}^{J'''} \sum_{\nu'=1}^{2J'-3} \tau^{-\nu-\nu'} \widetilde{\mathfrak{L}}_{\nu'} \psi_\nu\Big) \leqq \tag{9.3}$$

$$\leqq C_M\, \rho^{M - J'''/2 + \epsilon J'''(1+l/2) + 2\epsilon(J'-1) + \epsilon l/4}\, e^{-\operatorname{Im}(\rho W)}$$

We also have, by (8.28) :

$$-\operatorname{Im}(\rho W) \leqq Kt^2 + \rho^\delta h_0(\rho),$$

so that (9.3) has the following consequence :

$$\left(\frac{\partial}{\partial t}\right)^{p_0} \left(\frac{\partial}{\partial y}\right)^{p_1} \Big(g e^{i\rho W} \sum_{\substack{\nu=0 \\ \nu+\nu'>J'''}}^{J'''} \sum_{\nu'=1}^{2J'-3} \tau^{-\nu-\nu'} \widetilde{\mathfrak{L}}_{\nu'} \psi_\nu\Big) \leqq \tag{9.4}$$

$$\leqq C'_M\, \rho^{M+J'-J'''\delta}\, e^{\rho^\delta h_0(\rho)}.$$

We have used the following consequences of (8.23) :

$$\epsilon(2 + l/4) \leqq 1, \quad \frac{1}{2} - \epsilon\left(1 + \frac{l}{2}\right) = \delta.$$

On the other hand, again by (8.41) and (8.42) (see also (8.8)) :

$$\left|\left(\frac{\partial}{\partial t}\right)^{p_0}\left(\frac{\partial}{\partial y}\right)^{p_1}(e^{i\rho W}\mathcal{L}_g v_0)\right| \leqq C_M \rho^{M+1}\sum_{\nu=0}^{J'''}\rho^{-\nu\delta}e^{-\mathrm{Im}(\rho W)}\,. \tag{9.5}$$

Since $\mathcal{L}_g$ vanishes identically in any open set where $g \equiv 1$, this is to be considered only *on the support of the gradient of* g, where we have, by virtue of (8.22)

$$\rho\,\mathrm{Im}\,W \geqq c\rho\,|y|^2 + h(\rho, t) + \rho^\delta h_0(\rho) - \rho^\delta h_0(\rho) - Kt^2$$
$$\geqq c_3\rho^\delta - \rho^\delta h_0(\rho) - Kt^2\,.$$

If we combine this with (9.5) we obtain :

$$\left|\left(\frac{\partial}{\partial t}\right)^{p_0}\left(\frac{\partial}{\partial y}\right)^{p_1}(e^{i\rho W}\mathcal{L}_g v_0)\right| \leqq C'_M \rho^{M+1} e^{-c_3\rho^\delta+\rho^\delta h_0(\rho)}\,. \tag{9.6}$$

Clearly, for ρ sufficiently large, we will have $\exp(-c_3\rho^\delta) \leqq \rho^{-J'''\delta}$. We reach the conclusion that the first term, in the right-hand side of (6.32), does not exceed a constant times

$$\rho^{m+M+J'-2-J'''\delta}e^{\rho^\delta h_0(\rho)}\,. \tag{9.7}$$

We seek now an upper bound for the second term in the right-hand side of (6.32). Here the argument is somewhat subtler, due to the fact that we have no control on the order of differentiation M'.

Recalling that $\phi = \psi\exp(i\rho^{\frac{1}{2}}w_1)$ by (8.1) and setting $\widetilde{W} = w_2 + \rho^{-\frac{1}{2}}w_1$ (cf. (8.18)), we see that the second term in question does not exceed a constant times

$$\rho^{-J}\sup_{\mathrm{supp}\,g}\,(1+\rho^{\frac{1}{2}}|\partial\widetilde{W}|)^{M'}\,e^{-\rho\,\mathrm{Im}\widetilde{W}}. \tag{9.8}$$

(To check this it suffices to estimate

$$\sum_{|\alpha|\leqslant M'}\rho^{-|\alpha|/2}\{|e^{i\rho w_2}D^\alpha\phi| + |D^\alpha(e^{i\rho w_2}\phi)|\} \leqslant$$
$$\text{const.}\sum_{|\alpha|\leqslant M'}\rho^{-|\alpha|/2}\{|e^{i\rho w_2}D^\alpha(e^{i\rho^{\frac{1}{2}}w_1})| + |D^\alpha(e^{i\rho\widetilde{W}})|\}\,.)$$

In order to estimate (9.8) we observe that

$$(1+\rho^{\frac{1}{2}}\,|\partial(\mathrm{Re}\,\widetilde{W})|) \leqslant \text{const.}\,(1+\rho^{\frac{1}{2}}\,|\partial(\mathrm{Re}\,w_2)|), \tag{9.9}$$

$$(1 + \rho^{\frac{1}{2}} |\partial(\operatorname{Im}\widetilde{W})|) \leqslant \text{const.}\,(1 + \rho^{\frac{1}{2}}\{ |\partial_y(\operatorname{Im} w_2)| + |y|\} + \rho^{\frac{1}{2}} |a_t(t, \rho)|)\,, \tag{9.10}$$

where $a(t, \rho) = \operatorname{Im}\widetilde{W}|_{y=0} + \rho^{\delta-1} h_0(\rho)$. It is a consequence of (6.19)-(6.20) that $|\partial w_2| \leqslant$ const. $(|y| + |t|^k)$. On the other hand, by (9.2) we have, on supp g,

$$|t|^k \leqslant \text{const.}\ \rho^{-\epsilon k} \leqslant \text{const.}\ \rho^{-1/2} \quad \text{(cf. (8.23)).}$$

Applying (8.28) we conclude that

$$\rho^{\frac{1}{2}}(|\partial(\operatorname{Re} w_2)| + |\partial_y(\operatorname{Im} w_2)| + |y|) \leqslant \text{const.}\ \{1 + \rho \operatorname{Im}\widetilde{W} + \rho^\delta h_0(\rho)\}^{1/2}. \tag{9.11}$$

In estimating $\rho^{\frac{1}{2}} |a_t(t, \rho)|$, it is convenient to make the change of variables $s = \rho^\epsilon t$. Set $a^\#(s, \rho) = \rho^{1-\delta} a(t, \rho)$ (ϵ and δ are defined in (8.23)) ; note also that $a^\#(s, \rho) = a^\#(s, 0) + 0(\rho^{-\epsilon})$. Since $a^\#(s, \rho)$ remains $\geqslant 0$ in a fixed neighborhood of the point $s = s_m(\rho)$, such as (8.27), there is a constant $C > 0$, independent of ρ, such that, in this neighborhood,

$$|a_s^\#(s, \rho)| \leqslant C\ a^\#(s, \rho)^{1/2}.$$

Reverting to the variable t, this implies (when t remains in a set such as (9.2))

$$|a_t(t, \rho)| \leqslant C\ \rho^{\epsilon - \frac{1-\delta}{2}}\ a(t, \rho)^{1/2}.$$

But $\epsilon \leqslant (1 - \delta)/2$ (for $k \geqslant 1$) and thus $|a_t| \leqslant C\sqrt{a}$. In the notation of (8.28), $a = \rho^{-1}(h(\rho, t) + \rho^\delta h_0(\rho))$, and therefore

$$\rho^{\frac{1}{2}} |a_t(t, \rho)| \leqslant \text{const.}\{1 + \rho \operatorname{Im} W + \rho^\delta h_0(\rho)\}^{1/2}. \tag{9.12}$$

Finally, if we combine (9.11) and (9.12), we see that

the second term in the right-hand side of (6.32) *does not exceed a constant times* $\rho^{-J} \exp(\rho^\delta h_0(\rho))$. (9.13)

We seek now an upper bound for the third term in the right-hand side of (6.32). We observe that (in supp g)

$$\sum_{|\alpha|\leqq M} |D^\alpha(e^{i\rho w}H)| \leqq \text{const. } \rho^M \sum_{|\alpha|\leqq M} \rho^{-|\alpha|} |e^{i\rho w_2} D^\alpha H|. \quad (9.14)$$

We apply (9.14) to $H = D^\beta\varphi$. We see thus that the third term in question does not exceed a constant times

$$\rho^{m+M} \sup_{\text{supp}\, g} \left\{ (|y| + |t|)^{J''} e^{-\rho \operatorname{Im} w_2} \sum_{|\alpha|\leqq M+J'} \rho^{-|\alpha|} |D^\alpha \varphi| \right\} \quad (9.15)$$

If we combine this with (8.1), (8.5), (8.41) and (8.42), we see that (9.15) is bounded by a constant times

$$\rho^{m+M+1} \sup_{\text{supp}\, g} \{(|y| + |t|)^{J''} e^{-\rho \operatorname{Im} W}\}. \quad (9.16)$$

From (8.28) we derive that, on supp g,

$$-\operatorname{Im} \rho W \leqq -c\rho |y|^2 + \rho^\delta h_0(\rho) + Kt^2 \cdot \quad (9.17)$$

On the other hand, on supp g (since $\epsilon \leqq 1/2$) :

$$\begin{aligned} (|y| + |t|)^{J''} e^{-c\rho|y|^2} &\leqq \text{const. } \rho^{-\epsilon J''}(1 + \rho^\epsilon |y|)^{J''} e^{-c\rho|y|^2} \\ &\leqq \text{const. } \rho^{-\epsilon J''}. \end{aligned} \quad (9.18)$$

If we take (9.17) and (9.18) into account, it follows that (9.16) is bounded by a constant times

$$\rho^{m+M+1-\epsilon J''} e^{\rho^\delta h_0(\rho)}. \quad (9.19)$$

Therefore, requiring that

$$J'''\delta \geqq m + M + J' + J - 3 \text{ and } \epsilon J'' \geqq m + M + J$$

we derive from (9.7), (9.13) and (9.19) that the right-hand side of (6.32) is bounded by a constant times

$$\rho^{-(J-1)} e^{\rho^\delta h_0(\rho)}. \quad (9.20)$$

Next we consider the left-hand side in (6.32). We have

$$\iint v f dy dt = \iint e^{i\rho W} v_0 f g dy dt.$$

We shall choose

$$f(y, t) = F(\rho y, \rho t - \rho^{1-\epsilon} s_m), \; F \in C_c^\infty(\mathbf{R}^{n+1})$$

(for the meaning of s_m see Fig. 8-01 and (8.25)). There is a constant $R > 0$ such that $f(y, t) = 0$ unless

$$|y| \leqq R/\rho, \ |t - \rho^{-\epsilon} s_m| \leqq R/\rho.$$

Therefore, if ρ is sufficiently large, the support of f will lie in as small a neighborhood of the origin as we wish to choose. We make then the change of variables

$$y = \rho^{-1} \tilde{y}, \ t = \rho^{-\epsilon} s_m + \rho^{-1} s.$$

We see that

$$\iint vfdydt = \rho^{-2} \iint e^{i\rho\tilde{W}} \tilde{v}_0 F(\tilde{y}, s)\, g_0(s_m + \rho^{\epsilon-1} s)\, g_1(\rho^{-1}\tilde{y})\, d\tilde{y}\, ds$$

where

$$\tilde{W}(\tilde{y}, s, \rho) = W(\rho^{-1}\tilde{y}, \rho^{-\epsilon} s_m + \rho^{-1} s, \rho),$$

$$\tilde{v}_0 = \sum_{\nu=0}^{J'''} \psi_\nu(\rho^{-1}\tilde{y}, \rho^{-\epsilon} s_m + \rho^{-1} s)\, \rho^{-\nu/2}.$$

We have

$$|\operatorname{Im}(\rho\tilde{W}) + \rho^\delta h_0(\rho)| \leqslant \frac{c}{\rho} |\tilde{y}|^2 + K_1' \rho^{-2\epsilon}$$
$$+ \{h(\rho, \rho^{-\epsilon} s_m + \rho^{-1} s) + \rho^\delta h_0(\rho)\},$$

(cf. (8.28)). We may write with the notation of Section 8,

$$h(\rho, \rho^{-\epsilon} s_m + \rho^{-1} s) + \rho^\delta h_0(\rho) = \rho^\delta \{h_1(\rho, s_m + \rho^{\epsilon-1} s) - \rho^\delta h_0(\rho)\}.$$

Since the right-hand side has a minimum at $s = 0$, its value does not exceed a constant times $\rho^{\delta - 2(1-\epsilon)} \leqslant \rho^{-2\epsilon}$. Thus we obtain :

$$|\operatorname{Im}(\rho\tilde{W}) + \rho^\delta h_0(\rho)| \leqslant \text{const. } \rho^{-2\epsilon}. \tag{9.21}$$

Next we look at

$$\operatorname{Re}(\rho\tilde{W}) = \rho \operatorname{Re} \tilde{w} + \operatorname{Re} \tau \tilde{w}_1$$
$$= \langle \eta^2, \tilde{y} \rangle + O(|\tilde{y}|\, B(\rho^{-\epsilon} s_m + \tau \mathcal{R}(\rho^{-1}\tilde{y}, \rho^{-\epsilon} s_m + \rho^{-1} s),$$

where $\mathcal{R}$ denotes the real part of w_1.

We have

$$|\mathcal{R}(\rho^{-1}\tilde{y}, \rho^{-\epsilon}s_m + \rho^{-1}s) - \mathcal{R}(0, \rho^{-\epsilon}s_m)| \leqq \text{const. } 1/\rho,$$

therefore :

$$|\mathrm{Re}(\rho\tilde{\mathrm{W}}) - < \eta^2, \tilde{y} > - \tau\mathcal{R}(0, \rho^{-\epsilon}s_m)| \leqq \text{const. } \rho^{-\epsilon}. \tag{9.22}$$

At last we study $\tilde{v}_0$. We return to the expression (8.35) of the ψ_ν in terms of the χ_ν (keeping in mind, however, that the variable denoted by s on pages 269, 271, is what we denote here by t : it is different from what we are denoting presently by s). We may write

$$\psi_\nu(y, t) = t^{-q/4} e^{\tilde{\mu}(y, t)} \chi_\nu(y, t),$$

where $\tilde{\mu}(y, t) = -\mu(z(y, t), t)$ (cf. (8.34)). Thus :

$$v_0(y, t) = t^{-q/4} e^{\tilde{\mu}(y, t)} \sum_{\nu=0}^{J'''} \chi_\nu(y, t)\rho^{-\nu/2}.$$

Now

$$t^q = (\rho^{-\epsilon}s_m + \rho^{-1}s)^q = \rho^{-\epsilon q}s_m^q(1 + O(\rho^{-\epsilon})).$$

On the other hand, $\psi_0 \equiv 1$, and if $\nu > 0$, we derive from $(8.38)_\nu$ when $p_0 = p_1 = 0$,

$$|\chi_\nu| \leqq \text{const. } \rho^{\epsilon\nu(1+l/2)},$$

whence

$$|\chi_\nu\rho^{-\nu/2}| \leqq \text{const. } \rho^{-\nu\delta}. \tag{9.24}$$

Since by (8.34) it is clear that

$$\tilde{\mu}(\rho^{-1}\tilde{y}, \rho^{-\epsilon}s_m + \rho^{-1}s) \to 0$$

as $\rho \to +\infty$, we derive from (9.23) and (9.24) that, when $\rho \to +\infty$,

$$\rho^{-\epsilon q/4}\tilde{v}_0 \to \gamma = \{s_m^q\}^{-1/4} \neq 0.$$

Finally, if we combine this fact with (9.21) and (9.22) we reach the conclusion that, when $\rho \to +\infty$,

$$\rho^{2-\epsilon q/4} e^{-\rho^\delta h_0(\rho) - i\tau\mathcal{R}(0, \rho^{-\epsilon}s_m)} \iint vf dy dt$$

converges to

$$\gamma \iint e^{i<\eta^2,\tilde{y}>} F(\tilde{y}, s)\, d\tilde{y}\, ds = \hat{F}(-\eta^2, 0)\gamma.$$

Here $\hat{F}$ is the Fourier transform of F and we may choose F so as to have, say,

$$\hat{F}(-\eta^2, 0) = 2/\gamma.$$

Consequently, for ρ sufficiently large, we will have

$$|\iint vfdydt| \geqq \rho^{-2} e^{\rho^{\delta} h_0(\rho)}. \tag{9.25}$$

Observing that

$$\sup_{\mathbf{R}^{n+1}} \sum_{|\alpha| \leqq M} |D^{\alpha} f| \leqq \text{const. } \rho^{M}, \tag{9.26}$$

we combine (9.20), (9.25) and (9.26). If we choose $J > M + 3$ and let ρ go to $+\infty$, we reach a conclusion which contradicts (6.32).

10. Situations in which the lower-order terms have little influence : determination of the amplitude function.

In this and in the next two sections, we shall reason under Hypothesis (7.2). As before, we take (6.32) as our starting point and try to find a suitable approximate solution of the equation (7.1). We are going to take

$$\varphi = \zeta h, \tag{10.1}$$

with $\zeta \in C^{\infty}(V)$ (V is the open neighborhood of the origin entering in (5.15)) equal to one in a subneighborhood of the origin, and where h is an analytic function of (y, t) near $(0, 0)$. The choice of h will constitute the essential step in the rest of the proof of Th. 1.1. If we make the substitution (10.1) in (6.42), and if we apply Leibniz formula to the product ζh, we see that each time the cut-off function ζ is differentiated, we get a function which vanishes identically in some neighborhood of the origin. We reach the following conclusion :

$$|\iint fv\,dydt|/\sup \sum_{|\alpha|\leq M} |D^\alpha f| \leq \tag{10.2}$$

$$C\rho^{m-2} \sup_{V_\zeta} \sum_{|\alpha|\leq M} |D^\alpha\{e^{i\rho w}(\mathcal{L}^2 + \mathcal{M} - \rho\sigma - \sum_{j=1}^{J'-2} \rho^{-j}\mathcal{Q}_j)h\}| +$$

$$+ C'\rho^{-J} \sup_{V_\zeta} \sum_{|\alpha|\leq M'} \rho^{-|\alpha|/2} \{|D^\alpha(e^{i\rho w_2}h)| + |e^{i\rho w_2} D^\alpha h|\} +$$

$$+ C'' \sup\left\{(|y| + |t|)^{J''} \sum_{\substack{|\alpha|\leq M \\ |\beta|\leq J'}} \rho^{m-|\beta|} |D^\alpha(e^{i\rho w} D^\beta h)|\right\}.$$

In (10.2) V_ζ denotes the support of ζ. We have the right to take it as small as we wish but not however, to shrink it infinitely many times. In particular, its diameter should not go to zero as $\rho \nearrow +\infty$.

We show now how the analytic function h is chosen. We take

$$h = \sum_{j=0}^{J'''} \rho^{-j} h_j, \tag{10.3}$$

where J''' is a large positive integer to be eventually chosen and where $h_j = h_j(y, t, \rho)$ are solutions of the following equations

$$\mathcal{L}^2 h_j + \mathcal{M} h_j - \rho\sigma h_j = \sum_{j'=0}^{j-1} \mathcal{Q}_{j-j'} h_{j'} \tag{10.4}$$

with the agreement that the right-hand side in (10.4) is identically zero when $j = 0$ and that $\mathcal{Q}_{j-j'} = 0$ if $j - j' > J' - 2$. We shall further require

$$h_0 = 1,\ h_j = 0 \textit{ for } j > 0 \textit{ at } y = 0,\ t = 0. \tag{10.5}$$

It is easily checked that

$$\left\{\mathcal{L}^2 + \mathcal{M} - \rho\sigma - \sum_{j=1}^{J'-2} \rho^{-j}\mathcal{Q}_j\right\} h = -\sum_{\substack{j=0 \\ j+j'>J'''}}^{J'''} \sum_{j'=1}^{J'-2} \rho^{-j-j'} \mathcal{Q}_{j'} h_j. \tag{10.6}$$

We shall repeat an argument already used in Section 8.

Since all the coefficients in Eq. (10.4) are analytic functions of y, t and since we are seeking analytic solutions h_j, we have the right to (and shall) consider t and the y^j as *complex* variables, varying in an open neighborhood U^C of the origin in C^{n+1}. We can then make a holomorphic change of variables (y, t) into (z, t) such that, in the new coordinates, the differential operator $\mathcal{L}$, defined in (6.26), becomes D_t. The equations (10.4) become :

$$D_t^2 h_j + a(z, t) D_t h_j - \rho\sigma(z, t) h_j + \Phi(z, t, D_z) h_j = \sum_{j'=0}^{j-1} {}_{j-j'} h_{j'}, \tag{10.7}$$

where $\Phi(z, t, D_z)$ is a first-order differential operator in the z-variables whose coefficients, like all coefficients in (10.7), are holomorphic functions of (z, t) in U^C. In order to fulfill the requirement (10.5) we shall require :

$$h_0 = 1,\ h_j = 0 \textit{ for } j > 0 \textit{ when } t = 0. \tag{10.8}$$

We shall also impose the additional "initial" condition :

$$D_t h_j = 0 \textit{ for all } j = 0, 1, \ldots, J''', \textit{ when } t = 0. \tag{10.9}$$

For each j we have a noncharacteristic Cauchy problem, depending on the solutions of the same problem for $j' < j$. By the theorem of Cauchy-Kovalevska we know that there is a unique solution h_j, holomorphic with respect to (z, t) in a neighborhood of the origin (which we may assume to be U^C possibly after shrinking the latter). We recall that h_j depends on ρ. We are interested in getting an estimate for the h_j and their derivatives when ρ approaches $+\infty$. In view of thist we have the right to simplify somehow Eq. (10.7). Indeed, if we replace h_j by

$$h_j \exp\left\{ -\frac{1}{2} \int_0^t a(z, t')\, dt' \right\}, \tag{10.10}$$

the functions (10.10) satisfy the same equations (10.7), (10.8), (10.9). except that the coefficient $a(z, t)$ must be replaced by zero and that inessential modifications must be made in $\Phi(z, t, D_z)$ and in the $\mathfrak{Q}_{j-j'}$. For the sake of simplicity, we shall heretofore assume that $a(z, t) \equiv 0$.

Now, however, (in contrast to what happened in Section 8) we shall need an estimate of $|y - z|$. Consider the equations :

$$\frac{dy}{dt} = - it^k \beta_\eta(y, t, w_y(y, t)), \; y|_{t=0} = z. \tag{10.11}$$

They have a unique solution $y = y(z, t)$, holomorphic with respect to z, t in a neighborhood of the origin in $\mathbf{C}^{n+1}$. For $|t|$ small, the Jacobian matrix of the y's with respect to the z's is close to the identity and therefore $z \to y$ defines a holomorphic diffeomorphism. Let $z = z(y, t)$ denote the inverse diffeomorphism. Then $(y, t) \to (z, t)$ can be chosen as the change of variables transforming $\mathcal{L}$ into D_t. Since $y - z = - i \int_0^t \beta_\eta(y, s, w_y(y, s))\, s^k\, ds$, we obtain at once

$$|y - z| \leqslant \text{const. } t^{k+1}. \tag{10.12}$$

In (10.7) we have written $\sigma(z, t)$ where we should have written $\sigma(y(z, t), t)$. We know, by (6.37), that

$$|\sigma(y, t) - t^q \sigma_0(t) - t^q \sigma_1(t) \cdot y| \leqslant \text{const. } (|y|^2 + t^{2p}).$$

We recall that $k = 2p + 1$. Consequently, if we set

$$\sigma_2(z, t) = \sigma(y(z, t), t) - t^q \sigma_0(t) - t^q \sigma_1(t) \cdot z, \tag{10.13}$$

and take (10.12) into account, we reach the conclusion that

$$|\sigma_2(z, t)| \leqq \text{const. } (|z|^2 + |t|^{2p}). \tag{10.14}$$

From here on, unless otherwise specified, we return to the notation $\sigma(z, t)$ (incidentally, note that the notation (10.13) is at odds with the notation (7.5)-- we shall not use the latter in the remainder of the article. We rewrite Eq. (10.7) as follows :

$$\partial_t^2 h_j + \rho t^q(\sigma_0(t) + \sigma_1(t) \cdot z) h_j = \Phi(z, t, \mathrm{D}_z) h_j - \rho \sigma_2(z, t) h_j - \sum_{j'=0}^{j-1} \mathcal{Q}_{j-j'} h_{j'}. \tag{10.15}$$

As we said, we shall work in the present and in the following section under Hypothesis (7.2). We shall therefore be in a position to apply Lemma 7.2. We shall subdivide the study into two parts, according to whether (7.8) or (7.9) holds.

11. Situations in which the lower-order terms have little influence : estimate of the amplitude function.

This section is devoted to obtaining an estimate for the h_j and their derivatives when ρ approaches $+\infty$. We shall reason under Condition (7.9). If instead, Condition (7.8) holds, it is not difficult to see that the method described below also applies, leading to the same results (in particular, when $\sigma \equiv 0$ or $q = +\infty$, the argument is even simpler). We recall that (7.9) implies that

$$\sigma(z, t) = O(|z|^2 + |t|^{2r}). \tag{11.1}$$

We start analyzing some of the ingredients in Eq. (10.15). Let us write

$$\Phi(z, t, D_z) = A(z, t) \cdot D_z + A_0(z, t) \tag{11.2}$$

where the C^n valued function A and the C valued function A_0 are holomorphic in both variables in U^C. We recall that $A(z, t)$ is simply the "coefficient" of D_z in the differential operator $\mathfrak{M}$ (defined in (6.27)) in the new coordinates (z, t). Consequently, up to the non-vanishing factor Q, it is equal to the coefficient of D_z in the expression of the third invariant $\mathcal{R}_2^w$. A comparison between (2.5) and (2.6) shows that the latter is equal to

$$\mathfrak{M}^\natural_\zeta(z, t, w^\natural_z, w^\natural_t) \cdot D_z$$

where $\mathfrak{M}^\natural$ is the subprincipal part of P (see $(2.5)_{***}$), $\zeta \in C_n$ is the z-covariable and $w(z, t) = w(y(z, t), t)$. Notice that (6.1), (6.2) and (10.11) imply at once that

$$w^\natural(z, t) = \langle \eta^2, z \rangle + i|z|^2/2, \tag{11.3}$$

is independent of t.

If we apply (5.11) and (5.12) (and take into account the fact that $q = 2r < 2p = k - 1$, we see that

$$\mathfrak{M}(y, t, \eta, 0) = t^{2r}\, \tilde{\psi}(y, t, \eta) \quad (\text{in } \mathcal{U}'') \tag{11.4}$$

(When Hypothesis (7.8) replaces (7.9), we may replace $2r$ by $2p$.) It is very easy to derive from (11.4) that

$$\mathfrak{M}_\zeta(0\,,t\,,\eta^2\,,0) = O(|t|^r) \tag{11.5}$$

which implies at once that

$$\mathfrak{M}_\zeta(z\,,t\,,w_z\,,w_t) = O\,(|z| + |t|^r). \tag{11.6}$$

From (11.6) we then obtain :

$$A(z\,,t) = O\,(|z| + |t|^r). \tag{11.7}$$

(Again, when (7.8) replaces (7.9), we may take p instead of r). We begin by estimating h_0. Write

$$\psi = h_0. \tag{11.8}$$

The equations (10.15), (10.8) and (10.9) give :

$$\psi_{tt} + \rho t^q(\sigma_0(t) + \sigma_1(t)\cdot z)\ \psi - A(z\,,t)\cdot\psi_z - A_0(z\,,t)\,\psi + \rho\sigma_2(z\,,t)\,\psi = 0 \tag{11.9}$$

$$\psi = 1,\ \psi_t = 0 \textit{ at } t = 0. \tag{11.10}$$

Consider the solution $g(t)$ of the following Cauchy problem :

$$g_{tt} - (\operatorname{Re} A_0(z\,,t))\,g = 0,\ g(0) = 1,\ g_t(0) = 0. \tag{11.11}$$

Of course, g is analytic and real-valued in some interval $|t| < T$. We may evan choose T small enough so that $g > 1/2$ in $]-T\,,T[$. Let us thus then set $\psi = gf$. We have :

$$\psi_{tt} - (\operatorname{Re} A_0(z\,,t))\,\psi = gf_{tt} + 2g_tf_t.$$

Let us make the change of variables

$$s = \int_0^t \frac{dt'}{g(t')^2},\ |t| < T,$$

which implies :

$$\psi_{tt} - (\operatorname{Re} A_0(z\,,t))\,\psi = g^{-3}f_{ss}.$$

Since all our conditions are invariant under real analytic changes of variables, it means that we may assume that

$A_0(z\,,t)$ *is purely imaginary* (possibly zero) *for* $z\,,t$ *real.* (11.12)

We shall adapt the proof of the Ovcyannikov theorem (see [8], Section 1) and apply it to the successive Cauchy problems :

$$\psi^j_{tt} + \{\rho t^q \sigma_0(t) - A_0(z, t)\} \psi^j + \rho |z|^2 \psi^j = \tag{11.13}$$

$$= A(z, t) \cdot \psi^{j-1}_z - \rho t^q \sigma_1(t) \cdot z \psi^{j-1} + \lambda_1 \psi^{j-1} + \lambda_0 \psi^j ;$$

$$\psi^j = 1, \ \psi^j_t = 0 \ at \ t = 0, \tag{11.14}$$

where by (10.14), we write $\rho \sigma_2(z, t) = \rho |z|^2 - \lambda_1 - \lambda_0$,

$$\lambda_1 = O(\rho |z|^2) \ and \ \lambda_0 = O(\rho |t|^{2p}). \qquad (11.14)_*$$

(When working under Hypothesis (7.8) we may simply replace $\rho t^q \sigma_0(t)$ in the left-hand side of (11.13) by ρt^{2p}). We shall tacitly agree that $\psi^j \equiv 0$ when $j < 0$. Let us set $C(z, t, \rho) = \rho t^q \sigma_0(t) - A_0(z, t)$.

We shall consider Cauchy problems of the following kind :

$$u_{tt} + \{C(z, t, \rho) + \rho |z|^2\} u = f(z, t, \rho) + \lambda_0 u \tag{11.15}$$

$$u = u_0, \ u_t = 0 \ at \ t = 0 \ (u_0 \text{ is either 0 or 1}). \tag{11.16}$$

In view of (7.9) and (11.12) we have

$$C(z, t, \rho) = c_q \rho t^{2r} (1 + t a(t)) + i a_0(z, t),$$

where a_0 and a are analytic functions in U^C (which for convenience is taken as $\{(z, t) \in C^{n+1}, |z| < R_0, |t| < T\}$) and $\{t \in C, |t| < T\}$ respectively ; a_0 is *real* but a is not necessarily so. Let us write

$$C(z, t, \rho) = C_0(t, \rho) + i C_1(z, t, \rho).$$

Note that, in virtue of (7.9),

$$C_0(t, \rho) = \rho c_q t^{2r} (1 + O(t)),$$

$$C_1(z, t, \rho) = \rho c_1 t^{p+r} (1 + O(t)) + a_0(z, t).$$

Let us momentarily set

$$g = g(t, \rho) = (1 + C_0)^{-1/2}$$

We recall that $c_q > 0$. We multiply (11.15) by $g^2 \bar{u}_t$ and take $2\mathrm{Re}$ of both members. We obtain :

$$g^2 2\,\mathrm{Re}\; u_{tt}\bar{u}_t + 2\,\mathrm{Re}\; u\bar{u}_t + \rho\,|z|^2 g^2 2\,\mathrm{Re}\; u\bar{u}_t = \tag{11.17}$$
$$= 2\,\mathrm{Re}\,(1 - iC_1)\,g^2 u\bar{u}_t + 2\,\mathrm{Re}\; g^2 f\bar{u}_t + 2\mathrm{Re}\;\lambda_0 g^2 u\bar{u}_t.$$

We integrate both members of (11.17) from 0 to t. We obtain :

$$|gu_t|^2 + |u|^2 + \rho\,|zgu|^2 - \int_0^t |u_t|^2\,(g^2)'\,dt' -$$
$$-\rho\,|z|^2 \int_0^t ||u|^2 (g^2)'\,dt' \leqq u_0^2 + 2\int_0^t |(1 - iC_1)\,g^2|\,|uu_t|\,dt' +$$
$$+ 2\int_0^t |gf|\;|gu_t|\,dt' + \text{const.} \int_0^t (1 + \rho\,|t'|^{2p})\,g^2\;|uu_t|\,dt'.$$

First of all, we note that g^2 is a decreasing function of $|t|$ in a fixed (i.e., independent of ρ) interval $|t| < T$. On the other hand, if $T > 0$ is sufficiently small, we have $|t|^r g \leqq 2/\tau$ where $\tau = \rho^{1/2}$. Whence

$$|gu_t|^2 + |u|^2 + \rho\,|zgu|^2 \leqq u_0^2 + 2\int_0^t |(1 - iC_1)|\,g^2\;|uu_t|\,dt' + \tag{11.19}$$
$$+ 2\int_0^t |gf|\;|gu_t|\;dt' + \text{const.} \int_0^t (1 + \tau\,|t'|^p)^2 g^2\;|uu_t|\;dt'.$$

We have

$$g \leqq \text{const.}\;(1 + \tau\,|t|^r)^{-1} \leqq \text{const.}\; g,$$
$$1 + |C_1| \leqq \text{const.}\;(1 + \tau\,|t|^p)\;(1 + \tau\,|t|^r),$$

whence, by (11.19), since $r \leqq p$ ($r < p$ if (7.9) is hypothetized),

$$|gu_t|^2 + |u|^2 + \rho\;|zgu|^2 \leqq \tag{11.20}$$
$$u_0^2 + M_0 \int_0^t |u|\;|gu_t|\;(^1 + \tau\,|t'|^p)\,dt' + M_1 \int_0^t |gf|\;|gu_t|\,dt'.$$

1. *Estimates for* ψ^0

If $u = \psi^0$ we may take $f \equiv 0$ and $u_0 = 1$ in the above considerations. We derive at once from (11.20)

$$|g\psi_t^0|^2 + |\psi^0|^2 + \rho\;|zg\psi^0|^2 \leqq \tag{11.21}$$
$$\leqq 1 + M_2 \int_0^t (|g\psi_t^0|^2 + |\psi^0|^2 + \rho\;|zg\psi^0|^2)\;(1 + \tau\;|t'|^p)\,dt',$$

which in turn implies

$$|g\psi^0_t|^2 + |\psi^0|^2 + \rho\,|zg\psi^0|^2 \leqq \exp\left(M_2 \int_0^t (1 + \tau\,|t'|^p)\,dt'\right)$$
$$\leqq M_3 \exp(M_4 \tau\,|t|^{p+1}).$$

2. *Estimates for* ψ^j, $j > 0$.

In this case we take $u = u^j = \psi^j - \psi^{j-1}$ in (11.15) - (11.16) and $u_0 = 0$,

$$f(z, t, \rho) = A(z, t) \cdot u^{j-1}_z - \rho t^q \sigma_1(t) \cdot z u^{j-1} + \lambda_1 u^{j-1}. \tag{11.23}$$

We derive from (11.20) :

$$w_j^2 \leqq M_5 \int_0^t \{|u^j|\,(1 + \tau\,|t'|^p) + |gf|\}\, w_j\,dt', \tag{11.24}$$

where

$$w_j(z, t, \rho) = \sup_{0 \leqq t' \leqq t} \{(|gu^j_t|^2 + |u^j|^2 + \rho\,|zgu^j|^2)^{1/2}\,(z, t', \rho)\}.$$

Thus (11.24) implies

$$w_j \leqq M_5 \int_0^t \{|u^j|\,(1 + \tau\,|t'|^p) + |gf|\}\,dt',$$

and finally, by (11.7), $(11.14)_*$, (11.23) and the fact that $|u^j| \leqq w_j$,

$$w_j \leqq M_6 \int_0^t w_j (1 + \tau\,|t'|^p)\,dt' +$$
$$M_7 \{|a| \int_0^t |g|\,|u^{j-1}_z|\,dt' + \frac{1}{\tau} \int_0^t |u^{j-1}_z|\,dt' + \tau\,|z| \int_0^t w_{j-1}\,dt'\}. \tag{11.25}$$

Let us set

$$\Omega(t, \rho) = M_6 \int_0^t (1 + \tau\,|t'|^p)\,dt'. \tag{11.26}$$

Then (11.25) can be rewritten, in short,

$$w_j \leqq \int_0^t w_j \Omega'\,dt' + \int_0^t F\,dt'.$$

Let then w be the solution of $w = \int_0^t w\Omega' dt' + \int_0^t F dt'$. We must have $w_j \leqq w$ (recall that $w_j \geqq 0$). But we know the exact expression of w :

$$w(t) = \int_0^t e^{\Omega(t) - \Omega(t')} F(t')\,dt',$$

whence :

$$v^j(z, t) \leqq M_7 \int_0^t \{ |z| \, e^{-\Omega(t')} |g u_z^{j-1}| + \qquad (11.27)$$

$$+ \tau^{-1} e^{-\Omega(t')} |u_z^{j-1}| + \tau |z| v^{j-1} \} \, dt'$$

where we have set

$$v^j(z, t) = e^{-\Omega(t)} w_j(z, t). \qquad (11.28)$$

We apply Cauchy's inequality :

$$|u_z^{j-1}(z, t)| \leqq \frac{1}{r_j} \sup_{|z-z'|<r_j} |u^{j-1}(z', t)|. \qquad (11.29)$$

As a matter of fact, let us set (for $R < R_0$) :

$$V^j(R, t) = \sup_{|z|<R} v^j(z, t). \qquad (11.30)$$

We derive from (11.27) :

$$V^j(R, t) \leqq M_8 \left(\frac{1}{\tau r_j} \frac{R}{R + r_j} + \frac{1}{\tau r_j} + \tau R \right) \int_0^t V^{j-1}(R + r_j, t')\,dt' \qquad (11.31)$$

which in turn implies :

$$V^j(R, t) \leqq M_9 \left(\frac{1}{\tau r_j} + \tau R \right) \int_0^t V^{j-1}(R + r_j, t')\,dt'. \qquad (11.32)$$

Let us set

$$V^j(R, t) = W^j(R) M_9^j t^j / j\,! \,.$$

We derive from (11.32)

$$W^j(R) \leqq \left(\frac{1}{\tau r_j} + \tau R \right) W^{j-1}(R + r_j). \qquad (11.33)$$

Let us underline the fact that the ψ^j (defined by (11.13) - (11.14)) are analytic functions of z, for $|z| < R_0$. In view of (11.22) we may take $W^0(R) = M$, a constant independent of R. We are then going to choose, for $j > 0$,

$$r_j = (\tau j)^{-1}.$$

From all this and from (11.33) we derive :

$$W^j(R) \leqq M(j + \tau R)\left(j - 1 + \tau R + \frac{1}{j}\right)\left(j - 2 + \tau R + \frac{1}{j} + \frac{1}{j-1}\right). \tag{11.34}$$

Let us set $S = \sup \frac{1}{j}\left(1 + \frac{1}{2} + \ldots + \frac{1}{j}\right)$. Then :

$$W^j(R) \leqq Mj^j\left(1 + S + \frac{\tau R}{j}\right)^j \leqq M(1 + S)^j j^j e^{(1+S)^{-1}\tau R}. \tag{11.35}$$

Since $j^j \leqq$ const. $3^j j!$, we find

$$V^j(R, t) \leqq M_{10} M_{11}^j |t|^j \exp(M_{12}\tau R)$$

If we go back to the definition of $V^j(R, t)$ and to that of $v^j(z, t)$ we obtain

$$|u^j(z, t)| \leqq w_j(z, t) \leqq M_{10}(M_{11}|t|)^j \exp(M_{12}\tau R + \Omega(t)), \tag{11.36}$$

$$|z| < R < R_0, \ |t| < T.$$

It is important to observe that the constants M_{10}, M_{11}, M_{12} can be taken independently of R, and that $R < R_0$ can be arbitrary. We shall take $R = |z|$. On the other hand, we have, for the solution $\psi = h_0$ of (11.9) - (11.10),

$$h_0 = \lim_{j \to +\infty} \psi^j = \sum_{j=0}^{+\infty} u^j.$$

We derive from (11.36) :

$$|h_0(z, t)| \leqq M_{10}(1 - M_{11}|t|)^{-1} \exp(M_{12}\tau |z| + \Omega(t)). \tag{11.37}$$

If we choose T small enough we obtain :

$$|h_0(z, t)| \leqq \text{const.} \exp(M_{12}\tau |z| + \Omega(t)). \tag{11.38}$$

It is not difficult to show that, for every $j = 0, 1, \ldots,$

$$|D_z^\alpha D_t^{\alpha_0} h_j(z, t)| \leqq C_{j,\alpha,\alpha_0}\, \rho^{(j+\alpha_0)/2} \exp(M_{12}\tau |z| + \Omega(t)) \tag{11.39}$$

$$|z| < R_0, \ |t| < T.$$

Let us indicate how to prove (11.39) when $j = 0$. For $\alpha_0 = 0$ or 1, the combination of the method described above with the Cauchy's inequalities in the z-variables gives the result. For $\alpha_0 > 1$ it suffices to use the differential equation (10.15), differentiated as many times as we wish with respect to both z and t.

From there on we use induction on $j = 0, 1, \ldots$ (note that the differential operator $\mathcal{Q}_{j-j'}$ entering in Eq. (10.15) is of order $j - j' + 2$). We essentially repeat the argument of the previous pages, in particular, we avail ourselves of Estimate (11.20), but with a different choice of the datum f -- as dictated by Eq. (10.15). Finally, if we revert to the (y, t) coordinates, it is very easy to show that (see (10.11) and (11.26)) :

$$|D_y^\alpha D_t^{\alpha_0} h_j(y, t)| \leqq C'_{j,\alpha,\alpha_0}\, \rho^{(j+\alpha_0)/2} \exp(M_{12}\tau\, |y| + \Omega(t)), \qquad (11.40)$$

$$|y| < R'_0, \ |t| < T_0.$$

12. End of the proof of theorem 1.1 when the lower-order terms have little influence.

We return to inequality (10.2) and show that it leads to a contradiction.

First of all, we choose the support V_ζ of ζ contained in the intersection of $\mathbf{R}^{n+1}$ with the set $\{(y, t) \in \mathbf{C}^{n+1} \ ; |y| < R'_0, |t| < T_0\}$. This insures that the inequalities (11.40) are valid for all $j = 0, 1, \ldots, J'''$ and all $\rho > 1$, when $(y, t) \in V_\zeta$. Next we determine an upper bound for the *first* term in the right-hand side of (10.2). We apply (9.14) (in V_ζ) with H equal to the left-hand side in (10.6). Let us keep in mind that the order of $\mathcal{Q}_{j'}$ is $j' + 2$. By virtue of (11.40) we have

$$\sum_{|\alpha| \leqq M} \rho^{-|\alpha|} \left| e^{i\rho w_2} D^\alpha \left\{ \sum_{j,j'} \rho^{-j-j'} \mathcal{Q}_{j'} h_j \right\} \right| \leqq$$

$$\leqq \text{const.}\ \rho^{1-(j+j')/2}\, e^{-\rho \operatorname{Im} w_2 + M_{12}\tau|y| + \Omega(t)} \qquad (12.1)$$

The summation in the left-hand side of (12.1) ranges over the same indices $j,\ j'$ as the one in the right-hand side of (10.6). We reach easily the conclusion that the first term, in the right-hand side of (10.2), does not exceed a constant times

$$\rho^{m+M-1-J'''/2} \sup_{V_\zeta} \{e^{-\rho \operatorname{Im} w_2 + M_{12}\tau|y| + \Omega(t)}\}. \tag{12.2}$$

We seek now an upper bound for the second term in the right-hand side of (10.2). Here as in Section 9, the argument is somewhat subtler, due to the fact that we have no control over the order of differentiation M'. We see easily (cf. estimate of (9.8)) that the second term in question does not exceed a constant times

$$\rho^{-J} \sup\,(1 + \tau\,|\partial w_2|)^{M'} e^{-\rho \operatorname{Im} w_2} \sum_{|\beta| \leqq M'} \rho^{-|\beta|/2}\,|D^\beta h|. \tag{12.3}$$

If we recall that h is given by (10.3) and take (11.40) into account, we reach the conclusion that (12.3) is bounded by a constant times

$$\rho^{-J} \sup_{V_\zeta} \{(1 + \tau\,|\partial w_2|)^{M'} e^{-\rho \operatorname{Im} w_2 + M_{12}\tau|y| + \Omega(t)}\}. \tag{12.4}$$

We seek now an upper bound for the third term in the right-hand side of (10.2). We begin by applying (9.14) (in V_ζ) with H equal to $D^\beta h$. We see thus that the third term in question does not exceed a constant times

$$\rho^{m+M} \sup_{V_\zeta}\Big\{(|y| + |t|)^{J'''} \sum_{|\alpha| \leqq M+J'} \rho^{-|\alpha|}\,|D^\alpha h|\Big\}. \tag{12.5}$$

If we combine this with (10.3) and (11.40) we see that (12.5) is bounded by a constant times

$$\rho^{m+M} \sup_{V_\zeta} \{(|y| + |t|)^{J''} e^{-\rho \operatorname{Im} w_2 + M_{12}\tau|y| + \Omega(t)}\}. \tag{12.6}$$

Finally, we seek a common upper bound for (12.2), (12.4) and (12.6). The basic fact is that the phase function w, or the function w_2 if one prefers, satisfies the inequality (6.18), which means that :

(12.7) *if* V_ζ *is sufficiently small, there is a constant* $c > 0$ *such that, for all* $(y\,,\,t) \in V_\zeta$,

$$2c\,(|y|^2 + t^{k+1}) \leqq \operatorname{Im} w_2(y\,,\,t). \tag{12.8}$$

On the other hand, it is clear that for a suitable choice of $C_0 > 0$ we have (see (11.26) and recall that $k = 2p + 1$) :

$$M_{12}\tau\,|y| + \Omega(t) \leqq c\rho\,(|y|^2 + t^{k+1}) + C_0 \tag{12.9}$$

whence, by combining (12.8) and (12.9),

$$c\rho\,(|y|^2 + t^{k+1}) - C_0 \leqq \rho\,\mathrm{Im}\,w_2 - M_{12}\tau\,|y| - \Omega(t). \tag{12.10}$$

This implies at once that (12.2) is $\leqq \rho^{m+M-1-J'''/2}$. In what concerns (12.4) we observe that (see (6.21) and (12.8))

$$|\partial w_2| \leqq \text{const.}\,(|y| + |t|^{(k+1)/2}), \tag{12.11}$$

and consequently,

$$(1 + \tau\,|\partial w_2|)^{M'} e^{-c\rho(|y|^2+t^{k+1})} \leqq \text{const.} \tag{12.12}$$

whence it follows that (12.4) is $\leqq$ const. ρ^{-J}. By the same token we have

$$(|y| + |t|)^{J''}\, e^{-c\rho(|y|^2+t^{k+1})} \leqq \text{const.}\ \rho^{-J''/(k+1)}, \tag{12.13}$$

which implies at once that (12.6) is $\leqq$const. $\rho^{m+M-J''/(k+1)}$. Therefore, requiring that both J'' and J''' be $\geqq (k+1)(J+M+m)$ insures that the right-hand side of (10.2) will be $\leqq$ const. ρ^{-J}. Recalling that

$$v = \zeta h e^{i\rho w},$$

we have deduced from (10.2) that

$$|\textstyle\iint f v\, dy dt| \leqq \text{const.}\ \rho^{-J} \sup_{|\alpha|\leqq M} |D^\alpha f|. \tag{12.14}$$

We may now conclude the argument exactly as in the proof of Th. 6.1.1, [2], by choosing

$$f(y, t) = \rho^{n+1} F(\rho y, \rho t),$$

where $F \in C_c^\infty(\mathbf{R}^{n+1})$ satisfies :

$$\iint e^{i<\eta^2, y>} F(y, t)\, dy dt = 1. \tag{12.15}$$

(Observe that, for ρ large enough, the support of f will lie in V_ζ). Then, as $\rho \to +\infty$, the left-hand side of (12.14) converges to 1 (by virtue of (6.18)), whereas its right-hand side converges to zero, as soon as

$$J > M + n + 1. \tag{12.16}$$

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Manuscrit reçu le 22 mars 1973
accepté par B. Malgrange.

Fernando CARDOSO ,
Federal University of Pernamburco
Recife (Brazil)
and
the Institute for Advance Study
Princeton, New Jersey (USA)

François TREVES ,
Rutgers University
Department of Mathematics
New Brunswick
New Jersey 08903 (USA)